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From Dressed Particle to Dressed Mode in Plasmas

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Abstract

A theoretical method to analyze the strong turbulence in far-nonequilibrium plasmas is discussed. In this approach, a test mode is treated being dressed with interactions with other modes. In this article, nonlinear dispersion relation of the dressed mode and statistical treatment of turbulence is briefly reviewed. Application to the problem of abrupt onset oh global MHD mode is explained. Foundation based on the Mori method (projection operator method) is also described. Analogue to the method of dressed particle, which has given Balescu-Lenard collision operator for inter-particle collisions, is mentioned.

1 Introduction

Plasmas have been one of the main subjects of modern physics. This is because that almost all the matter, the presence of which is known to mankind, is in the plasma state and the understanding of the physics of plasmas constitutes foundations for our perception of the nature. In addition, plasmas have revealed challenging problems. One important issue is that the charged particles in plasmas interact with others through the long-range interaction of electromagnetic fields which are at the same time governed by the motion of plasma particles. This feature is known as the collective interactions. The other stimulating issue is that the plasmas are often far-away from the thermodynamical equilibrium. [In a standard terminology, one may use the words "thermal equilibrium" and "thermal fluctuations" to describe the state which is equilibrated at a given temperature. In order to avoid the confusion of the fluctuation of thermal energy and fluctuations at thermal equilibrium, the words "thermodynamical equilibrium" and "fluctuations at thermodynamical equilibrium" are employed here.] Fluctuating electromagnetic fields or fluctuating component of plasma parameters are far from those predicted for the thermodynamical fluctuations and do not at all satisfy the equi-partition law. The strong non-equilibrium nature of fluctuations comes from instabilities and turbulence, and influences the nature of plasmas. In the preceding article [1], some aspect of turbulence theory has been reviewed, highlighting the origin by Balescu and further evolution. This article is an extension of it, with supplement of recent progresses in the approach of the dressed test mode method..

The collective nature of plasmas influences the collisions of charged particles. The analysis of collisional process in plasmas is essential: this is because the inter-particle collisions are the origin of irreversibility and dissipations. The rate of dissipation has a basic importance for the analysis of the transport processes. This collective nature was successfully formulated by Balescu [2] and Lenard [3], as is now known as Balescu-Lenard collision operator. It has been clearly demonstrated that charged particles are "not alone" in plasmas. When one particle encounters with the other, it is not a collision of "bare" particles. In stead, each particle is dressed with interactions between many other particles through electromagnetic fields. The interaction between two "bare" particles is screened by many other particles, and the collision is formulated as those between "dressed" particles. The inter-particle collision is an origin of the transport processes and this formula has been a foundation of the analysis of transport coefficients. For instance, the collisional transport is basic to the cross-field transport in strongly-magnetized plasmas, which have been intensively investigated for the motivation to realize controlled thermonuclear fusion. The cross-field transport owing to the binary collision of charged particle is called "classical transport" (or "neoclassical transport" in toroidal configurations) and has been subject to long and intensive investigations. For this transport coefficient, the monograph by Balescu [4] provides a systematic deduction, and forms a firm basis of plasma transport processes, together with other literature [5].

Far-nonequilibrium property of plasmas has required a breakthrough in understanding the fluctuations. Instabilities, which are caused by inhomogeneities, boundaries, or anisotropy of distribution function, drive fluctuations into the level which is much higher than the thermodynamical equilibrium fluctuations. The subject of strong turbulence has been a main issue in the plasma theory. In strong turbulence, the growth of a mode that is labelled among a large number of excited modes is different from what has been predicted by linear stability. Proper theoretical treatment of the interaction of excited mode with other fluctuations has been (and will be) a central theme. (See [6] for an illustrative description of the problem and [7] for the approach developed by Balescu.) A method of dressed test mode in a strongly-turbulent magnetized plasmas has been proposed [8]. Recent development of this method is reported briefly in this article.

2 Model

2.0.1 Example for the case of reduced set of equations

The method of dressed mode is illustrated by use of an example of a reduced set of equations. The reduced set of equations has the form

$$\frac{\partial}{\partial t} f + \mathcal{L}^{(0)} f = \mathcal{N}(f,\ f) + \tilde{S}_{th} \tag{1}$$

where f denotes the set of fluctuating field variables. (See [9] for a survey.) For instance, $f^T = (\Phi,\ n)$ for Hasegawa-Wakatani model [10], $f^T = (\phi,\ J,\ p)$ for three-field model [8], or $f^T = (n,\ \phi,\ \Phi, v_{||},\ p_e,\ p_i,\ A_{||})$ for Yagi-Horton model [11]. (ϕ: electrostatic potential, J: current along the strong magnetic field, p: pressure, n: density, etc.) These have been used for the study of nonlinear dynamics of resistive drift mode turbulence, current-diffusive mode turbulence, and for a comprehensive study of many instabilities, respectively. The linear operator $\mathcal{L}^{(0)}$ is an $N \times N$ matrix for the

$N-$field model and controls the linear modes. $\mathcal{N}(f,\ f)$ is the nonlinear terms, e.g., $\mathcal{N}(f,\ f) = -\left(\nabla_\perp^{-2}\left[\phi, \nabla_\perp^2\phi\right], [\phi, J], [\phi, p]\right)^T$, for the case of $f^T = (\phi,\ J,\ p)$. The term $\tilde{S}_{th}$ stands for the thermodynamical excitations induced by the interaction with a heat bath.

Theoretical models have been developed to separate the nonlinear interaction term into two terms:

$$\mathcal{N}(f,\ f) = \mathcal{N}_{coherent}(f,\ f) + \tilde{S} \tag{2}$$

where $\mathcal{N}_{coherent}(f,\ f)$ is the coherent part, which changes with the phase of the test mode, f_k, $\tilde{S}$ and is the incoherent part (noise part). Explicit forms of $\mathcal{N}_{coherent}(f,\ f)$ and $\tilde{S}$ are given by modelling. Various models for the coherent and incoherent parts have been analyzed. For detailed discussion, see e.g. [9, 12, 13]. In the method of dressed test mode, a test mode f_k is chosen, and a following modelling is taken: The term $\mathcal{N}_{coherent}(f,\ f)$ is modelled as an effectively-linear term of f_k renormalizing nonlinear interactions with background turbulent fluctuations, and $\tilde{S}$ is a random noise. Recently, authors has applied the Mori method of projection operator [14] to separate a memory function and fluctuating force in the nonlinear term [15]. By this extension, more firm basis for the dressed test method was given. A short illustration is presented in the appendix.

In an actual application of the dressed-test mode method to the plasma turbulence, a couple of approximations are often used for analytic insight: a Markovian approximation by which the memory function is replaced by a damping term, and a diagonalization approximation of $\mathcal{N}_{coherent}(f,\ f)$. The diagonal terms in $\mathcal{N}_{coherent}(f,\ f)$ are approximated by the diffusion terms with the turbulent viscosity (μ_N for ion viscosity, μ_{Ne} for electron viscosity, and χ_N for thermal diffusivity), or by the eddy-damping coefficients (γ_v for ion momentum, for γ_e parallel electron momentum, γ_p and for thermal energy), as $\mathcal{N}_{coherent}(f,\ f)_k = \left(\mu_N \nabla_\perp^2 f_1,\ \mu_{N_e}\nabla_\perp^2 f_2,\ \chi_N \nabla_\perp^2 f_3\right)^T$or $\mathcal{N}_{coherent}(f,\ f)_k = -\left(\gamma_v f_1,\ \gamma_e f_2,\ \gamma_p f_3\right)_k^T$.

Within this diagonal approximation, the renormalized operator is given by

$$\mathcal{L}_{ij} = \mathcal{L}_{ij}^{(0)} + \gamma_i \delta_{ij} \tag{3}$$

and one has a renormalized reduced set of equations (with a thermodynamical noise source) as

$$\frac{\partial}{\partial t} f_k + \mathcal{L} f_k = \tilde{S}_k + \tilde{S}_{th,\ k} \tag{4}$$

where k denotes the test mode [13,15].

2.1 Dressed modes

Equation (4) shows that the amplitude of the fluctuation $|f_k|$ becomes large in the vicinity of the pole of the renormalized operator $\mathcal{L}$. Thus the nonlinear dispersion relation

$$\det\left(\lambda \mathbf{I} + \mathcal{L}\right) = 0 \tag{5}$$

describes the characteristic feature of the turbulence, where $\mathbf{I}$ is a unit tensor, and $-\lambda$ is the eigenvalue of the operator $\mathcal{L}$. The sign of λ is defined so that $\mathcal{R}e\lambda$ is positive when the test mode perturbation does not increase. The decorrelation rate is given by $\mathcal{R}e\lambda$. This dispersion relation includes the (coherent part of) nonlinear interactions with back ground fluctuations, and the eigenmode corresponding to the nonlinear eigenvalue is called dressed mode.

In order to solve the Langevin equation (4), an ansatz for a large number of degrees of freedom in the random modes, N, is introduced. The renormalized term γ_j in $\mathcal{L}$ arises from the statistical sum from N components, so that its variation in time becomes $O(N^{-1/2})$ less than that of f_k. Therefore, in solving f_k, $\mathcal{L}$ is approximated to be constant in time in the limit of $N \to \infty$. The general solution is formally given as

$$f(t) = \sum_m \exp\left(-\lambda_m t\right) f^{(m)}(0) + \int_0^t \exp\left[-\mathcal{L}(t-\tau)\right] \tilde{S}(\tau) d\tau \tag{6}$$

where $-\lambda_m$ ($m = 1, 2, 3...$ and $\mathcal{R}e\lambda_1 < \mathcal{R}e\lambda_2 < \mathcal{R}e\lambda_3 < ...$) represent the eigenvalues of the renormalized matrix $\mathcal{L}$. ($f^{(m)}(0)$ represents the initial value which is transformed into a diagonal basis.)

2.2 Statistical theory

The incoherent part acts as a nonlinear noise. Taking an example of three-field model, the statistical analysis is explained [13, 16]. The matrix $\exp\left[-\mathcal{L}(t-\tau)\right]$ in equation (6) are decomposed as

$$\left\{\exp\left[-\mathcal{L}(t-\tau)\right]\right\}_{ij} = A_{ij}^{(1)} \exp\left[-\lambda_1(t-\tau)\right] + A_{ij}^{(2)} \exp\left[-\lambda_2(t-\tau)\right] + A_{ij}^{(3)} \exp\left[-\lambda_3(t-\tau)\right],$$

where explicit forms of $\mathbf{A}^{(m)}$ are given in [13]. By introducing a projected noise source,

$$S^{(m)}(\tau) = \left(1,\ -ik_{||}k_\perp^{-2}\left(\bar{\gamma}_e - \lambda_m\right)^{-1},\ -ik_y\kappa k_\perp^{-2}\left(\bar{\gamma}_p - \lambda_m\right)^{-1}\right) \cdot \left\{\tilde{S}(\tau) + \tilde{S}_{th}(\tau)\right\},$$

where κ is the magnetic field gradient and the superscript (m) denotes m-th eigenmode, one can estimate the noise source as

$$\left\langle S^{(1)} * S^{(1)} \right\rangle \simeq C_0 \gamma_v A_{11}^{-2} \left\langle f_{1,\ k}^{(1)} * f_{1,\ k}^{(1)} \right\rangle + \text{ thermal excitations},$$

where C_0 is a numerical factor of the order of unity. With this estimate, the long time average of the fluctuation amplitude is given as

$$\left\langle f_{1,\ k}^{(1)} * f_{1,\ k}^{(1)} \right\rangle = \frac{C_0 \gamma_v}{2\mathcal{R}\lambda_1} \left\langle f_{1,\ k}^{(1)} * f_{1,\ k}^{(1)} \right\rangle + \text{thermal excitations} \tag{7}$$

This is one form of extended fluctuation dissipation relation for the non-equilibrium plasmas. In this formula, the effects of turbulence are renormalized in γ_v and λ. The formula $\mathcal{R}\lambda_1 = C_0\gamma_v/2$ describes the stationary state of strong turbulence.

3 Applications

3.1 Nonlinear instability and subcritical excitation

The method of dressed mode has been applied to interchange mode turbulence [8]. When there is a dissipation that impedes the free electron motion along the magnetic field line, the interchange mode becomes unstable. This mechanism allows the nonlinear instability. When the electrons respond to the test mode (interchange mode) in the presence of the back-ground turbulent fluctuations, electrons are dressed with reactions from back-ground fluctuations. Electrons are 'heavy' owing to the presence of turbulence, and electrons do no longer freely cancel the charge separation associated with the mode. There arises a nonlinear link of mechanisms that excites fluctuations: (1) fluctuations impede the free motion of electrons through cross-field diffusion, (2) this electron diffusion increases the growth rate (3) the increased growth rate further enhances the fluctuation level. An explosive growth of fluctuations takes place until the fluctuation level becomes high enough so that the ion viscosity stabilizes the mode. Plasma turbulence is self-sustained, not necessarily being driven by linear instability [8, 17].

By use of this method, a subcritical excitation and anomalous transport in plasma can be analyzed. A nonlinear marginal stability condition has been derived for current diffusive interchange mode (CDIM) as [18]

$$\frac{G_0}{s^{4/3}}\frac{\left(\mu_{eN}+\mu_{ec}\right)^{2/3}\left(c/a\omega_p\right)^{4/3}}{\left(\chi_N+\chi_c\right)\left(\mu_N+\mu_c\right)^{1/3}}=\mathcal{F}_c \tag{8}$$

where G_0 is a normalized pressure gradient $G_0 = a^2\nabla \ln p_0 \cdot \nabla \ln B$ and s is a magnetic shear parameter, and the length, time, and the scalar and vector potentials are normalized to the plasma radius a, poloidal Alfven transit time $\tau_{Ap} = a/\upsilon_{Ap} = R/\upsilon_A$, Ba^2/R and $B\upsilon_A a^2/R$, respectively. A critical Itoh number $\mathcal{F}_c$, is of the order of unity. (Suffix c for μ, μ_e, and χ indicates the collisional transport process.) This formula shows that the turbulence is self-sustained even in a linearly stabile region $G_0 < G_c$. At the critical pressure gradient G_*, the turbulent transport coefficient is subject to a subcritical excitation. Figure 1 illustrates a theoretical prediction of fluctuation level as a function of pressure gradient, G_0. Explicit multifold form of electrostatic potential perturbation $\tilde{\phi}(G_0)$ is seen. A subcritical excitation of turbulence is predicted to occur if G_0 exceeds the critical value G_*. The subcritical excitation and self-sustaining of turbulence are confirmed by direct numerical simulations [19].

3.2 Turbulence transition and transition probability

The result of the current-diffusive turbulence shows that the fluctuations have cusp catastrophe owing to the two excitation mechanisms (i.e., inhomogeneity that induces instability and thermodynamical excitations). The statistical transition can take place among the turbulent states and the transition probability can be calculated.

The renormalized Langevin equation is reduced to the one for a course-grained quantity [13]. The total fluctuating energy, which is the quantity integrated over some finite-size volume of size L, $\mathcal{E} \equiv \frac{1}{2}\sum_m k_\perp^2 \phi_k^2$ is taken as an examples. By introducing an average dissipation rate, $\Lambda \equiv 2\sum_m \lambda_{1,\ m} k_\perp^2 \phi_k^2/\mathcal{E}$, the Langevin equation for the total fluctuating energy is given as

$$\frac{\partial}{\partial t}\mathcal{E} + 2\Lambda\mathcal{E} = g\omega(t) \tag{9}$$

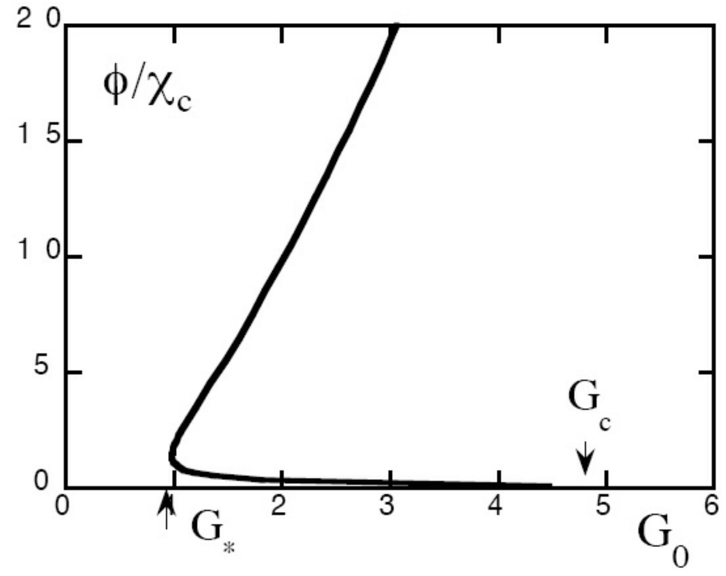

Figure 1: Fluctuation level as a function of the pressure gradient. Strong turbulence exists below the critical pressure gradient against the linear instability, $G_0 < G_c$. Transition to the turbulent state takes place at $G_0 = G_*$. On horizontal axis, G_0, G_* and G_c are divided by $s^{4/3}a^{2/3}\delta^{-2/3}\chi_c^{2/3}$, $\delta = c/\omega_p$[18].

where $\omega(t)$ denotes the white noise and $g^2 = 4\hat{T}\gamma_m\mathcal{E} + \sum_k \left(\sum_{j=1}^{3} A_1 g_{j,k}\right)^2 k_\perp^4 \phi_k^2$. ($g_{j,k}$: the amplitude of $\tilde{S}_k$, $\tilde{S}_{j,k} = g_{j,k}\omega(t)$, and $\hat{T} = 2\mu_0 B_p^{-2} k_B T$: the normalized temperature, γ_m: the mean decorrelation rate at thermodynamical equilibrium.) The associated effective potential $S(\mathcal{E})$

$$S(\mathcal{E}) = \int^{\mathcal{E}} \frac{4\Lambda\mathcal{E}}{g^2} d\mathcal{E} \tag{10}$$

is introduced. This renormalized potential plays a central role in the statistical property of fluctuations. First, the probability density function (PDF) of fluctuation energy in a stationary state is given by

$$P_{st}(\mathcal{E}) = \bar{P} g^{-1} \exp\{-S(\mathcal{E})\}$$

The minima of $S(\mathcal{E})$ denote the probable states. In the case that a hysteresis exists, $S(\mathcal{E})$ has multiple minima, separated by local maximum. Thermodynamical fluctuation state (horizontal axis of Fig.1) and turbulent state (upper branch of Fig.1) are denoted by A and B; the lower branch of Fig.1 that is an unstable marginal state is denoted by C. Second, the transition probability between different turbulent state can be given by $S(\mathcal{E})$ [20].

The transition probability from the thermodynamical branch to the turbulent state is given as

$$r_{A\to B} = \frac{\sqrt{\Lambda_C \gamma_m}}{\sqrt{\pi}} \exp\{-S(\mathcal{E})\} \tag{11}$$

$\Lambda_C = \mathcal{E} d\Lambda / d\mathcal{E}$ at $\mathcal{E} = \mathcal{E}_C$. This is an extension of the Arrhenius law to the system far from thermodynamical equilibrium. For the case of CDIM turbulence, the probability of transition from thermodynamical fluctuation to the turbulent fluctuation is given, near the critical gradient for linear instability $G_0 \simeq G_c$, as

$$r_{A\to B} \tilde{} \frac{\gamma_m}{\sqrt{\pi}} \left(\frac{\mu_{ec}}{2}\right)^{-2b_1} k_0^{-4b_1/3} \left(\hat{T}\gamma_m \frac{3}{16C_0}\right)^{2b_1/3} \left(1 - \frac{G_0}{G_0}\right)^{-b_1} \tag{12}$$

where $b_1 = \left(k_0\gamma_m/\sqrt{3}16C_0\right)^{2/3} \hat{T}^{-1/3} \left(L/a\right)^2$. Important feature is that the probability is expressed in terms of the power law $r_{A\to B} \propto (1 - G_0/G_c)^{-b_1}$.

The phase boundary for the ensemble average is given by the formula

$$S(\mathcal{E}_A) = S(\mathcal{E}_B) \tag{13}$$

This is an extension of the Maxwell's construction.

Comparison between the rules in the far-non-equilibrium system and those near thermodynamical equilibrium is made in the table 1.

This method can be applied to various problems. Extensions to the cases with many kinds of instabilities are presented in [21-23]. Statistical excitation of stable and long-wavelength fluctuations has also been discussed, in conjunction with the nonlocal transport processes [21]. The dynamics of the transition processes in plasma turbulence described by the nonlinear stochastic equation is studied in detail. It was shown that intermittent or global transitions between metastable states can appear [24]. The conditions for the generation of these transitions and their statistical characteristics are determined. Detailed survey of the problem is given in a recent review [25].

	Near Thermodynamical Equilibrium	**Far-non-equilibrium**
Basic Assumption	Stosszahl Ansatz; $1/\Omega$ -expansion	Large degree of freedom with positive Lyapunov exponent
Damping	Molecular viscosity $\gamma_c = \mu_c k_\perp^2$	Nonlinear (eddy) damping $\gamma_N \sim \tilde{\phi} k_\perp^2 / B$
Micro vs Macro	$\mu_{\mathrm{micro}} = \mu_{\mathrm{macro}}$ Onsagar's Ansatz	Scale-dependent
Excitation		
(random)	Thermal excitation	Nonlinear drive
(coherent)	-	Instability drive
Decorrelation Rate	γ_c	Nonlinear decorrelation λ_1
Balance	FD Theorem Einstein's relation	Extended FD Theorem $I \sim \dfrac{\textit{nonlinear noise}}{\textit{nonlinear decorrelation}}$
Partition	Equi-partition $E_k \sim T\ k$	Nonlinear Balance $E_k \sim \left\| \nabla p_0 \right\| k^{-3}$
Probability	Boltzmann	Integral of renorm. dissipation
Distribution function	$P(\mathcal{E}) \sim \exp\left(-\mathcal{E}/k_B T\right)$	$P(\mathcal{E}) \sim \exp\left(-S(\mathcal{E})\right)/g$ power law tail
Min./Max. principle	Maximum Entropy/ Minimum entropy production rate	$S(\mathcal{E})$ minimum
Phase boundary	Maxwell's construction	$S(\mathcal{E}_A) = S(\mathcal{E}_B)$
Transition probability	$\ln(K) \sim -\Delta Q/T$ Arrhenius law	$K \propto \exp\left(-S(\mathcal{E}_{saddle})\right)$ Power law
Transport Matrix	Onsagar's symmetry	Not necessarily symmetric
Interference of fluxes	Curie's principle	Interferences between heat, particle and momentum
Transport coefficients	Independent of gradient	Depend on gradient

Table 1: Comparison between the rules near thermodynamical equilibrium and in the far-non-equilibrium system

3.3 Onset of global modes

The method of dressed test mode can also be applied to the onset of global (MHD) modes in the presence of background turbulence. Subcritical excitation for global MHD mode by fluctuating force (owing to microscopic fluctuations) was analyzed [26], and a lifetime of a state (free from the onset of subcritical global MHD perturbations) was calculated [26].

The model is briefly outlined here. A stochastic equation for the amplitude of neoclassical tearing mode (NTM) has been derived. The model and basis are explained in ref.26 and references therein. By taking into account of the fluctuating force on the global NTM owing to the ambient microscopic fluctuations, the stochastic equation is given as

$$\frac{\partial}{\partial t}A + \eta\Lambda A = g\omega(t) \tag{14}$$

where $A \equiv \tilde{A}_* g^2 R/B r_s^3 q'$ is the normalized amplitude of the global (m, n) -Fourier component of helical vector potential perturbation $\tilde{A}_*$ at the mode rational surface, $r = r_s$, η is the inverse of resistive diffusion time $\eta = \eta_{||}\mu_0^{-1} r_s^{-2}\tau_{Ap} = R_M^{-1}$, where $\eta_{||}$ stands for a parallel resistivity, and R_M is the Lundquist number (magnetic Reynolds number), $-\Lambda$ is the nonlinear growth rate ($-\Lambda > 0$ if unstable) and $g\omega(t)$ is the random kick (by ambient microscopic fluctuations) where g is the magnitude and $\omega(t)$ indicates white-noise. Note that the fluctuation force is not necessary to be a white noise but can be coloured. Equation (14) is the stochastic equation for the global MHD mode which is dressed by background microscopic turbulence. Here, the time is normalized to poloidal Alfven transit time, $\tau_{Ap} = qR/v_A$ (v_A: Alfven velocity) and the length to r_s.

An explicit form of the nonlinear growth rate is given by

$$-\Lambda = 2\Delta A^{-1/2} - \frac{C_1}{W_1^2 + A^2} + \frac{C_2}{W_2 + A} \tag{15}$$

within the neoclassical transport theory, where the first, second and third terms of RHS stand for the effects of current density gradient, polarization drift and bootstrap current, respectively. The term W_1 represents the cut-off due to the banana orbit effect, and is modelled as $W_1 = \rho_b^2 r_s^{-2}$, W_2 represents the cut-off determined by the cross-field energy transport, and coefficients C_1 and C_2 are given as $C_1 = 2a_{bs}\beta_p\varepsilon^{1/2}\rho_b^2 r_s^{-2} L_q^2 L_p^{-2}$ and $C_2 = 2a_{bs}\beta_p\varepsilon^{1/2} L_q^2 L_p^{-2}$. ($\rho_b$ is the banana width, L_q and L_p are the gradient scale lengths of safety factor and pressure, respectively, ε is the inverse aspect ratio and a_{bs} is a numerical constant.) The parameter Δ controls the linear stability of tearing mode when induced by the current density gradient. That is, the tearing mode is linearly stable if $\Delta' < 0$. However, the mode can be nonlinearly unstable even if it is linearly stable. Namely, when the amplitude A takes finite values, $-\Lambda$ can be positive even if $\Delta' < 0$, because C_1 and C_2 can be positive. The marginal stability condition $\Lambda = 0$ can have three solutions at $A \simeq 0$, $A = A_m$ and $A = A_s$ ($A_m < A_s$), where A_m and A_s are the threshold and saturation amplitudes, respectively. Near the linear stability boundary, $\Delta' \simeq 0$, they can be estimated as $A_m = C_1 C_2^{-1}$ and $A_s \simeq C_2^2/4\Delta'^2$.

The transition from the state $A \simeq 0$ to the state $A = A_s$ is the nonlinear onset of the NTM (subcritical excitation). The transition from $A = A_s$ to $A \simeq 0$ is the subcritical elimination of the NTM. These nonlinear excitations can be induced by the fluctuating force owing to the ambient microscopic turbulence. The transition probability from $A \simeq 0$

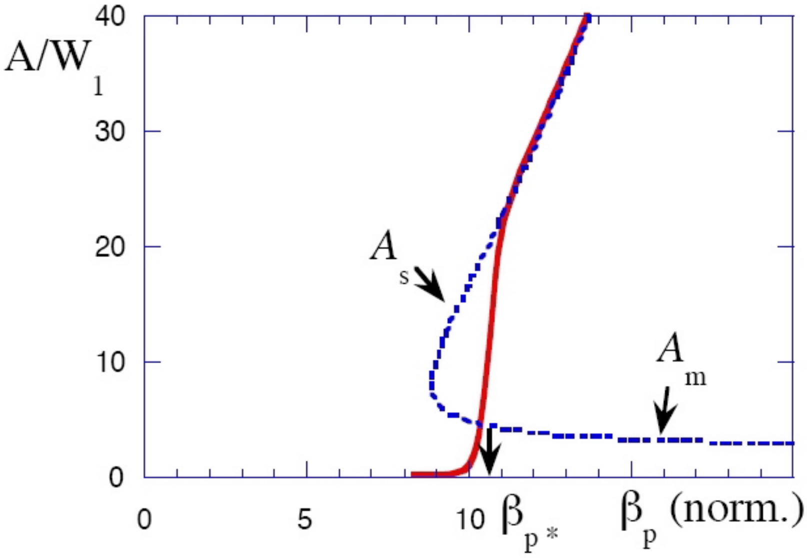

Figure 2: Amplitude of NTM as a function of the plasma pressure. Solid line shows the statistical average $\langle A \rangle$. A thin dotted line indicates the threshold A_m and saturation amplitude A_s for the deterministic model. Normalized β_p is $C_2/\left(-\Delta' W_1^{1/2}\right)$, i.e., $(2a_{bs}\varepsilon^{1/2}L_q r_s/\rho_b L_p(-\Delta'))\beta_p$. (Parameters are: $W_1 = W_2$, $C_1/2C_2W_1 = 1$, $\Gamma C_2 W_1 = 5$.) [26]

to $A = A_s$ (the excitation probability, r_{ex}) and that from $A = A_s$ to $A \simeq 0$ (the decay probability, r_{dec}) are calculated by substituting Eq.(15) into Eqs.(10) and (11).

The long time average is given as $\langle A \rangle = (A_s r_{ex} + \langle A_0 \rangle r_{dec})(r_{ex} + r_{dec})^{-1}$. $\langle A \rangle$ approaches to A_s if $r_{ex} > r_{dec}$ holds. It reduces to $\langle A_0 \rangle$, if $r_{ex} < r_{dec}$ holds. The phase boundary for the statistical average is determined by the condition $r_{ex} = r_{dec}$. Apart from a logarithmic dependence, the condition is given by $S(A_s) = 0$. Figure 2 shows the statistical average $\langle A \rangle$, together with threshold and saturation amplitudes (A_m and A_s), as a function of β_p. $\langle A \rangle$ drastically changes across the condition $\beta_p = \beta_{p*}$, a formula of which is derived from Eq.(13) [26].

The stochastic equation is formulated including the subcritical excitation mechanism of NTM. The rate of transition and statistical average of amplitude are derived, and the phase boundary in plasma parameter space, β_{p*} or Δ'_* , is obtained. Linearly stable systems are prone to nonlinear instability if $S(A_s) < 0$ holds. The formula is applied to either cases of micro fluctuations or of other random MHD activities. Experimental database for the presence of NTM must be compared with the result of phase boundary derived from the statistical theory. The rate of stochastic transition depends on the microfluctuation level and is evaluated for example cases. However, the boundary is given by $S(A_s) = 0$ and is insensitive to the magnitude of micro fluctuations. It is plausible that the stochastic transition without the trigger by large MHD events (e.g., sawtooth or fish-bone instabilities) can be observed in high temperature tokamak plasmas if the condition $\beta_p > \beta_{p*}$ is satisfied.

4 Summary

In this article, a brief review was made for a recent development of the theory of plasma turbulence. Based on the previous article [1], which was dedicated to Prof. Balescu, very recent progresses were explained. Property of turbulent plasma was formulated by a method of dressed modes. The coherent part of nonlinear interactions was included in a nonlinear dispersion relation, which allowed analyses of subcritical turbulence or nonlinear saturation states. The incoherent part contributed to the stochastic noise term, and a statistical theory was constituted. Two fundamental issues of plasmas, i.e., the collective phenomena and non-equilibrium property, were investigated by this method.

Two examples (subcritical excitation and onset of global MHD mode) were explained here. Other issue is the role of mesoscale fluctuations (e.g., zonal flows), which also have screening effects on microfluctuations. The turbulence dressed by mesoscale fluctuations has been discussed in [28] in detail. In this article, the plasma inhomogeneity is treated as a given control parameter. In reality, it evolves with turbulence. The structural formation and turbulent transport are discussed in literature and monograph [8, 9, 29]. Along this line of thought, a project Grant-in-Aid for Specially Promoted Research "Research on Structural Formation and Selection Rules in Turbulent Plasmas" (S-I Itoh, principal investigator) has been initiated from FY 2004 [30].

Starting from the concept of dressed particle, which was developed by Balescu, research of far-nonequilibrium plasmas now includes the method of dressed modes. This direction will provide a prosperous path to explore the further progress of modern physics.

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6 Dedication

This article is dedicated to the memory of Prof. R. Balescu, with cordial thanks for his elucidating comments and discussions in the course of research which is presented here and in the International Advisory Board Panel of Specially Promoted Research Project.

7 Appendix: Separation of Memory Function

A basis of the dressed test mode method, which is developed upon Mori method, is briefly explained in this appendix.

7.1 Mori's principle and memory function

In the study of nonlinear equation

$$\partial f_k/\partial t + \mathcal{L}_k^{(0)} f_k = \sum{}' M_{kpq} f_p f_q = \mathcal{N}_k \tag{A1}$$

where $\sum{}'$ means that the summation is taken over $\mathbf{p}$ and $\mathbf{q}$ with constraints $\mathbf{k} = \mathbf{p} + \mathbf{q}$, Mori has shown the methodology to deduce the memory function $\Gamma_k(s)$, by which Eq.(A1) can be rewritten as

$$\partial f_k/\partial t + \mathcal{L}_k^{(0)} f_k = i\Omega_k f_k + \mathcal{R}_k(t) - \int_0^t ds \Gamma_k(s) f_k(t-s) \tag{A2}$$

where Ω_k is the nonlinear frequency shift and $\mathcal{R}_k$ is a rapidly-changing fluctuating force term. (A boldface is not used for vector and tensor fields for brevity of expression.) The systematic method is as follows. The dynamical equation (A1) is rewritten as

$$\partial f_k/\partial t = \Xi f_k \tag{A3a}$$

$$\Xi = \sum_p \dot{f}_p \frac{\partial}{\partial f_p} = \sum_p \left(-\mathcal{L}_p^{(0)} f_p + \mathcal{N}_p\right) \frac{\partial}{\partial f_p} \tag{A3b}$$

The projection operators $\mathcal{P}$ and $\mathcal{Q}$ are introduced ($\mathcal{P} + \mathcal{Q} = \infty$). $\mathcal{P}$ represents the projection

$$\mathcal{P}Y = \left\langle Y(t) f^\dagger(0) \right\rangle \left\langle f(0) f^\dagger(0) \right\rangle^{-1} f(0) \tag{A4a}$$

where

$$\left\langle A_l(t) A_m^\dagger(0) \right\rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T ds A_l(t+s) A_m^\dagger(s) \tag{A4b}$$

By use of this projection operator, the memory function $\Gamma_k(s)$, the fluctuating force term $\mathcal{R}_k(t)$ and nonlinear frequency shift Ω_k are given as

$$\Gamma_k(s) = \left\langle \mathcal{R}_k(t) \mathcal{R}_k^\dagger(0) \right\rangle \left\langle f(0) f^\dagger(0) \right\rangle^{-1} \tag{A5a}$$

$$\mathcal{R}_k(t) = \exp(t\mathcal{Q}\Xi)\, \mathcal{Q}\, \mathcal{N}_k \tag{A5b}$$

$$i\Omega_k = \left\langle \dot{f}_k f_k^\dagger \right\rangle \left\langle f_k f_k^\dagger \right\rangle^{-1} \tag{A5c}$$

respectively. The fluctuating force is orthogonal, i.e.,

$$\left\langle \mathcal{R}_k(t) f_k^\dagger(0) \right\rangle = 0 \tag{A5d}$$

7.2 Evaluation of memory function by use of Mori's principle

In the method of dressed-test mode, a renormalized dispersion relation has been derived for any choice of test modes. In order to discuss a statistical basis of the eddy-damping term in the dressed-test-mode method, the form of which has been assumed a priori in I-V, the Mori method is applied to the interacting terms recurrently.

The nonlinear terms of equation (A1) include the contribution of other terms f_{k_1}, which obeys a similar equation like Eq.(A1), i.e.,

$$\partial f_{k_1}/\partial t + \mathcal{L}_{k_1}^{(0)} f_{k_1} = \sum{}' M_{k_1,p,q} f_p f_q = \mathcal{N}_{k_1} \tag{A6}$$

We here introduce a notation $\hat{\mathcal{N}}_{k_1}$ by separating one term $M_{k_1,k,q} f_k f_q$, $(\mathbf{q} = \mathbf{k}_1 - \mathbf{k})$ out from $\mathcal{N}_{k_1}$ as

$$\mathcal{N}_{k_1} = \hat{\mathcal{N}}_{k_1} + M_{k_1,k,q} f_k f_q \tag{A7}$$

The Mori's principle is applied to $\hat{\mathcal{N}}_{k_1}$, and $\hat{\mathcal{N}}_{k_1}$ is rewritten by use of the nonlinear frequency shift, the fluctuating force term and the memory function as

$$\hat{\mathcal{N}}_{k_1} = i\hat{\Omega}_{k_1} f_{k_1} + \hat{\mathcal{R}}_{k_1}(t) - \int_0^t ds \hat{\Gamma}_{k_1}(s) f_{k_1}(t-s) \tag{A8a}$$

where

$$\left\langle \hat{\mathcal{R}}_{k_1}(t) f_{k_1}^{\dagger}(0) \right\rangle = 0 \tag{A8b}$$

holds. In this equation, the notation ˆ denotes the component without the contribution from f_k. Substituting Eqs.(A7) and (A8) into Eq.(A6), Eq.(A6) is rewritten as

$$\partial f_{k_1}/\partial t + \hat{H}_{k_1} f_{k_1} = \hat{\mathcal{R}}_{k_1}(t) + M_{k_1,k,q} f_k f_q \tag{A9a}$$

with

$$\hat{H}_{k_1} f_{k_1} = \mathcal{L}_{k_1}^{(0)} f_{k_1} - i\hat{\Omega}_{k_1} f_{k_1} + \int_0^t ds \hat{\Gamma}_{k_1}(s) f_{k_1}(t-s) \tag{A9b}$$

We next introduce a Green's function as

$$\frac{\partial}{\partial t} \hat{g}_{k_1}(t,t') + \hat{H}_{k_1}(t) \hat{g}_{k_1}(t,t') = \delta(t-t') \tag{A10}$$

with $\hat{g}_{k_1}(t,t') = 1$, where $\delta(t-t')$ is the Dirac's delta function. By use of $\hat{g}_{k_1}(t,t')$, the solution of Eq.(A9a) is formally given as

$$f_{k_1}(t) = \zeta_{k_1}(t) + \int_0^t dt' \hat{g}_{k_1}(t,t') M_{k_1,-k_2,k} f_{-k_2}(t') f_k(t') \tag{A11a}$$

with

$$\zeta_{k_1}(t) = \hat{g}_{k_1}(t,0) f_{k_1}(0) + \int_0^t dt' \hat{g}_{k_1}(t,t') \hat{\mathcal{R}}_{k_1}(t') \tag{A11b}$$

where we write $\mathbf{k}_2 = \mathbf{k} - \mathbf{k}_1$. In Eq.(A11a), $\zeta_{k_1}(t)$ represents the response against the fluctuating force terms which are originated from $\hat{\mathcal{N}}_{k_1}$.

We assume that the time-scale separation is relevant. That is, the fluctuating force term $\hat{\mathcal{R}}_k(t')$ ($\hat{\mathcal{R}}_{k_1}$as well) is very rapidly changing in time and so is $\zeta_{k_1}(t)$ which is the response against $\hat{\mathcal{R}}_{k_1}$, and the observed quantities such as the correlation function $\langle f_k(t) f_{-k}(t')\rangle$ change in a time scale which represents the observed evolution of the system. In the mutual interaction between drift waves, a nonlinear force from a pair of modes changes faster than the correlation time of a test mode. In addition, number of pairs which contribute to the nonlinear force of the test mode is of the order of $L^2\rho_i^{-2}$ (ρ_i: ion gyroradius, L: characteristic scale length of global inhomogeneity). Therefore the total nonlinear force is considered to change rapidly in comparison with the correlation time of the test mode. The time-scale separation which is employed here means that the rapidly varying quantities like $\hat{\mathcal{R}}_k(t')$ is homogenous in time, within a time scale that describes the temporal change of $\langle f_k(t) f_{-k}(t')\rangle$. Under this circumstance, homogeneity of rapidly varying fluctuating force is used and the correlation functions are written as

$$\langle f_{k_1}(t) f_{-k_1}(t')\rangle = C_{k_1}(t-t') \tag{A12}$$

With this argument, the correlation function $\langle f_{k_1}(t)\zeta_{-k_1}(t')\rangle$ is replaced by $\langle f_{k_1}(t) f_{-k_1}(t')\rangle$.

The argument which gives Eq.(A11) provides a similar expression for f_{k_2}. Substituting these expressions into Eq.(A6) and so on, Eq.(A1) is rewritten as

$$\partial f_k/\partial t + \mathcal{L}_k^{(0)} f_k = \int_0^t dt' \left\{ \sum{}' 2 M_{k,k_1,k_2} M_{k_2,k,-k_1} \hat{g}_{k_2}(t,t') C_{k_1}(t-t') \right\} f_k(t') + \hat{\mathcal{R}} \tag{A13}$$

This is a stochastic equation for the dressed test mode. In this equation, the first term in the RHS sands for the memory function, and the fluctuating force is given as

$$\begin{aligned}
\hat{\mathcal{R}} &= \sum{}' M_{k,k_1,k_2}\zeta_{k_1}\zeta_{k_2} \\
&+ \int_0^t \sum{}' dt' M_{k,k_1,k_2} M_{k_2,k,-k_1} \hat{g}_{k_2}(t,t') \left\{ \zeta_{k_1}(t) f_{-k_1}(t') - \langle f_{k_1}(t) f_{-k_1}(t')\rangle \right\} f_k(t') \\
&+ \int_0^t \sum{}' dt' M_{k,k_2,k_1} M_{k_1,k,-k_2} \hat{g}_{k_1}(t,t') \left\{ \zeta_{k_2}(t) f_{-k_2}(t') - \langle f_{k_2}(t) f_{-k_2}(t')\rangle \right\} f_k(t') \\
&+ \sum{}' m_{k,k_1,k_2} \int_0^t dt' \hat{g}_{k_2}(t,t') f_{-k_2}(t') f_k(t') \int_0^t dt'' \hat{g}_{k_2}(t,t'') f_{-k_1}(t'') f_k(t'')
\end{aligned} \tag{A14}$$

In previous work, only the first term in the RHS of Eq.(A14) was kept for the expression of $\hat{\mathcal{R}}$. Thus, the systematic application of the Mori method has given a corrections to the fluctuating force, clarifying the boundary of the previous intuitive derivation for the fluctuating force.

A recurrent relation for the Green's function is derived from Eq.(A13). From Eq.(A13), the equation for the Green's function of the k_2-mode is successively introduced as

$$\begin{aligned}
&\left\{ \frac{\partial}{\partial t} + \mathcal{L}_{k_2}^{(0)} \right\} \hat{g}_{k_2}(t,t') - \\
&\quad \int_0^t dt'' \left\{ \sum{}' M_{k_2,k_3,k_4} M_{k_4,k_2,-k_3} \hat{g}_{k_4}(t,t'') C_{k_3}(t-t'') \right\} \hat{g}_{k_2}(t'',t') = \delta(t-t')
\end{aligned} \tag{A15}$$

7.3 Continued fraction expressions

When the homogeneous approximation in time holds, $g(t,t') = g(t-t')$, the Laplace transformation is introduced as

$$G_{k_2}(p) = \int_0^t d\tau \exp(-p\tau)\hat{g}_{k_2}(\tau) \tag{A16}$$

and is given as

$$G_{k_2}(p) = \frac{1}{p + \mathcal{L}_{k_2}^{(0)} - \sum{}' 2M_{k_2,k_3,k_4} M_{k_4,k_2,-k_3} U_{k_3,k_4}(p)} \tag{A17}$$

where $U_{k_3,k_4}(p)$ is a Laplace transform of the function $\hat{g}_{k_4}(s)\, C_{k_3}(s)$. When the change of the Green's function is faster than the decay of the correlation function, the term $U_{k_3,k_4}(p)$ in Eq.(A17) is given as $U_{k_3,k_4}(p) = C_{k_3} G_{k_4}(p)$, and Eq.(A17) takes a form

$$G_{k_2}(p) = \frac{1}{p + \mathcal{L}_{k_2}^{(0)} - \sum{}' 2M_{k_2,k_3,k_4} M_{k_4,k_2,-k_3} C_{k_3} G_{k_4}(p)} \tag{A18}$$

The turbulent memory function $\Gamma(\tau)$ for the dressed test mode is evaluated from Eq.(A13). Its Laplace transform $\bar{\Gamma}(p)$ is given, under the assumption which is used for deriving Eq.(A18), as

$$\bar{\Gamma}(p) = \sum{}' V_{k,k_1,k_2} G_{k_2}(p) \tag{A19}$$

where

$$V_{k,k_1,k_2} \equiv M_{k,k_1,k_2} M_{k_2,k,-k_1} C_{k_1}(s=0) \tag{A20}$$

Similarly, the continued fraction expression is obtained as

$$\bar{\Gamma}_k(p) = \sum{}' \frac{V_{k,1,2}}{p_2 + \sum{}' \frac{V_{2,3,4}}{p_4 + \left(\sum{}' \frac{V_{4,5,6}}{p_6 + \sum{}' \frac{V_{6,7,8}}{p_8 + \left(\sum{}' \frac{V_{8,9,10}}{p_{10} + \ldots} \right)}} \right)}} \tag{A21}$$

where $p_n = p + \mathcal{L}_{k_n}^{(0)}$ and $(k,1,2,...)$ is an abbreviation of $(k,k_1,k_2,...)$.

The eddy damping rate $\bar{\Gamma}_{0,k_j}$, at which the dressed test mode of k_j decays due to turbulent damping, has a relation with the Laplace transform $\bar{\Gamma}_{k_j}(p)$ as

$$\bar{\Gamma}_{0,k_j} = \bar{\Gamma}_{k_j}(p=0) \tag{A22}$$

$k_j = (k,k_1,k_2,...)$. The expression for $\bar{\Gamma}_{0,k_j}$ by use of the continued fraction is deduced from Eq.(A21). In the case of the matrix operator $\mathcal{L}_k^{(0)}$, the least-stable eigenvalue λ_k is used in the continued fraction as

$$\bar{\Gamma}_{0,k}(p) = \sum{}' \frac{V_{k,1,2}}{\lambda_{k_2}^{(0)} + \sum{}' \frac{V_{2,3,4}}{\lambda_{k_4}^{(0)} + \left(\sum{}' \frac{V_{4,5,6}}{\lambda_{k_6}^{(0)} + \sum{}' \frac{V_{6,7,8}}{\lambda_{k_8}^{(0)} \left(+ \sum{}' \frac{V_{8,9,10}}{\lambda_{k_{10}}^{(0)} + \ldots} \right)}} \right)}} \tag{A21}$$

Equation (A23) provides the formula for the eddy-damping rate for the dressed test mode. Using this renormalized response function, the fluctuation spectrum is obtained as is explained in the main text.

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