Linear Stochastic Processes with Multiplicative Noise

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Affine evolution equations with random coefficients

- Discrete time random affine processes used in image compression
- Example: the stationary distribution function of the simplest one dimensional processes generate complex fractal structures
- The stationary PDF of general affine processes has heavy tail asymptotic form

References

- Stochastic processes with heavy tail (asymptotic self similarity)

  - 1. H. Takayasu et all, P.R.L. **79**, 966 (1997) (discrete time,1 white multiplicative and 1 additive noise, analytic)
  - 2.N. Fuchikami, PRE, **61**, 1060 (1999). ()
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  - 4. Tian-You Hu, Ka-Sing Lau, Adv. App. Math **27,** 1, (2001).(Simplest possible discrete time linear process, w. noise add. Term, constant multiplicative term)
  - 5.G. Steinbrecher, B. Weyssow, PRL. **92,** 125003, (2004).(arbitrary number of additive and multiplicative colored noise, limiting cases: white noise, fractional Brownian Motion)
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Random linear (RL) models, discrete time

- Colored noise: fluctuations near stable equilibrium.
- Modelling instabilities: randomise the multiplicative term.
- Discrete time 1d colored noise=fractal equilibrium PDF [ref.4] Higher dimensions: more complex. The continuous time models are more easy.
- Heavy tail in discrete time models [ref. 1,2] with noisy in additive and multiplicative terms. Analytic results only for white noise .

Unexpected complexity of random linear (RL) models, simple example:

- The stationary PDF is the Cantor set for this discrete time system (Tian-You Hu, Ka-Sing Lau, Adv.Appl. Math. **27** 2001, 1):

$$x_{n+1} = x_n / 3 + f_n$$

$$f_n = i.i.d;\ f_n = \{0, 2/3\}$$

$$with\ prob. = 1/2$$

Moreover: RL models with fractal stationary PDF used in "image compression":

- Iterated function systems: $\mathbf{x}(t+1)=A(t)\mathbf{x}(t)+\mathbf{b}(t)$; $A(t)$, $\mathbf{b}(t)$ stationary random sequences, $\mathbf{x}(t)$ =>fractal images .
- J. E. Hutchinson, Indiana Univ. Math. J. **30** (1981) 713.
- M. Barnsley, "Fractals everywhere", Academic Press, Boston MA, 1993.

Discrete time RL models: condensed matter, SOC physics.

- H. Takayasu, Phys. Rev. Lett. 63, 2563 (1989). H. Takayasu, &all A-H. Sato, and M. Takayasu, Phys. Rev. Lett. 79, 966 (1997); A-H. Sato, H. Takayasu, and Y. Sawada, Phys. Rev. E 61, 1081 (2000).
- Explains the heavy tail of stationary PDF, by RW on the 1d affine group.

RL models of complex systems.

- Classics: 1-d Black-Scholes model, Nobel prize in economy. [B. Øksendal, Stochastic Differential Equations, Springer, 2000].
- “Auto-Regressive” models:
- Sato, A.-H., Phys. Rev. **E 69**, 047101-1 - 047101-4, (2004).
- D. Nagakura, Statistics and Probability Letters, **79,** 2467-2483, (2009)

Higher dimensions, “free, (non commutative) probability”.

- H. Furstenberg, Non-Commuting random products, Trans. Amer. Math. Soc. 108, (1963), 193-229.
- A. Brandt, The stochastic equation Y_{n+1}=A_{n}Y_{n}+B_{n} with stationary coefficients. Adv. Appl. Prob. 18, (1986),211-220.
- H. Kesten, Random Difference Equations and Renewal Theory for Products of Random Matrices. Acta Math. 131, (1973), 207-248.
- E. Le Page, Théorèmes de renouvellement pour les produits de matrices aléatoires. Equations aux différentes aléatoires. Séminaires des probabilités, Rennes, 1983.

Random linear models. Continuous time, white noise

- Black-Scholes: dx=[-a*dt+b*dw(t)]x(t), used in economy; $a>0$. No stationary PDF.
- With additive independent white noise term, stationary PDF with algebraic decay. Simple calculations.
- Problem: effect of small intensity noise with large correlation time (colored noise)?

Random linear model, continuous time, colored noise

- The process x(t) given by linear SDE, randomised O.-U. process.[Model of boiling water fission reactor]
- Ref: G. Steinbrecher, B. Weyssow. PRL, **92,** 125003, (2004)
- Rand. process: dx(t)=-(a dt-dY(t)) x(t)+dU(t)+e dt.

$$N \text{ colored noise, } y_1(t),\ldots,y_N(t) \quad \text{corelation time} \quad \tau_i; \quad dy_i(t) = -\frac{1}{\tau_i} y_i(t) + \sqrt{\frac{2}{\tau_i}} dw_i(t)$$

$$w_i(t) = \text{independent Wiener processes}, \; \left\langle dw_i(t) dw_j(t) \right\rangle = \delta_{i,j} dt$$

$$\text{Define}: \; Y(T) = \sum_{i=1}^{N} b_i \int_0^T y_i(t) dt \; ; U(T) = \sum_{i=1}^{N} c_i \int_0^T y_i(t) dt; \quad \text{noises}$$

$$\text{RANDOMISED} \quad O.-U. \quad \text{PROCESS}$$

$$dx = -(a\,dt - dY(t))\,x(t) + dU(t) + e\,dt; \quad a > 0; \quad e = \text{arbitrary} \quad \text{const.}$$

Stability criteria for stochastic systems

- Definitions and example of L^p stability.
- Linear SDE, L^p stability and heavy tail
- Linear Hamiltonian systems with parametric noise: Stochastic analogue of parametric resonance.
- Central Limit Theorem, case of correlated summands.

References

- 1. T. Gard, Introduction to Stochastic Differential Equations (Marcel Dekker, N.Y., 1988). (!!!)
- 1.I.I. Gikhman, A. V. Skorokhod, Introduction to the theory of random processes. Saunders, Philadelphia, 1965. French translation from Russian in Ed. Mir, Moscow.
- P. Embrechts, M. Maejima: ‘Self similar processes’, Princeton Univ. Press, 2002
- R.J. Adler, J. E. Taylor, K. J. Worsley, ‘Random Field and Geometry’, Springer, 2007.

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- 1. C. W. Gardiner, Handbook of Stochastic Methods (for Physics, Chemistry…), Springer, (2003).
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Technical details

- Theory of stochastic processes=real "New Kind of Science" .
- Computing mean values over random evolutions (e. g. turbulence driven anomalous transport), computing quark-gluon plasma in QCD, phase transitions in Equilibrium Statistical Mechanics, the same problem: integrating over infinite dimensional spaces.
- Origin: Lebesgue, Daniell integrals. (G.E. Shilow, B. L. Gurevich, Integral, Measure, Derivative, Dover, 1977) .

Main problem: stochastic linear stability study

- The parametric resonance in deterministic linear systems give rise to exponential increase of the amplitude
- The resonance effect of the additive periodic forcing is linearly increasing at most
- For linear stable systems the additive Gaussian noise has no spectacular effects: the output is Gaussian coloured noise, no heavy tail, concentrated near stable point.

L^p stability

- The effect of multiplicative (parametric) noise on the stability of the linear system, is characterized by a scale of L^p norms, not an single norm with fixed p.

Definitions: L^p spaces

Let X_ω random variable, $\rho(x)$ its PDF

$$X_\omega \in L^p \text{ iff } \left\langle \left| X_\omega \right|^p \right\rangle_\omega < \infty \quad or$$

$$\int_{-\infty}^{\infty} |x|^p \rho(x) dx < \infty;$$

$$for \;\; p \geq 1 : \left\| X_\omega \right\|_p = \left[\left\langle \left| X_\omega \right|^p \right\rangle_\omega \right]^{\frac{1}{p}}$$

$$for \;\; 0 < p \leq 1 : \left\| X_\omega \right\|_p = \left\langle \left| X_\omega \right|^p \right\rangle_\omega$$

Remark: all Lp spaces are complete metric spaces

- With this convention, $d(X,Y)=\left\|X-Y\right\|_p$ has the properties of distance, defines limit.
- The addition and scalar multiplication are continuous with respect to norm
- Al L^p spaces are complete,=> calculus can be extended, including 0<p<1.

Definitions: convergence, boundedness, probability measure

- Convergent and bounded sequences in L^p .

$$\text{Let}\,\Phi,\Psi_{\mathrm{n}}\;(1\le n<\infty)\quad \text{random variables.}$$

$$\text{Def}:\Psi_{\mathrm{n}}\overset{\mathrm{L^p}}{\longrightarrow}\Phi\quad(\text{converg. in }\mathrm{L^p})\,\text{iff}\quad(0<\mathrm{p}\le\infty)$$

$$\lim_{n\to\infty}\left\langle\left|\Psi_{\mathrm{n}}-\Phi\right|^p\right\rangle=0;\,\text{Equivalent}:$$

$$\left\|\Psi_{\mathrm{n}}-\Phi\right\|_p\to 0,\,\text{when n}\to\infty$$

$$\text{Def}:\;\Psi_{\mathrm{n}}\text{ is bounded in }\mathrm{L^p}\text{ iff for all n}$$

$$\left\|\Psi_{\mathrm{n}}\right\|_p\le M\;\text{ for some M or}$$

$$\left\langle\left|\Psi_{\mathrm{n}}\right|^p\right\rangle<M$$

Remarks

- In the framework of probability theory, the norm is defined by mean value. Then results
- If p>q, then from convergence or boundedness in L^p results the corresponding property in L^q
- Increasing ‘p’ the ‘ topology of L^p became strictly stronger’.
- The L^p spaces with 0<p<1 are very unusual for math. too, they are used for very heavy tail effects

Example:Black-Scholes model, slightly adapted

- Similar to stability problem of x=0 equilibrium

$$\text{Ito form}: \frac{dx_\omega(t)}{dt} = (\pm a + \sigma\xi_\omega(t)) \circ x_\omega(t)$$

$$dx_\omega(t) = (\pm a dt + \sigma dw_\omega(t)) \circ x_\omega(t)$$

$$\text{Instability growth perturbed by } \xi_\omega(t) = \text{white noise}$$

$$dw_\omega(t) = \xi_\omega(t)dt;\ w_\omega(t) = \text{standard Wiener process}$$

$$\text{NYSE}: '+'; \text{We use}: \text{(stability)}: '-'; a > 0$$

B-S –the solution

ITO : use familiar calculus!!

$$x_\omega(t) = x_\omega(0)\exp\left[-at + \sigma w_\omega(t)\right]$$

$$x_\omega(0) = A = \text{deterministic}$$

$$\text{Recall}: \langle \exp(i\lambda w_\omega(t)) \rangle_\omega = \exp\left(-\lambda^2 t / 2\right)$$

$$\left\langle \left| x_\omega(t) \right|^p \right\rangle_\omega = |A|^p \exp(-apt) \langle \exp(p\sigma w_\omega(t)) \rangle_\omega$$

$$i\lambda = p\sigma; \Rightarrow \left\langle \left| x_\omega(t) \right|^p \right\rangle_\omega = |A|^p \exp\left(\left(p\sigma^2 / 2 - a\right)pt\right)$$

B-S model: Critical exponent p_{crit}

$$Denote: D = \frac{\sigma^2}{2};\ p_{crit} = \frac{a}{D};\ t_n = n\Delta t$$

$$\left\langle \left| x_\omega(t_n) \right|^p \right\rangle_\omega \propto \exp\left(n\left(p - p_{crit}\right)Dp\Delta t\right)$$

$$Denote: \Psi_n = x_\omega(t_n); \text{Consider } n \to \infty$$

$$p < p_{crit} : \left\langle \left| x_\omega(t_n) \right|^p \right\rangle_\omega \to 0; \Leftrightarrow \left\| \Psi_n \right\|_p \to 0$$

$$\Leftrightarrow \Psi_n \xrightarrow{L^p} 0; \Rightarrow \mathrm{x(t)} \equiv 0 \text{ is stable in } \mathrm{L^p}$$

$$For\ \ p > p_{crit} : \left\| \Psi_n \right\|_p \to \infty$$

$$\text{The } \mathrm{x(t)} \equiv 0 \text{ solution is unstable in } \mathrm{L^p}$$

Linear SDE, L^p stability and heavy tail

- Generalization to 1 dimensional SDE.
- 1 d, soluble model with heavy tail.
- Heavy tail and critical index
- Relation to noise-driven intermittency
- Higher dimensional linear SDE.

Approximation of the nonlinear problem:

- Study of the linear stability of the $u(\mathbf{r},t)=const$ solution, with coupling to rapidly varying fields.
- Approximate the rapid varying fields with their time average + random field
- Improvement to the classical averaging method (Gauss principle,…)

Generalization of B-S model 1

In the BS model, for the multiplicative process

$\mathrm{Y}_{\omega}(\mathrm{t}) = \sigma\, w_{\omega}(t)$; Because $\left\langle \left[\mathrm{w}_{\omega}(t)\right]^{2} \right\rangle_{\omega} = t$

we have $\left\langle \left[\mathrm{Y}_{\omega}(t)\right]^{2} \right\rangle_{\omega} = 2Dt$

Consider a more general driving process:

Exists D = 'diffusion coefficient' s.t.

$$\left\langle \left[\mathrm{Y}_{\omega}(t)\right]^{2} \right\rangle_{\omega} \underset{t\to\infty}{=} 2Dt + o(t)$$

Generalization of B-S model 2

- G. Steinbrecher, B. Weyssow, Phys. Rev. Lett. **92**, 125003 (2004): Linear equation, driving noises from sum of colored noise
- G. Steinbrecher, X. Garbet, B. Weyssow, “Large time behavior in random multiplicative processes”, arXiv:1007.0952v1, math.PR, (2010): non linear additive term, general noise.

Generalized B-S model: p_{crit}=a/D

Consider a more general SDE

$$dx_\omega(t) = (-adt + dY_\omega(t))x_\omega(t) + U_\omega(t)dt$$

$$a = \text{instability thereshold}$$

$$Y_\omega(t) = \text{parametric, multiplicative noise}$$

$$U_\omega(t) = \text{ additive noise.}$$

$$\text{Result: Exists L}^{\text{p}}\text{ limit: } Z_\omega, \text{for p} < \text{p}_{\text{crit}} = \frac{\text{a}}{\text{D}}:$$

$$\left\langle \left| x_\omega(t) - Z_\omega \right|^p \right\rangle_\omega \propto \exp\left(\left(p - p_{crit}\right)Dpt\right)$$

Remark

- The rate of convergence/divergence is exponential
- For high p no convergence
- Additive noise has no influence on p_{crit} .
- For low ‘a’ or high D, convergence only in the exotic L^p spaces.’Very heavy tail’.
- Edge turbulence on DIII-D: p_{crit} <0.1.(Steinbrecher, Weyssow, 2004)

The heavy tail of limit PDF

$$\text{Let } \rho_\infty(x) \text{ the PDF of the limit RV} : Z_\omega$$

$$\text{prob}\left(x \le Z_\omega \le x + dx\right) = \rho_\infty(x)dx$$

$$\left\langle \left|Z_\omega\right|^p \right\rangle_\omega = \int_{-\infty}^{\infty} |x|^p \rho_\infty(x)dx < \infty \text{ IFF } p < p_{\text{crit}}$$

$$\text{When } p > p_{\text{crit}} \quad \left\langle \left|x_\omega(t)\right|^p \right\rangle_\omega \to \infty$$

$$\text{We have, for } |x| \to \infty : \ \rho_\infty(x) \propto |x|^{-1-\beta}$$

$$\text{HT exponent} = \beta = p_{crit} = \frac{a}{D}$$

Example 1

$$dx(t) = (-adt + \sigma_1 dw_1(t)) \circ x(t) + \sigma_2 dw_2(t);$$

$w_1(t),\ w_2(t)$: independent, Wiener .

ITO form. Notation : $a' = a - \sigma_1^2 / 2$

$$dx = (-a' dt + \sigma_1 dw_1(t)) x(t) + \sigma_2 dw_2(t);$$

Addition of independent normal variables

$$\sigma_1 x dw_1(t) + \sigma_2 dw_2(t) \overset{distribution}{=} \sigma(x) dw(t)$$

$$\sigma(x)^2 = (\sigma_1 x)^2 + (\sigma_2)^2; \text{ (Errors, Gauss)}$$

$$dx(t) = V(x,t)dt + \sigma(x)dw(t)$$

$$\text{Fokker – Planck :}$$

$$\partial_{\mathrm{t}}\rho(x,t)+\partial_x\left[V\rho(x,t)\right]=\partial_x{}^2\left[\frac{1}{2}\sigma^2\rho(x,t)\right]$$

$$V(x,t)=-a'x;$$

$$\partial_{\mathrm{t}}\rho(x,t)=0;\ \ \text{Stationary solution}$$

$$\rho(\mathrm{x})=\frac{\mathrm{const}}{\left(\sigma_1{}^2x^2+\sigma_2{}^2\right)^{\frac{a}{2D}}}\approx|x|^{-p_{crit}}$$

Higher dimensional linear stochastic differential equation SDE

- Analytic results: only for white noise.
- It is possible to compute the L^p norms only for even values of p
- It is possible to obtain closed set of linear equations for correlation functions of any order.
- Second order correlation functions physically interesting

K. Ito formula, Wiener process

ITO SDE

$$dx_{\omega,i}(t) = V_i(\mathbf{x}_\omega(t),t)\,dt + \sum_{a=1}^{A} \sigma_{i,a}(\mathbf{x}_\omega(t),t)\,dw_{\omega,a}(t)$$

From ITO formula results:

$$\frac{d}{dt}\left\langle f(\mathbf{x}_\omega(t),t)\right\rangle_\varpi = \left\langle \frac{\partial f}{\partial t} + \sum_{k=1}^{N} \frac{\partial f}{\partial x_k} V_k(\mathbf{x}_\omega,t) + \sum_{k=1}^{N}\sum_{l=1}^{N} D_{k,l}(\mathbf{x}_\omega,t)\frac{\partial^2 f}{\partial x_k \partial x_l}\right\rangle_\omega$$

$$D_{k,l}(\mathbf{x},t) = \frac{1}{2}\sum_{a=1}^{N} \sigma_{k,a}(\mathbf{x},t)\sigma_{l,a}(\mathbf{x},t)$$

Apply to linear system, N dimensions,

Einstein summation convention

The linear SDE 1

$$dx_i = L_{i,m} x_m dt + r_{i,m,a} x_m dw_{\omega,a}(t)$$

Summation with m, a

$L_{i,m}$, $r_{i,m,a}$ constants

$$\left\langle w_{\omega,a}(t) w_{\omega,b}(t') \right\rangle_{\omega} = \delta_{a,b} \min(t,t')$$

The linear SDE 2

$$\text{First order moments}:$$

$$\mathrm{f}(\mathbf{x},\mathrm{t}) \equiv \mathrm{x}_{\mathrm{k}}$$

$$\text{Denote}: \left\langle \mathrm{x}_{\omega,\mathrm{k}}(\mathrm{t}) \right\rangle_{\omega} = X_k(t);$$

$$\text{Results}:$$

$$\frac{dX_k(t)}{dt} = L_{k,m} X_m :$$

The linear SDE 3

Second order moments:

$$\mathrm{f}(\mathbf{x},\mathrm{t}) \equiv x_i x_k$$

$$\text{Denote}: \left\langle x_{\omega,i}(t) x_{\omega,k}(t) \right\rangle_{\omega} = Y_{ik}(t);$$

$$S_{i,m;k,n} = r_{i,m,a}\, r_{k,n,a}$$

Results:

$$\frac{dY_{i,k}(t)}{dt} = L_{i,m} Y_{m,k} + L_{k,n} Y_{i,n}$$

$$+ S_{i,m;k,n} Y_{m,n}$$

The linear SDE 4

Third order moments:

$$\mathrm{f}(\mathbf{x},\mathrm{t}) \equiv x_i x_j x_\mathrm{k}$$

Denote: $\left\langle x_i x_{,j} x_{,k} \right\rangle_\omega = U_{i,j,k}(t);$

Results:

$$\frac{dU_{i,j,k}(t)}{dt} = L_{i,m}U_{m,j,k} + L_{j,n}U_{i,n,k} +$$

$$L_{k,p}U_{i,j,p} + S_{i,m;j,n}U_{i,j,k} + \ldots\mathrm{perm}$$

Linear SDE driven by Markov process

Markov process, M states

$$\frac{dp_a(t)}{dt} = \sum_{k=1}^{M} R_{a,b}\, p_b(t)$$

In state 'a' *we have*

$$dx_i = L_{i,k;a} x_k$$

$$Denote: X_{i,a}(t) = \langle x_i(t) \rangle_a$$

$$Y_{i,j;a}(t) = \langle x_i(t) x_j(t) \rangle_a$$

Closed evolution equations

$$\frac{dX_{i;a}}{dt} = L_{i,k;a} X_{k;a} + R_{a,b} X_{i,b}$$

$$\frac{dY_{i,j;a}}{dt} = L_{i,k;a} Y_{k,j;a} + L_{j,k;a} Y_{i,k;a} + R_{a,b} Y_{i,j,b}$$

Example: LC circuit with random capacitance, driven by dichotomous Markov chain

$$\partial_t x_1 = x_2; \qquad \partial_t x_2 = -\omega_a^2 x_1$$

$$\partial_t p_1 = \lambda (p_2 - p_1)$$

$$\partial_t p_2 = -\lambda (p_2 - p_1)$$

Random LC circuit, results

- First order moments are bounded
- Second order moments increase exponentially in time=>energy transfer from the noise to oscillator.
- The process is universal, no fine tunning
- K. Lindenberg, V. Seshadri, B. J. West, Phys. Rev.A **22,** 2171 (1980)

Linear partial SDE

- Method: discretization, apply previous method, back to continuum limit
- Example: Stochastic destabilization of the random Klein-Gordon equation, Langmuir waves G. Steinbrecher, X. Garbet, 2009.
- Result: The lowest eigenmodes are destabilized first.
- Model for inverse cascade

The linear system results from the approximation of the nonlinear problem:

- Study of the linear stability of the u(**r**,t)=const solution, with coupling to rapidly varying fields.
- Approximate the rapid varying fields with their time average + random field
- Improvement to the classical averaging method (Gauss principle,…)

Problem: stability of Langmuir waves

- Perturbation: temporal white noise with possible spatial correlations
- Perturbation models the background charge density fluctuation of plasma near equilibrium
- Formally: Klein-Gordon equation with random mass term

Results: critical white noise intensity

- In thermodynamical limit, 3 d, non zero **k**, and small spatio-temporal white noise intensity, there exist a threshold, such that for smaller noise intensity, the mode **k** remains stable(!). (Naïve expectation=> complete destabilization)
- In all other cases (1d, 2d) all of the modes are destabilised by arbitrary weak spatial and temporal white noise.

Time evolution of the moments, only 2 order, possible for higher. Warning!

- With driving white noise, the time evolution of the integer order moments can be found, for general linear SDE .
- Even if the second order moment converges, it is possible that higher order moments diverges exponentially; S.G., B.W., 2004. Shaw, W. T., 2008.
- The critical order of diverging/converging moments is equal to the heavy tail exponent =>

Remarks

- The fundamental mode **k=0** is most sensitive to the stochastic perturbations. The eigenvalue is real and positive.
- The poles in the l.h.s of the eigenvalue equation lies on the imaginary axis. For small perturbations the eigenvalues are near poles.
- The time reversal symmetry in the eigenvalue equation is broken.
- Definition of stability: different Banach norms are inequivalents

Results in the thermodynamic limit:

- The fundamental mode **k=0 is always destabilized**, for arbitrary small noise.
- The destabilization for non zero **k** is dimension dependent.
- For d=1 & d=2, arbitrary small noise destabilizes all of the modes

New result, dimension 3. Critical noise intensity

- For non zero **k**, and small noise intensity there exist a threshold, such that, for smaller noise intensity the mode **k** remains stable.
- For small |**k**| the value of this threshold of noise intensity has the asymptotic form:

$$\sigma_{crit} = O(|\mathbf{k}|^{3/2})$$

Long range spatial correlations

- If the driving multiplicative process is the spatial and temporal white noise, with small intensity, then the long wavelength modes are destabilized first. => Long range spatial correlations dominates, if the driving noise is weak.
- This result was derived by the linear approximation.

The new effect of parametric spatial and temporal white noise:

- In 3 dim and thermodynamical limit, low noise intensity, the modes with high |**k**| **remains stable.** The threshold for the noise intensity, such that above that threshold the mode **k** is destabilized, increases with |**k**|. Short wavelength modes are the most robust.
- Spatial white noise with small intensity, by parametric amplification, generates long range spatial correlations.

Long range spatial correlations in 3d

- When the driving multiplicative process is the spatial and temporal white noise, then for small intensity, the long wavelength modes are destabilized first. => Long range spatial correlations dominates for weak noise.
- The shortest wavelength modes are more robust under white noise perturbations

REMARK

- In the opposite extreme case, when the driving (temporal white) noise has complete spatial correlations, the all of the modes are destabilized and the problem is trivial.
- The method used here can be extended to the study of the dissipative systems, and the of the higher order moments.

Conclusions

- The divergence of the second order moments is a signature of the heavy tail with exponent less then 2, in the stationary PDF of the field variables. Results that in the case of the multiplicative noises the linear approximation has a restricted range of applicability.