A reduction method for solving nonlinear PDEs

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Abstract

In this paper we are dealing with a few direct methods for finding solutions of various type of nonlinear PDEs. The focus will be put on two methods, both proposed by our group. The first one consider the use of an auxiliary equation, with well known solutions, in terms of which will be expressed the solutions of more complicated PDEs. More precisely, we will use the procedure called functional expansion". The second procedure we are using here allows to reduce the order of differentiability by using a so called "potential representation". The two methods will be exemplified on few well known models of PDEs.

1 Introduction

There is not a general approach allowing to solve nonlinear equations, despite of their huge importance in physics and other fields, their solutions offering us a better understanding of the phenomenas. Classes of solutions are given by specific techniques and approaches as for example:

- Symmetry method and similarity reduction,
- Inverse scattering method,
- Hirota bilinear method
- Lax pair operators, etc.

In this presentation we will focus on with a *direct approach* based on the *auxiliary equation technique*. It supposes to look for a specific class of solutions (traveling wave solutions) of a quite complicated nonlinear equation in terms of a simpler auxiliary equation with already known solutions. Here are the main steps of the classical algorithm for finding such solutions for a PDE of the form:

$$\Delta(u(t,x,y,...),u_t,u_x,u_y,...)=0.$$

1) The PDE is reduced to an ODE by introducing the "wave transformation":

$$\xi = f(t,x,y,...)$$

By that $u(t, x, y, ...) \to U(\xi)$ and:

$$\Delta(u) = 0 \to F(U, U', U", ...) = 0.$$

2) An ODE with known solutions $G(\xi)$ is considered as auxiliary equation.

3) The solutions of $F(U) = 0$ are expressed in terms of those of the auxiliary equation. Usually, the expression $U = U(G)$ is chosen as an expansion (a serie) and many such expansions were considered by various authors:

- expansions directly in terms of the solutions $G(\xi)$ (*ex: tanh method*), or following powers of G'/G
- expansions with constant coefficients or with coefficients depending on ξ;
- expension in Taylor type series or, more general, in Laurent series (with terms at negative powers).

Each of the previous mentioned steps could generate various problems and many improvments have been proposed. From our perspective, we focused on a possible way of simplification the ODE $F(U) = 0$, by reducing its differentiability order. My proposal was to atach to the solution $U(\xi)$ of $F(U) = 0$ a *potential function* or *a flow function* given by:

$$U'(\xi) = V(U)$$

The ODE $F(U) = 0$ becomes a simpler equation of the form $F'(V) = 0$. We will exemplify this approach on few important models of nonlinear equations from Mathematical Physics.

2 The general method

We consider nonlinear partial differential equations in the general form:

$$u_t = \Delta(u, u_x, ..., u_{mx}); u_{mx} = \frac{\partial^m x}{\partial x^m} \tag{1}$$

which are encountered in different fields of mathematics, physics, chemistry and biology. The given partial nonlinear equation (1) can be converted into an ordinary differential equation introducing the transformation $u \equiv U(\xi)$, where ξ is given by $\xi = x \pm \lambda t$.

$$F(\xi, U, U', ..., U^{(m)}) = 0; U^{(m)} = \frac{\mathrm{d}^m U}{\mathrm{d}\xi^m} \tag{2}$$

The main idea of the method we are proposing consists in attaching to the "master" equation a supplementary, flux type equation, of the form:

$$U' = V(U). \tag{3}$$

The quantity $V(U)$ can be a polynomial or a function of $U(\xi)$. The method is a "reduction method", leading to an equation in $V(U)$ with a reduced order of differentiability. We will apply the method both for the BBM equation, generalized Boussinesq equation, Tzitzteica equation and Whitman-Broer-Kaup equations.

3 Examples

3.1 Benjamin-Bona-Mahony

The BBM equation describes the uni-directional propagation of small-amplitude long waves on the surface of the water in a channel and also for hydromagnetic and acoustic waves. For a quantity $u(x,t)$ described in a 2-dimensional space, its mathematical form is:

$$u_t - u_{xxt} + u_x(1+u^n) = 0 \tag{4}$$

for $n=1$ the equation represents the BBM equation itself. We note that for this case BBM it is an alternative of the Korteweg-de-Vries equation. It is well known that the KdV equation is a basic model in nonlinear wave theory and has been regarded as the classical model for studying soliton phenomena. The KdV and BBM are two typical examples associated to effects of dissipation and disspersion. The similarity between the two equations becomes more obvious when we implement the change of variables, looking for traveling wave solutions. In this case the attached ODEs are similar.

for $n=2$ the equation represents the modified BBM equation

As far as our approach, we will reduce (**??**) to an ordinary differential equation by using a wave transformation of the independent variables of the form:

$$\xi = x - \lambda t. \tag{5}$$

Using this wave transformation and adopting the notation $u(x,t) = U(\xi)$, the equation (**??**) becomes:

$$\lambda U''' - U'(\lambda - 1 - U^n) = 0 \tag{6}$$

$n = 1$

By integrating the ODE once with respect to ξ, will result the next equation:

$$\lambda U'' + U(1 - \lambda + \frac{1}{2}U^2) = 0 \tag{7}$$

We will try to find traveling wave solutions of (6) , considering the supplementary requirement $U' = V(U)$

$$\lambda V(U)\frac{\mathrm{d}V}{\mathrm{d}U} + U(1 - \lambda + \frac{1}{2}U^2) = 0 \tag{8}$$

The above equation can be solved, assuming the integration constant 0:

$$V(U) = \pm\frac{U}{\lambda\sqrt{3}}\sqrt{\lambda(3\lambda - 3 - U)} \tag{9}$$

$$\xi = \frac{2\lambda\sqrt{3}\tanh^{-1}(\frac{\sqrt{3\lambda(\lambda-1)-\lambda U}}{\sqrt{3\lambda(\lambda-1)}})}{\sqrt{3\lambda(\lambda-1)}} \tag{10}$$

Finnaly we get the solution of BBM equation:

$$u(x,t) = \frac{3(\lambda - 1)}{\cosh^2(\frac{1}{2\lambda}(x - \lambda t)\sqrt{\lambda(\lambda - 1)})} \tag{11}$$

$n = 2$

By integrating (6) once with respect to ξ, will result the next equation:

$$\lambda U'' + U(1 - \lambda + \frac{1}{3}U^3) = 0 \tag{12}$$

$$\lambda V(U)\frac{\mathrm{d}V}{\mathrm{d}U} + U(1 - \lambda + \frac{1}{3}U^3) = 0 \tag{13}$$

$$V(U) = \pm\frac{1}{\lambda\sqrt{6}}\sqrt{-\lambda U(-6\lambda + 6 + U^2)} \tag{14}$$

$$\zeta = \frac{\lambda}{\sqrt{\lambda(\lambda - 1)}} \ln\left(\frac{12\lambda(\lambda - 1) + 2\sqrt{6\lambda(\lambda - 1)}\sqrt{6\lambda(\lambda - 1) - \lambda U^2}}{U}\right) \tag{15}$$

$$u(x,t) = \frac{24\lambda(\lambda - 1)\left(\cosh A + \sinh A\right)}{\cosh 2A + \sinh 2A + 24\lambda^2(\lambda - 1)} \tag{16}$$

where $A = \frac{(x - \lambda t)\sqrt{\lambda - 1}}{\sqrt{\lambda}}$

3.2 Nerve pulse propagation Equation (Generalized Boussinesq)

We will consider the equation describing the propagation for the sound in cylindrical bio-membranes. It is quite similar to the equation which describes the propagation of nerve pulses through neurons. Propagation is accompanied by a significant change of density, so we may say that elastic constants are sensitive functions of density. The equation has the form:

$$\frac{\partial^2 \Delta\rho}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(\Delta\rho)\frac{\partial \Delta\rho}{\partial x}\right) - h\frac{\partial^4 \Delta\rho}{\partial x^4} \tag{17}$$

The function $c^2(\Delta\rho)$ describes the square of the pulse velocity in the membrane while the last term, proportional with the fourth-order derivative of the density variation, describes dispersion, with h a positive parameter indicating dispersion's magnitude. More general, we can write:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(A(u)\frac{\partial u}{\partial x}) - h\frac{\partial^4 u}{\partial x^4} \Longleftrightarrow u_{2t} - A'(u)u_x^2 - A(u)u_{2x} + hu_{4x} = 0 \tag{18}$$

Here prime denotes the derivative with respect to u. The dependence of the square of pulse velocity up the density's variation is usually considered as having the form:

$$c^2(u) \equiv A(u) = \alpha u^2 + \beta u + c_0^2 \tag{19}$$

where c_0 is a constant denoting the sound velocity through air, while parameters α, β describe nonlinear elastic properties of the membrane. The case $\alpha = 0$ corresponds to the already known Boussinesq equation, while the case $\alpha \neq 0$ is a most general situation.

3.2.1 Case $\alpha \neq 0$

We will consider now the equation:

$$u_{2t} - (2\alpha u + \beta)u_x^2 - (\alpha u^2 + \beta u + c)u_{2x} + hu_{4x} = 0 \tag{20}$$

where $c = c_0^2$. Using the wave transformation and adopting the notation $u(x,t) = U(\xi)$, the above equation becomes:

$$\left(\lambda^2 - c - \alpha U^2 - \beta U\right) U'' - (2\alpha U + \beta)U'^2 - hU^{(4)} = 0 \tag{21}$$

By integrating twice with respect to ξ, will result the equation:

$$\lambda^2 U - \frac{\alpha}{3}U^3 - \frac{\beta}{2}U^2 - cU - hU'' = 0 \tag{22}$$

using $U' = V(U)$ and we obtain a new equation for V :

$$\lambda^2 U - \frac{\alpha}{3}U^3 - \frac{\beta}{2}U^2 - cU - hV(U)\frac{\mathrm{d}V(U)}{\mathrm{dU}} = 0 \tag{23}$$

The solution is:

$$V(U) = \pm\frac{\sqrt{-6hU(\alpha U^2 + 2\beta U - 6\lambda^2 + 6c}}{6h} \tag{24}$$

Considering the solutions for $V(U)$ we get the solution U of the ODE:

$$U(\xi) = \frac{144h(c-\lambda^2)(\cosh B + \sinh B)}{24h\beta(\cosh B + \sinh B) + \cosh B + \sinh B + 144h^2\beta^2 + 864\alpha h^2(\lambda^2 - c)} \tag{25}$$

where

$$B = \frac{(x - \lambda t)\sqrt{-h(c-\lambda^2)}}{h}. \tag{26}$$

3.2.2 Case $\alpha = 0$

After the double integration with respect to ξ, we will have:

$$\lambda^2 U - \frac{\beta}{2}U^2 - cU - hU'' = 0 \tag{27}$$

using $U' = V(U)$ and we obtain a new equation for V :

$$\lambda^2 U - \frac{\beta}{2}U^2 - cU - hV(U)V'(U) = 0 \tag{28}$$

$$V(U) = \pm \frac{\sqrt{-3hU(\beta U - 3\lambda^2 + 3c}}{3h} \tag{29}$$

$$U(\xi) = -\frac{3\left(c - \lambda^2\right)}{\beta \cosh\left(\frac{\xi\sqrt{-h(c-\lambda^2)}}{2h}\right)} \tag{30}$$

$$u(x,t) = -\frac{3\left(c - \lambda^2\right)}{\beta \cosh\left(\frac{(x-\lambda t)\sqrt{-h(c-\lambda^2)}}{2h}\right)} \tag{31}$$

3.3 Whitham–Broer–Kaup equations

Let us consider the Whitham–Broer–Kaup (WBK) equations:

$$\begin{aligned} u_t + \mathrm{u}u_x + h_x + \beta u_{2x} &= 0 \\ h_t + (hu)_x + \alpha u_{3x} - \beta h_{2x} &= 0 \end{aligned} \tag{32}$$

which is a complete integrable model ntroduced by Whitham, Broer and Kaup which describes the dispersive long wave in shallow water.The field of horizontal velocity is represented by $u(x,t)$, and $h(x,t)$ is the height that deviates from the equilibrium position of the liquid, and α and β are constants that represent different disspersive power.

Using the wave transformation $\xi = x - \lambda t$ and adopting the notations $u(x,t) = U(\xi),\ h(x,t) = H(\xi)$ the above equations become:

$$\begin{aligned} -\lambda U' + UU' + H' + \beta U'' &= 0 \\ -\lambda H' + H'U + HU' + \alpha U''' - \beta H'' &= 0 \end{aligned} \tag{33}$$

Integrating, the first equation of the system, once in respect to ξ we can express $H(\xi)$

$$H = \lambda U - \frac{1}{2}U^2 - \beta U' \tag{34}$$

Using the last relation, we introduce it in the secon equation of the system and it will result a second order differential equation:

$$(\beta^2 - \alpha)U'' - \lambda^2 U + \frac{3\lambda}{2}U^2 - \frac{1}{2}U^3 = 0 \tag{35}$$

using $U' = V(U)$ and we obtain a new equation for V :

$$(\beta^2 - \alpha)V(U)V'(U) - \lambda^2 U + \frac{3\lambda}{2}U^2 - \frac{1}{2}U^3 = 0 \tag{36}$$

From the above equation we find:

$$V(U) = \frac{U\left(U - 2\lambda\right)}{2\sqrt{\beta^2 + \alpha}} \tag{37}$$

Using the specific form for $V(U)$ from (37) we can find the solution of equation (35):

$$U = \ln(2\lambda) - \ln(\exp(\frac{-\lambda\xi}{\sqrt{\beta^2+\alpha}} - 1)) - \frac{-\lambda\xi}{\sqrt{\beta^2+\alpha}} \tag{38}$$

Coming back in (34) we can find the form of $H(\xi)$ and easily find the solutions of WBK equations:

$$\begin{aligned} u(x,t) &= -\frac{2\lambda(\cosh A + \sinh A)}{\cosh B + \sinh B - (\cosh A + \sinh A)} \\ h(x,t) &= -\frac{2\lambda^2(\sqrt{\beta^2+\alpha}+\beta)(\cosh C + \sinh C)}{\sqrt{\beta^2+\alpha}(\cosh B + \sinh B - \cosh A - \sinh A)^2} \end{aligned} \tag{39}$$

where $A = \frac{\lambda^2 t}{\sqrt{\beta^2+\alpha}}, B = \frac{\lambda x}{\sqrt{\beta^2+\alpha}}, C = \frac{\lambda(x+\lambda t)}{\sqrt{\beta^2+\alpha}}$.

3.4 Generalized Tzitzeica-Dodd-Bullough-Mikhailov Type

Now we consider the following nonlinear evolution equation:

$$u_{xt} = \alpha e^{mu} + \beta e^{nu} \tag{40}$$

where α, β are two non-zero real numbers and m, n are two integers. We call it generalized Tzitzeica-Dodd-Bullough- Mikhailov equation because it contains Tzitzeica equation, Dodd-Bullough-Mikhailov equation and Tzitzeica-Dodd-Bullough equation. The Dodd-Bullough-Mikhailov equation and Tzitzeica-Dodd-Bullough equation appeared in many problems varying from fluid flow to quantum field theory. When $\alpha = 1$, $\beta = -1$, $m = 1$, $n = -2$ or $\alpha = -1$, $\beta = 1$, $m = -2$, $n = 1$ the above equation becomes the classical Tzitzeica equation as follows:

$$u_{xt} = e^u - e^{-2u} \tag{41}$$

which was originally found in the field of geometry in 1907 by G. Tzitzeica and appeared in the fields of mathematics and physics alike. Using the wave transformation $\xi = x - \lambda t$ and adopting the notation $u(x,t) = U(\xi)$, the above equation becomes:

$$-\lambda U'' - e^u + e^{-2u} = 0 \tag{42}$$

using $U' = V(U)$ and we obtain a new equation for V :

$$-\lambda V(U)\frac{\mathrm{d}V(U)}{\mathrm{dU}} - e^u + e^{-2u} = 0 \tag{43}$$

$$V(U) = \pm\frac{1}{\lambda}\sqrt{-\lambda(2e^U + e^{-2U} - 3)} \tag{44}$$

$$U = -\ln(2) - 2\ln\left(\cos\frac{\xi\sqrt{3}}{2\sqrt{\lambda}}\right) + \ln\left(2\cos^2\frac{\xi\sqrt{3}}{2\sqrt{\lambda}}\right) \tag{45}$$

$$u(x,t) = -\ln(2) - 2\ln\left(\cos\frac{(x-\lambda t)\sqrt{3}}{2\sqrt{\lambda}}\right) + \ln\left(2\cos^2\frac{(x-\lambda t)\sqrt{3}}{2\sqrt{\lambda}}\right) \tag{46}$$

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