Existence properties for problems with double-phase

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Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. Consider the following double-phase problem with singular reaction

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = a(z)u(z)^{-\eta} + f(z, u(z)) \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = 0,\ 1 < q \leqslant p,\ 0 < \eta < 1,\ u > 0. \end{cases} \tag{1}$$

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For every $r \in (1, \infty)$, we denote by Δ_r the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}\left(|Du|^{r-2} Du\right) \text{ for all } u \in W_0^{1,r}(\Omega).$$

Now we introduce the hypotheses on the data of problem (1).

H_0: $a \in C_0^1(\overline{\Omega})$, $a(z) > 0$ for all $z \in \Omega$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant a(z)(1 + x^{p-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$;

(ii) $\hat{\lambda}_1(p) \leqslant \liminf_{x \to +\infty} \frac{f(z,x)}{x^{p-1}}$ uniformly for a.a. $z \in \Omega$;

(iii) if $F(z,x) = \int_0^x f(z,x)ds$ then there exists $\tau \in (q,p)$ such that

$$0 < \beta_0 \leqslant \liminf_{x \to +\infty} \frac{pF(z,x) - f(z,x)x}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist $\mu \in (1,q)$ and $\delta, \vartheta > 0$ such that

$$C_0 x^\mu \leqslant f(z,x)x \leqslant \mu F(z,x) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant \delta, \text{ some } C_0 > 0,$$

$$a(z)\vartheta^{-\eta} + f(z,\vartheta) \leqslant -\hat{C} < 0 \text{ for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

By a solution of problem (1), we mean a function $u \in W_0^{1,p}(\Omega)$ such that $u^{-\eta}h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and

$$\langle A_p(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega \left[a(z)u^{-\eta} + f(z,u)\right] h dz \text{ for all } h \in W_0^{1,p}(\Omega).$$

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \frac{1}{1-\eta}\int_\Omega a(z)(u^+)^{1-\eta}dz - \int_\Omega F(z,u^+)dz$$

for all $u \in W_0^{1,p}(\Omega)$.
The presence of the singular term implies that $\varphi(\cdot)$ is not C^1 and so we cannot use the results of critical point theory directly on this functional. We need to find ways to bypass the singularity and deal with a C^1-functional.

The main result establishes the following multiplicity property for problem (1).

Theorem

If hypotheses H_0, H_1 hold, then problem (1) *admits at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, u_0 \neq \hat{u}, u_0(z) < \vartheta \textit{ for all } z \in \overline{\Omega}.$$

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This is a joint paper with Professor Nikolaos S. Papageorgiou and Professor Vicenţiu Rădulescu.