Hirota bilinear method for constructing integrable discretizations of semidiscrete solitonic equations

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Abstract

Hirota bilinear method for constructing integrable discretizations of solitonic equations is presented. For illustrating the full procedure we reunite the results presented in literature for several semidiscrete integrable forms of KdV, mKdV, multicomponent Ablowitz-Ladik and general Volterra system.

Keywords: lattice Volterra system; lattice KdV; lattice mKdV; lattice Ablowitz-Ladik; Hirota bilinear formalism; soliton solutions

1 Introduction

Building integrable discretizations of partial differential or differential-difference equations is a challenging topic in the field of integrable systems, subject of interest for many scientists in the last decades. Constructing integrable full discretizations is important for numerical simulation, since the integrable structure of the equation is preserved. There are several integrability criterion like complexity growth [1], singularity confinement [2], cube consistency [3] or the existence of the Lax pairs [4, 5, 6, 7], but in this paper we focus on the existence of the multi-soliton solution, the integrability criterion in the Hirota bilinear formalism [8, 9, 10, 11, 12]. The method proposed by Hirota proved to be a very effective tool in constructing integrable discretizations for an important number of solitonic equations [13, 14].

In this review paper, we are going to present several integrable discretizations of differential-difference equations/systems [16, 17, 18, 19], constructed with the Hirota method. The paper is organized as follows: after a brief introduction, in Section 2 we present the three steps of the Hirota bilinear method

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for building integrable discretizations. In Section 3, we exemplify the application of the Hirota method steps on several well known equations/systems, each presented in a different subsection: discrete KdV, discrete mKdV, discrete multicomponent Ablowitz-Ladik and discrete general Volterra system. In Section 4 we summarize our conclusions.

2 The Hirota bilinear method for integrable discretization

The main motivation for deriving fully discretized versions of nonlinear equations is that the discrete integrable systems are fundamental compared to the continuous ones, as the dynamics is way richer and they can lead to cellular automaton forms. The Hirota bilinear method for building integrable discretizations has three steps [8, 9, 10, 11]. The starting point is a semidiscrete bilinear form of an integrable system/equation.

Step 1. The first step consists in the direct discretization of the Hirota operator D_t and the replacement of t with $m\delta$. At first, one has to to replace the ordinary derivatives in

$$D_t f \cdot g \equiv \frac{df}{dt} g - f \frac{dg}{dt}$$

with the finite differences

$$\Delta_m f \cdot g \equiv [\Delta_t f] g - f[\Delta_t g], \quad \Delta_t f = [f(t+\delta) - f(t)]/\delta.$$

Afterwords, all t that appear in the equation must be replaced with $m\delta$. Above f, g are the tau functions, t is the continuous time variable, m is the discretized time variable and δ is the discretisation step on the time axis;

Step 2. In the second step one must impose the gauge invariance to the bilinear equations obtained:

$$T(\Delta_m, \Delta_n, ...) f \cdot g = 0,$$

in other words, the bilinear equations must be invariant with respect to the transformations:

$$f \to f e^{\theta n + \sigma m}, \quad g \to g e^{\theta n + \sigma m},$$

where θ and σ are arbitrary constants.

Step 3. In the third step two problems must be solved: computing the multi-soliton solution and, the most difficult part, recovering the nonlinear form of the analyzed system/equation.

3 Fully discretized equations with Hirota method

3.1 Discrete KdV equation

The semidiscrete form of KdV (Korteweg de Vries) equation:

$$\dot{u}_n = \frac{du_n}{dt} = u_n^2(u_{n+1} - u_{n-1})$$

was casted in the following bilinear form [16]:

$$\begin{aligned} &D_t G \cdot F = \overline{G}\underline{F} - \underline{G}\overline{F} \\ &G^2 - \overline{F}\underline{F} = 0, \end{aligned} \tag{1}$$

using the projective nonlinear substitution involving two τ functions namely $u = G/F$. The significance of the notation used in (1) is: $\overline{G} = G(n+1,t)$, $\underline{G} = G(n-1,t)$, $\overline{F} = F(n+1,t)$, $\underline{F} = F(n-1,t)$ (n is the discrete space variable with step 1, while t is the continuous time variable).

Applying step 1 (discretization of D_t and replacing t with δm) and step 2 (imposing gauge-invariance) the bilinear form (1) turns into:

$$\begin{aligned} &\widetilde{G}F - G\widetilde{F} = \delta(\overline{\widetilde{G}}\underline{F} - \underline{G}\overline{\widetilde{F}}), \\ &G\widetilde{G} - \overline{\widetilde{F}}\underline{F} = 0, \end{aligned} \tag{2}$$

where we used the notation $\widetilde{G} = G(\delta(m+1), n)$, $\overline{\widetilde{G}} = G(\delta(m+1), n+1)$, $\underline{G} = G(\delta m, n-1)$, $\widetilde{F} = F(\delta(m+1), n)$, $\overline{\widetilde{F}} = F(\delta(m+1), n+1)$, $\underline{F} = F(\delta m, n-1)$. The system (2) is completely integrable as it admits multi-soliton solution [20]. Taking $G = \overline{\widetilde{f}}\underline{f}$, $F = f\widetilde{f}$, the second equation turns into an identity whilst for the first equation the N-soliton solution in term of f is:

$$f = \sum_{\mu_{i,j}=0,1} \exp(\sum_i^N \mu_i \eta_i + \sum_{i<j} a_{ij}\mu_i\mu_j),$$

where:

$$\eta_i = k_i n + \omega_i m, \quad \sinh\omega_i = \delta \sinh k_i, \quad \exp a_{ij} = \left(\frac{e^{k_i} - e^{k_j}}{e^{k_i}e^{k_j} - 1}\right)^2.$$

To complete the third step one must find the nonlinear form. For that, one must divide the first equation of (2) with $F\widetilde{F}$, denote $X = G/F$ and use the second equation of (2). The fully integrable discrete KdV equation obtained in [12, 16] is:

$$\tilde{X} - X = \delta(\bar{\tilde{X}} - \underline{X})X\tilde{X}.$$

3.2 Discrete mKdV equation

The classical semidiscrete mKdV equation (self-dual nonlinear network) [21], [22]:

$$\dot{u}_n = (1 + u_n^2)(u_{n+1} - u_{n-1}) \tag{3}$$

or the potential semidiscrete mKdV equation proposed in [17]:

$$\dot{v}_n = 2v_n \frac{v_{n+1} - v_{n-1}}{v_{n+1} + v_{n-1}} \tag{4}$$

were casted in the following bilinear form:

$$\begin{aligned} D_t G \cdot F &= \overline{G}\underline{F} - \underline{G}\overline{F} \\ 2GF &= \overline{G}\underline{F} + \underline{G}\overline{F}, \end{aligned} \tag{5}$$

using the nonlinear substitutions $u_n = \frac{i}{2}\frac{d}{dt}\log\frac{G(n,t)}{F(n,t)}$ for (3) and $v_n = \frac{G(n,t)}{F(n,t)}$ for (4). Both forms of the semidiscrete mKdV presented above are completely integrable as they poses multi-soliton solution.

Replacing time derivatives in (5) with finite differences and imposing the bilinear gauge invariance, the following bilinear system was obtained:

$$\begin{aligned} \overline{G}F - G\overline{F} &= \delta(\widetilde{\overline{G}}\underline{F} - \underline{G}\widetilde{\overline{F}}) \\ 2GF &= \overline{G}\underline{F} + \underline{G}\overline{F}. \end{aligned}$$

The above system admits the following N-soliton solution:

$$\begin{aligned} F_n^m &= \sum_{\mu_1,\mu_2,..\mu_n \in \{0,1\}} \left(\prod_{i=1}^{N} a_i^{\mu_i} (p_i^n q_i^{m\delta})^{\mu_i} \right) \prod_{i<j}^{N} A_{ij}^{\mu_i \mu_j}, \\ G_n^m &= \sum_{\mu_1,\mu_2,..\mu_n \in \{0,1\}} \left(\prod_{i=1}^{N} b_i^{\mu_i} (p_i^n q_i^{m\delta})^{\mu_i} \right) \prod_{i<j}^{N} A_{ij}^{\mu_i \mu_j}, \end{aligned}$$

with the same phase factors and interaction terms as in the differential-difference case:

$$b_i = -a_i = 1$$

$$A_{ij} = \frac{\cosh(k_i - k_j) - 1}{\cosh(k_i + k_j) - 1} \quad , \qquad i < j = \overline{1, N}$$

but different dispersion relation:

$$q_i = \left(\frac{1-\delta p_i^{-1}}{1-\delta p_i}\right)^{1/\delta}, \quad i = \overline{1, N},$$

where $p_i = e^{k_i}$, $q_i = e^{\omega_i}$, $i = \overline{1, N}$ (k_i is the wave number and ω_i is the angular frequency).

The nonlinear form of semidiscrete mKdV is recoverd using the following notations $\omega = \frac{G}{F}$, $\Gamma = \frac{\underline{F}\widetilde{\overline{F}}}{F\widetilde{\overline{F}}}$:

$$\begin{aligned}
\widetilde{\omega} - \omega &= \delta\Gamma(\widetilde{\overline{\omega}} - b\underline{\omega}) \\
\overline{\Gamma} &= \frac{\omega + \underline{\omega}}{\overline{\overline{\omega}} + \widetilde{\omega}}\frac{\widetilde{\overline{\omega}}}{\omega}\Gamma
\end{aligned}$$

Eliminating Γ and $\overline{\Gamma}$ one can obtain the higher order nonlinear lattice mKdV equation [17]:

$$\frac{\widetilde{\omega} - \omega}{\widetilde{\overline{\omega}} - \overline{\omega}} = \frac{\widetilde{\overline{\omega}} - \underline{\omega}}{\widetilde{\overline{\overline{\omega}}} - \omega}\frac{\widetilde{\overline{\overline{\omega}}} + \widetilde{\omega}}{\omega + \overline{\omega}}\frac{\omega}{\widetilde{\overline{\omega}}}$$

3.3 Discrete coupled Ablowitz-Ladik system

The general system of coupled Ablowitz-Ladik with M equations [23]:

$$\begin{aligned}
i\dot{q}_1 &= (1 + |q_1|^2)(\overline{q_2} + \underline{q_M}) \\
i\dot{q}_2 &= (1 + |q_2|^2)(\overline{q_3} + \underline{q_1}) \\
i\dot{q}_3 &= (1 + |q_3|^2)(\overline{q_4} + \underline{q_2}) \\
\ldots\ldots & \ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots \\
i\dot{q}_{M-1} &= (1 + |q_{M-1}|^2)(\overline{q_M} + \underline{q_{M-2}}) \\
i\dot{q}_M &= (1 + |q_M|^2)(\overline{q_1} + \underline{q_{M-1}})
\end{aligned}$$

was casted in the Hirota bilinear form:

$$\begin{aligned}
i\mathbf{D_t} G_\mu \cdot F_\mu &= \overline{G_{\mu+1}}\underline{F_{\mu-1}} + \underline{G_{\mu-1}}\overline{F_{\mu+1}} \\
F_\mu^2 + |G_\mu|^2 &= \overline{F_{\mu+1}}\underline{F_{\mu-1}},
\end{aligned} \tag{6}$$

where $F_0 = F_M,\ F_{M+1} = F_1,\ G_0 = G_M,\ G_{M+1} = G_1$ (F_M are real functions, while G_M and complex valued functions), using the nonlinear substitutions: $q_\mu = G_\mu / F_\mu$, $\mu = \overline{1, M}$.

Using the Hirota bilinear formalism [11] in (6) and the Hirota-Tsujimoto approach [24], the fully integrable discretzation was obtained [18]:

$$
\begin{aligned}
i(\widetilde{G}_\mu F_\mu - G_\mu \widetilde{F}_\mu) &= \delta(\widetilde{\overline{G_{\mu+1}}}\underline{F_{\mu-1}} + \underline{G_{\mu-1}}\widetilde{\overline{F_{\mu+1}}}) \\
F_\mu^2 + |G_\mu|^2 &= \overline{F_{\mu+1}}\underline{F_{\mu-1}}, \quad \mu = \overline{1,M},
\end{aligned}
$$

where $F_\mu = F_\mu(m\delta, n)$, $\overline{F_\mu} = F_\mu(m\delta, n+1)$, $\underline{F_\mu} = F_\mu(m\delta, n-1)$, $\widetilde{F_\mu} = F_\mu((m+1)\delta, n)$, $\widetilde{\overline{F_\mu}} = F_\mu((m+1)\delta, n+1)$, $G_\mu = G_\mu(m\delta, n)$, $\overline{G_\mu} = G_\mu(m\delta, n+1)$, $\underline{G_\mu} = G_\mu(m\delta, n-1)$, $\widetilde{G_\mu} = G_\mu((m+1)\delta, n)$, $\widetilde{\overline{G_\mu}} = G_\mu((m+1)\delta, n+1)$. The N-soliton solution for multicomponent Ablowitz-Ladik system with branched dispersion relation was also constructed:

$$
\begin{aligned}
G_\mu &= \sum_{\nu=0,1} D_2(\underline{\nu}) \exp\left(\sum_{i=1}^{2N} \nu_i[\eta_i + (\mu-1)\log(\epsilon_i)] + \sum_{1\le i<j}^{2N} \nu_i\nu_j\phi_{ij}\right) \\
F_\mu &= \sum_{\nu=0,1} D_1(\underline{\nu}) \exp\left(\sum_{i=1}^{2N} \nu_i[\eta_i + (\mu-1)\log(\epsilon_i)] + \sum_{1\le i<j}^{2N} \nu_i\nu_j\phi_{ij}\right)
\end{aligned}
$$

where:

$$
e^{\phi_{ij}} = \begin{cases}
\frac{1}{2}\left(\frac{\epsilon_i^2+\epsilon_j^2}{2\epsilon_i\epsilon_j}\cosh(k_i+k_j^*) + \frac{\epsilon_i^2-\epsilon_j^2}{2\epsilon_i\epsilon_j}\sinh(k_i+k_j^*) - 1\right)^{-1} \\
\qquad \text{if } i = 1,...N \text{ and } j = N+1,...2N; \\
2\left(\frac{\epsilon_i^2+\epsilon_j^2}{2\epsilon_i\epsilon_j}\cosh(k_i-k_j) + \frac{\epsilon_i^2-\epsilon_j^2}{2\epsilon_i\epsilon_j}\sinh(k_i-k_j) - 1\right), \\
\qquad \text{if } i = 1,...N \text{ and } i = 1,...N \\
\qquad \text{or } i = N+1,...2N \text{ and } j = N+1,...2N;
\end{cases}
$$

and:

$$
\eta_j = k_j n + \omega_j \delta m + \eta_j^{(0)}, \quad \eta_{j+N} = \eta_j^*, \quad k_{j+N} = k_j^*, \quad j = 1,...N
$$

$$
\log(\epsilon_{j+N}) = \log(\epsilon_j)^*, \quad \omega_{j+N} = \omega_j^*, \quad \log(\epsilon_j) \in \{l\frac{2\pi i}{M}\}, \quad l = 1,...M.
$$

$$D_1(\underline{\mu}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \mu_{i+N}; \\ 0 & \text{otherwise;} \end{cases}$$

$$D_2(\underline{\mu}) = \begin{cases} 1, & \text{when } \sum_{i=1}^{N} \mu_i = 1 + \sum_{i=1}^{N} \mu_{i+N}; \\ 0 & \text{otherwise.} \end{cases}$$

Each of the N solitons can have any of the M branches of dispersion:

$$\exp \delta\, \omega_j(k_j) = \frac{i + \delta \epsilon_j^{-1} e^{-k_j}}{i - \delta \epsilon_j e^{k_j}}, \quad \epsilon_j \in \left\{ e^{l \frac{2\pi i}{M}} \right\},\ l = \overline{1, M},\ j = \overline{1, N}$$

where k_j is the wave number and j is the index of the soliton. The system is completely integrable since it admits the N soliton solution.

Considering the following notations $q_\mu = G_\mu / F_\mu$ and $\Gamma_\mu = \widetilde{\overline{F_{\mu+1}}}\,\underline{F_{\mu-1}} / F_\mu \widetilde{F}_\mu$ the fully discrete Ablowitz-Ladik was constructed [18]:

$$\begin{aligned} i(\widetilde{q}_\mu - q_\mu) &= \delta(\widetilde{\overline{q_{\mu+1}}} + \underline{q_{\mu-1}})\Gamma_\mu \\ \frac{\overline{\Gamma_{\mu+1}}}{\Gamma_\mu} &= \frac{1 + |\widetilde{\overline{q_{\mu+1}}}|^2}{1 + |q_\mu|^2}, \quad \mu = \overline{1, M}. \end{aligned}$$

Eliminating Γ_μ also a higher order fully discrete Ablowitz-Ladik system was constructed:

$$\left(\frac{\widetilde{\overline{q_{\mu+1}}} - \overline{q_{\mu+1}}}{\widetilde{\overline{\overline{q_{\mu+2}}}} + q_\mu} \right) \left(\frac{\widetilde{\overline{q_{\mu+1}}} + \underline{q_{\mu-1}}}{\widetilde{q}_\mu - q_\mu} \right) \left(\frac{1 + |q_\mu|^2}{1 + |\widetilde{\overline{q_{\mu+1}}}|^2} \right) = 1, \quad \mu = \overline{1, M}.$$

3.4 Discrete general Volterra system

The general differential-difference Volterra system with any number of coupled equations [25]:

$$\begin{aligned} \dot{q}_1 &= (c_0 + c_1 q_1 + c_2 q_1^2)(\overline{q_2} - \underline{q_M}) \\ \dot{q}_2 &= (c_0 + c_1 q_2 + c_2 q_2^2)(\overline{q_3} - \underline{q_1}) \\ &\ldots.. \\ \dot{q}_{M-1} &= (c_0 + c_1 q_{M-1} + c_2 q_{M-1}^2)(\overline{q_M} - \underline{q_{M-2}}) \\ \dot{q}_M &= (c_0 + c_1 q_M + c_2 q_M^2)(\overline{q_1} - \underline{q_{M-1}}), \end{aligned} \tag{7}$$

where $q_\nu = q_\nu(n,t)$, $\overline{q_\nu} = q_\nu(n+1,t)$, $\underline{q_\nu} = q_\nu(n-1,t)$ for any $\nu = \overline{1,M}$, was casted as:

$$\begin{aligned}
\dot{u}_1 &= (1+u_1^2)(\overline{u_2} - \underline{u_M}) \\
\dot{u}_2 &= (1+u_2^2)(\overline{u_3} - \underline{u_1}) \\
&\cdots\cdots \\
\dot{u}_{M-1} &= (1+u_{M-1}^2)(\overline{u_M} - \underline{u_{M-2}}) \\
\dot{u}_M &= (1+u_M^2)(\overline{u_1} - \underline{u_{M-1}}),
\end{aligned}$$

using the scalings and translations $u_\nu(n,t) = \frac{2c_2}{\sqrt{4c_0c_2-c_1^2}}(q_\nu(n,t) + \frac{c_1}{2c_2})$ with $\nu = \overline{1,M}$ [26]. In the above case $u_\nu(n,t) \to \alpha$ as $n \to \pm\infty$ and the value for $\alpha = c_1(4c_0c_2 - b_1^2)^{-1/2}$. For negative argument of the square root in α, the system turns into the defocusing form [26].

Using the nonlinear substitutions:

$$u_\nu = \alpha - \frac{\mathrm{i}}{2}\frac{\partial}{\partial t}\ln\frac{g_\nu(n,t)}{f_\nu(n,t)}, \quad \nu = \overline{1,M},$$

the Hirota bilinear form was obtained [19]:

$$\begin{aligned}
&\mathbf{D_t} g_\nu f_\nu = (1+\alpha^2)(\overline{g_{\nu+1}}\,\underline{f_{\nu-1}} - \underline{g_{\nu-1}}\,\overline{f_{\nu+1}}), \quad \nu = \overline{1,M} \\
&(1+\mathrm{i}\alpha)\overline{g_{\nu+1}}\,\underline{f_{\nu-1}} + (1-\mathrm{i}\alpha)\underline{g_{\nu-1}}\,\overline{f_{\nu+1}} = 2g_\nu f_\nu,
\end{aligned}$$

where $g_0 = g_M,\ g_{M+1} = g_1,\ f_0 = f_M,\ f_{M+1} = f_1$.

Applying the Hirota steps presented in Section 2, the bilinear equations for general Volterra system with any M coupled equations were obtained:

$$\begin{aligned}
&\widetilde{g}_\nu f_\nu - g_\nu \widetilde{f}_\nu = \delta(1+\alpha^2)(\widetilde{\overline{g_{\nu+1}}}\,\underline{f_{\nu-1}} - \underline{g_{\nu-1}}\,\widetilde{\overline{f_{\nu+1}}}), \\
&(1+i\alpha)\widetilde{\overline{g_{\nu+1}}}\,\underline{f_{\nu-1}} + (1-i\alpha)\underline{g_{\nu-1}}\,\widetilde{\overline{f_{\nu+1}}} = \widetilde{g}_\nu f_\nu + g_\nu \widetilde{f}_\nu, \quad \nu = \overline{1,M}.
\end{aligned}$$

The N-soliton solution was constructed:

$$\begin{aligned}
g_\nu &= \sum_{\mu_1\ldots\mu_N\in\{0,1\}} \left(\prod_{i=1}^{N}(\epsilon_i^{\nu-1}\beta_i p_i^n q_i^{\delta m})^{\mu_i}\prod_{i<j}^{N} A_{ij}^{\mu_i\mu_j}\right), \quad \nu = \overline{1,M} \\
f_\nu &= \sum_{\mu_1\ldots\mu_N\in\{0,1\}} \left(\prod_{i=1}^{N}(\epsilon_i^{\nu-1}\gamma_i p_i^n q_i^{\delta m})^{\mu_i}\prod_{i<j}^{N} A_{ij}^{\mu_i\mu_j}\right),
\end{aligned}$$

where:

$$p_j = e^{k_j}, \quad A_{ij} = \left(\frac{\epsilon_i e^{k_i} - \epsilon_j e^{k_j}}{\epsilon_i \epsilon_j e^{k_i+k_j} - 1}\right)^2, \quad i < j = \overline{1, N}$$
$$\beta_j = \frac{K}{2}\left(\frac{\epsilon_j e^{k_j} + 1}{\epsilon_j e^{k_j} - 1}\frac{\alpha}{1 + \delta(1 + \alpha^2)} + \mathrm{i}\right), \quad \gamma_j = \frac{K}{2}\left(\frac{\epsilon_j e^{k_j} + 1}{\epsilon_j e^{k_j} - 1}\frac{\alpha}{1 + \delta(1 + \alpha^2)} - \mathrm{i}\right),$$

with the M possible branches of dispersion for each of the N solitons:

$$q_j = e^{\omega_j} = \left[\frac{(\epsilon_j e^{k_j})^{-1}\delta(1 + \alpha^2) - 1}{\epsilon_j e^{k_j}\delta(1 + \alpha^2) - 1}\right]^{1/\delta}, \ \epsilon_j \in e^{\nu\frac{2\pi \mathrm{i}}{M}}, \ \nu = \overline{1, M}, \ j = \overline{1, N}.$$

In the final step, the nonlinear form of lattice Volterra system with M coupled equation was recovered:

$$\tan(\widetilde{\phi}_\nu - \phi_\nu) = \frac{\delta(1 + \alpha^2)\tan(\widetilde{\overline{\phi_{\nu+1}}} - \underline{\phi_{\nu+1}})}{1 + \alpha\tan(\widetilde{\overline{\phi_{\nu+1}}} - \underline{\phi_{\nu+1}})}, \quad \nu = \overline{1, M} \tag{8}$$

where $\phi_\nu = \frac{i}{2}\log(g_\nu/f_\nu)$ and $\phi_0 = \phi_M$, $\phi_{M+1} = \phi_1$.

It is important to notice that for $\alpha \to 0$, system (8) becomes the coupled lattice self-dual network of Hirota [22]:

$$\tan(\widetilde{\phi}_\nu - \phi_\nu) = \delta\tan(\widetilde{\overline{\phi_{\nu+1}}} - \underline{\phi_{\nu+1}}), \quad \nu = \overline{1, M}.$$

Taking $\alpha \to 0$ and $M = 1$ in (8), one finds the well known nonlinear form of discrete mKdV [27]:

$$\tan(\widetilde{\phi} - \phi) = \delta\tan(\widetilde{\overline{\phi}} - \underline{\phi}),$$

different from the one presented in Subsection 3.2, which was higher order.

4 Conclusions

In this review paper, we have presented the Hirota bilinear method proposed for deriving integrable discretizations of solitonic equations. The method was illustrated for differential-difference forms of well known equations and systems: KdV, mKdV, multicomponent Ablowitz-Ladik and Volterra systems with branched dispersion relation. The three steps involved can be summarized as it follows: starting form an integrable semidiscrete form of the studied equation, one has to discretize the differential Hirota bilinear operator, impose gauge invariance, build the multi-soliton solution and then recover the nonlinear form with the aid of some auxiliary functions. This approach may lead to higher order nonlinear equations, as we have seen in most cases analysed.

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