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Higher spin algebras as higher symmetries

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Abstract

The exhaustive study of the rigid symmetries of arbitrary free field theories is motivated, along several lines, as a preliminary step in the completion of the higher-spin interaction problem.

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1 Higher-spin interaction problem

Whereas covariant gauge theories describing arbitrary free massless fields on constant-curvature spacetimes of dimension n are firmly established by means of the unitary representation theory of the isometry groups, it still remains unclear whether non-trivial consistent self-couplings and/or cross-couplings among those fields may exist for $n > 2$, such that the deformed gauge algebra is non-Abelian. The goal of the present paper is to advocate that many informations on the interactions can be extracted from the symmetries of the free field theory.

The conventional local free field theories corresponding to unitary irreducible representations of the helicity group $SO(n-2)$ that are completely symmetric tensors have been constructed a while ago (for some introductions, see [1]). In order to have Lorentz invariance manifest and second order local field equations, the theory is expressed in terms of completely symmetric tensor gauge fields $h_{\mu_1\ldots\,\mu_s}$ of rank $s > 0$, the gauge transformation of which reads

$$\overset{(0)}{\delta_\xi}\, h_{\mu_1\mu_2\ldots\,\mu_s} = \overset{(0)}{\nabla}_{\mu_1}\, \xi_{\mu_2\ldots\mu_s} \,+\, \text{cyclic}\,, \tag{1}$$

where $\overset{(0)}{\nabla}$ is the covariant derivative with respect to the background Levi–Civita connection and "cyclic" stands for the sum of terms necessary to have symmetry of the right-and-side under permutations of the indices. The gauge parameter ξ is a completely symmetric traceless tensor field of rank $s-1$. In this field theory, the "spin" is equal to the rank s. For spin $s = 1$ the gauge field h_μ represents the photon with $U(1)$ gauge symmetry while for spin $s = 2$ the gauge field $h_{\mu\nu}$ represents the graviton with linearized diffeomorphism invariance. The gauge algebra of field independent gauge transformations such as (1) are of course Abelian.

Non-Abelian gauge theories for "lower spin" $s \leqslant 2$ are well known and essentially correspond to Yang-Mills ($s = 1$) and Einstein ($s = 2$) theories for which the underlying geometries (principal bundles and Riemannian geometry) were familiar to mathematicians before the construction of the physical theory. In contrast, the situation is rather different for "higher spins" $s > 2$ for which the underlying geometry (if any!) remains obscure. Due to this lack of information, it is natural to look for inspiration in the perturbative "reconstruction" of Einstein gravity as the non-Abelian gauge theory of a spin-two particle propagating on a constant-curvature spacetime (see *e.g.* [2] for a comprehensive review).

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Let us denote by $\overset{(0)}{S}[h_{\mu_1\ldots\mu_s}]$ the Poincaré-invariant, local, second-order, quadratic, ghost-free, gauge-invariant action of a spin-s symmetric tensor gauge field. In order to perform a pertubative analysis via the Noether method [3], the non-Abelian interaction problem for a collection h of higher (and possibly lower) spin gauge fields is formulated as a deformation problem.

Higher-spin interaction problem: *List all Poincaré-invariant local deformations*

$$S[h] = \overset{(0)}{S}[h] + \varepsilon\,\overset{(1)}{S}[h] + \mathcal{O}(\varepsilon^2)$$

of a positive sum

$$\overset{(0)}{S}[h] = \sum_s \overset{(0)}{S}[h_{\mu_1\ldots\mu_s}]$$

of quadratic actions such that the deformed local gauge symmetries

$$\delta_\xi h = \overset{(0)}{\delta_\xi} h + \varepsilon\,\overset{(1)}{\delta_\xi} h + \mathcal{O}(\varepsilon^2)$$

are already non-Abelian at first order in the deformation parameters ε and do not arise from local redefinitions

$$h \to h + \varepsilon\,\phi(h) + \mathcal{O}(\varepsilon^2)\,, \quad \xi \to \xi + \varepsilon\,\zeta(h,\xi) + \mathcal{O}(\varepsilon^2)$$

of the gauge fields and parameters.

2 Noether method

The assumption that the deformations are formal power series in some deformation parameters enables to investigate the problem order by order. The crucial observation of any perturbation theory is that the first order deformations are constrained by the symmetries of the undeformed system. In the present case, the Noether method scrutinizes the gauge symmetry of the action, $\delta_\xi S = 0$. At zero order, the latter equation is satisfied by hypothesis. At first order, it reads

$$\overset{(0)}{\delta_\xi}\overset{(1)}{S} + \overset{(1)}{\delta_\xi}\overset{(0)}{S} = 0\,. \tag{2}$$

This equation may be used to constrain the possible deformations by reinterpreting them as familiar objects of the undeformed gauge theory.

By definition, an *observable* of a gauge theory is a functional which is gauge-invariant on-shell, while a *reducibility parameter* of a gauge theory is a gauge parameter such that the corresponding gauge variation vanishes off-shell.

First-order deformations in terms of the undeformed theory:
• *First-order deformations of the action are observables of the undeformed theory.*
• *First-order deformations of the gauge symmetries evaluated at reducibility parameters of the undeformed gauge theory define symmetries of the undeformed theory.*

Proof: In (2) the infinitesimal variation $\overset{(1)}{\delta_\xi}\overset{(0)}{S}$ of the undeformed action is proportional to the undeformed Euler–Lagrange equations. This proves the fist part of the theorem. Reducibility parameters $\overline{\xi}$ of the undeformed gauge theory verify by definition $\overset{(0)}{\delta_{\overline{\xi}}} h = 0$. Inserting this fact into (2) with $\xi = \overline{\xi}$ gives $\overset{(1)}{\delta_{\overline{\xi}}}\overset{(0)}{S} = 0$, which is precisely the translation of the second part of the theorem. □

In the mathematical litterature, a *(conformal) Killing tensor* of a pseudo-Riemannian manifold is a symmetric tensor field ξ such that its symmetrized covariant derivative with respect to the Levi–Civita connection, $\nabla_{\mu_1}\xi_{\mu_2\ldots\mu_s}$+cyclic, vanishes (modulo a term proportional to the metric for conformal Killing tensors). Therefore, any reducibility parameter of the spin-s symmetric gauge field theory on the constant-curvature spacetime $\mathcal{M}$ is identified with a Killing tensor of rank $s-1$ of the manifold $\mathcal{M}$.

The space of Killing tensors on any constant-curvature spacetime is known to be finite-dimensional [4], thus the linear gauge symmetries (1) are irreducible.

These results suggest two strategies for addressing the higher-spin interaction problem. The more ambitious is the computation of *all* local observables of the free gauge theory associated to deformations of the gauge algebra. This result would provide the exhaustive list of first order vertices, but this computation is technically demanding and seems out of reach in the completely general case. Nevertheless, the BRST reformulation of the problem [5] allowed the complete classification of non-Abelian deformations in various particular cases (see *e.g.* the review [6] and refs therein). Actually, a more humble strategy is the computation of *all* rigid symmetries of the free irreducible gauge theory. It is of interest because the knowledge of these rigid symmetries would strongly constrain the candidates for gauge symmetry deformations. Indeed, the constant tensors appearing in the rigid symmetries could be compared with the complete list [4] of constant-curvature spacetime Killing tensors.

3 Free theory symmetries

Bosonic fields are usually described in terms of their components living in some subspace V of the space $\otimes(\mathbb{R}^n)$ of tensors on $\mathbb{R}^n$ (*e.g.* $V = \odot(\mathbb{R}^n)$ for symmetric tensor fields). The background metric of the constant-curvature space-time induces some non-degenerate bilinear form on V. This defines a non-degenerate sesquilinear form $\langle \mid \rangle$ on the space $L^2(\mathbb{R}^n)\otimes V$ of square-integrable fields taking values in the countable space V (the components). Let $\dagger$ stands for the adjoint with respect to the sesquilinear form $\langle \mid \rangle$.

Any quadratic action for bosonic fields ψ can be expressed as a quadratic form

$$\overset{(0)}{S}[\psi] \,=\, \frac{1}{2}\,\langle\,\psi\mid \mathrm{K}\mid\psi\,\rangle\,, \tag{3}$$

where the kinetic operator K is self-adjoint, $\mathrm{K}^\dagger = \mathrm{K}$. Because the sesquilinear form $\langle \mid \rangle$ is non-degenerate, the Euler-Lagrange equation extremizing the quadratic action is the linear equation

$$\frac{\delta \overset{(0)}{S}}{\delta\langle\,\psi\mid} \,=\, \mathrm{K}|\,\psi\,\rangle \,=\, 0\,. \tag{4}$$

Moreover, the quadratic form $\langle\,\psi\mid\mathrm{K}\mid\psi\,\rangle$ is degenerate if and only if the kinetic operator K is degenerate. This happens if and only if there exists a linear operator P (on $L^2(\mathbb{R}^n)\otimes V$) such that $\mathrm{K\,P}=0$. Infinitesimal gauge symmetries read

$$\overset{(0)}{\delta}_\xi \mid\psi\,\rangle \,=\, \mathrm{P}\mid\chi\,\rangle\,,$$

with gauge parameters χ. The Noether identity is $\mathrm{P}^\dagger\mathrm{K} = (\mathrm{K\,P})^\dagger = 0$.

A *symmetry of the quadratic action* (3) is an invertible linear pseudo-differential operator U preserving the quadratic form $\langle \mid \mathrm{K} \mid \rangle$. In other words,

$$\mathrm{U}^\dagger\,\mathrm{K}\,\mathrm{U} \,=\, \mathrm{K}\,.$$

The *group of off-shell symmetries* is the group of symmetries of the quadratic action endowed with the composition $\circ$ as product. A *symmetry generator of the quadratic action* (3) is a linear differential operator T which is self-adjoint with respect to the quadratic form $\langle \mid \mathrm{K} \mid \rangle$. More concretely,

$$\mathrm{K\,T} \,=\, \mathrm{T}^\dagger\mathrm{K}\,.$$

Any symmetry generator T defines a symmetry $\mathrm{U} = e^{i\mathrm{T}}$ of the quadratic action (3). If $\mathrm{T}=\mathrm{T}^\dagger$ then the linear operator T is a symmetry generator of the quadratic action if and only it commutes with K. The *real Lie algebra of off-shell symmetries* is the algebra of symmetry generators of the quadratic action endowed with i times the commutator as Lie bracket, $\{\ ,\ \} := i\,[\ ,\ ]$.

A *symmetry of the linear equation* (4) is a linear differential operator T obeying

$$\mathrm{K}\,\mathrm{T} = \mathrm{S}\,\mathrm{K}\,, \tag{5}$$

for some linear operator S. Such a symmetry T preserves the space KerK of solutions to the equations of motion. Any symmetry generator T of the action (3) is always a symmetry of the equation of motion (4) with $\mathrm{S} = \mathrm{T}^\dagger$ in (5). A symmetry T is *trivial on-shell* if $\mathrm{T} = \mathrm{RK}$ for some linear operator R. Such an on-shell-trivial symmetry is always a symmetry of the field equation (4), since it obeys (5) with $\mathrm{S} = \mathrm{KR}$. The algebra of on-shell-trivial symmetries obviously forms a left ideal in the algebra of linear differential operators endowed with the composition as multiplication. Furthermore, it is also a right ideal in the algebra of symmetries of the linear equation (4). The *complex associative algebra of on-shell symmetries* is the associative algebra of symmetries of the linear equation quotiented by the two-sided ideal of on-shell-trivial symmetries. The *complex Lie algebra of on-shell symmetries* is the algebra of on-shell symmetries endowed with the commutator as Lie bracket.

Notice that when K is non-degenerate, a linear operator $\mathrm{T} = \mathrm{RK}$ is a symmetry generator of the quadratic action (3) if and only if R is self-adjoint. Moreover, the Lie subalgebra of such on-shell-trivial symmetry generators is an ideal in the Lie algebra of off-shell symmetries.

4 Higher-spin algebras

Let $\mathfrak{g}$ be the Lie algebra corresponding to the finite-dimensional (conformal) isometry group G of the constant-curvature spacetime of dimension $n > 2$. For $n = 2$, the spacetime may be taken to be arbitrary and the conformal algebra is of course infinite-dimensional. If the free field theory is relativistic, then $\mathfrak{g}$ is linearly realized on the space $L^2(\mathbb{R}^n) \otimes V$ (respectively, KerK) of off-shell (resp. on-shell) fields. This induces a linear realization of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Endowed with the commutator as Lie bracket, this realization is nowadays refered to as *(conformal) on/off-shell higher-spin algebra of the constant-curvature spacetime.* The name comes from the fact that its generators are in "higher-spin" representations of the Lorentz group, and the algebra is said to be on or off shell wether the algebra is realized on the space of solutions of the Euler-Lagrange equations or not.

The isometry algebra $\mathfrak{g}$ of a constant-curvature spacetime is a module of the Lorentz subalgebra $\mathfrak{o}(n-1,1) \subset \mathfrak{g}$ for the adjoint representation. This module decomposes as the sum of two irreducible $\mathfrak{o}(n-1,1)$-modules: the "translations" are in the vector module $\cong \mathbb{R}^n$ while the boosts and rotations are in the antisymmetric module $\cong \wedge^2(\mathbb{R}^n)$. These representations are labeled by one-column Young diagrams of, respectively, one and two cells. The number of columns is associated with the spin.

Universal enveloping algebra of isometries: *The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the isometry algebra $\mathfrak{g}$ of an n-dimensional constant-curvature spacetime is an infinite-dimensional module of the general linear Lie algebra $\mathfrak{gl}(n)$, decomposing as an infinite sum of finite-dimensional irreducible $\mathfrak{gl}(n)$-modules labeled by the set of all Young diagrams with multiplicity one, the first column of which have length $\leqslant n$.*

Proof: The Poincaré-Birkhoff-Witt theorem states that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to the symmetric algebra $\odot(\mathfrak{g})$ as a vector space. As a ($\mathfrak{gl}(n)$-)module, the vector space $\mathfrak{g}$ is isomorphic to the sum $\mathbb{R}^n \oplus \wedge^2(\mathbb{R}^n)$ of irreducible modules. This leads to the following isomorphism of modules

$$\odot(\mathfrak{g}) \cong \Big(\odot(\mathbb{R}^n)\Big) \otimes \Big(\odot(\wedge^2(\mathbb{R}^n))\Big)\,. \tag{6}$$

The idea is to evaluate the right-hand-side of (6) using the available technology on Kronecker products of irreducible representations [7]. The module $\odot(\mathbb{R}^n)$ decomposes as the infinite sum of irreducible modules labeled by all one-row Young diagrams with multiplicity one. A formula of Littlewood for symmetric plethsyms implies that the module $\odot(\wedge^2(\mathbb{R}^n))$ decomposes as the infinite sum of irreducible modules with multiplicity one labeled by all Young diagrams with columns of even lengths. The Kronecker product in (6) decomposes as the infinite sum of all the Kronecker products between a one-row Young diagram and a Young diagram with columns of even lengths, each with multiplicity one. Using the Littlewood–Richardson rule, one may show that the result of this computation is the infinite sum of irreducible modules labeled with all possible Young diagrams, each with multiplicity one. The Young diagrams whose first column has length greater than n lead to vanishing modules, hence they do not appear in the series. $\square$

The higher-spin algebras are important in relativistic theories because they always appear as spacetime symmetry algebras in the free limit.

Spacetime symmetries of relativistic free theories: *If the Lie algebra of off/on-shell symmetries contains the (conformal) isometry algebra $\mathfrak{g}$ of some constant-curvature spacetime $\mathcal{M}$, then it also contains the (conformal) off/on-shell higher-spin algebra of $\mathcal{M}$.*

Proof: The Poincaré-Birkhoff-Witt theorem states that one can realize the universal enveloping $\mathcal{U}(\mathfrak{g})$ as Weyl-ordered polynomials in the elements of the Lie algebra $\mathfrak{g}$. The above theorem is proved by observing that any Weyl-ordered polynomial in on-shell symmetries is itself an on-shell symmetry. The same is true for symmetry generators. $\square$

Remark that both cases (off/on-shell) are respectively covered, but the fact that the corresponding Lie algebras are respectively real/complex is implicitly understood. As a corollary, this proves that any relativistic free theory has an infinite number of rigid symmetries, and therefore possess an infinite number of conserved currents via the Noether theorem. Notice that relativistic integrable models are precisely such that they possess an infinite set of commuting rigid symmetries corresponding to an infinite set of conserved charges in involution. Actually, the relationship between higher-spin algebras and integrable models appears to be very intimate.

Symmetries may be characterized by their action on the spacetime coordinates. A smooth change of coordinates is generated by a first-order linear differential operator. Therefore, a higher-order linear differential operator does not generate coordinate transformations. For instance, an isometry generator is a first-order linear differential operator corresponding to a Killing vector field, but the spacetime higher-symmetries are powers of such isometry generators, hence they are higher-order linear differential operators. They do not generate coordinate transformations and this explains why spacetime higher-symmetries are usually not considered in textbooks.

Let us focus on the first non-trivial example of free field theory: the quadratic action of a complex scalar field on an n-dimensional spacetime $\mathcal{M}$. In such case, the space $V = \mathbb{C}$ and the kinetic operator K can be taken to be a constant mass term plus the Laplacian on $\mathcal{M}$,

$$\overset{(0)}{\Box} = \overset{(0)}{\nabla}{}^{\mu} \overset{(0)}{\nabla}_{\mu} \; .$$

A scalar field is said to be *conformal* if its kinetic operator is the conformal Laplacian

$$\overset{(0)}{\Box} - \frac{n-2}{4\,(n-1)} \overset{(0)}{R} \, , \tag{7}$$

where R denotes the scalar curvature. The quadratic action and the linear equation are symmetric under the full conformal algebra $\mathfrak{o}(n,2)$ if and only if the scalar field is conformal and has conformal weight $1 - n/2$.

Higher symmetries of the conformal scalar field: *For the quadratic action of a complex conformal scalar field on a constant-curvature spacetime $\mathcal{M}$ of dimension $n \geqslant 2$, the following Lie algebras are isomorphic:*

• *The Lie algebra of off-shell symmetries quotiented by the ideal of on-shell-trivial symmetry generators,*
• *A real form of the Lie algebra of on-shell symmetries.*
• *The on-shell higher-spin algebra of the conformal group,*
• *The space of Weyl-ordered polynomials in the conformal Killing vector fields quotiented by the ideal generated by the conformal Laplacian, endowed with i times the commutator as Lie bracket. The symbols of these differential operators* T *may be represented by traceless symmetric tensor fields* $\bar{\xi}$

$$\mathrm{T} = \bar{\xi}^{\mu_1 \ldots \mu_r} \overset{(0)}{\nabla}_{\mu_1} \ldots \overset{(0)}{\nabla}_{\mu_r} + \textit{lower} + \textit{on-shell-trivial} \, ,$$

which are conformal Killing tensors.

In $n = 2$ dimensions the theorem is valid for an arbitrary spacetime manifold.

Proof: The theorem can be extracted from the results of [8] on flat spacetime of dimension $n > 2$ by taking into account that any constant-curvature spacetime $\mathcal{M}$ can be seen as a conics in the projective null cone of the ambient space $\mathbb{R}^{n,2}$. The two-dimensional case is easily adressed by using the left/right-moving coordinates. □

Notice that the on-shell higher-spin algebras of the constant-curvature spacetime for a conformal scalar field are proper subalgebras of the universal enveloping algebra of the conformal algebra $\mathfrak{g}$: they decompose as the infinite sum of irreducible $\mathfrak{o}(n,2)$-modules labeled by all two-row Young diagrams with multiplicity one, as reviewed in [9]. Remark also that the conformal on-shell higher-spin algebra of a two-dimensional spacetime for a massless scalar field is isomorphic to the direct sum of $\mathfrak{u}(1)$ and the two Lie algebras of differential operators for the left and right moving sectors respectively. Each such algebra of differential operators is isomorphic to the algebra $\mathcal{W}_\infty$ with zero central charge [10].

The deep connection between higher-spin algebras and integrable models is exhibited by the following result for $n = 2$.

Higher symmetries of the interacting scalar field: *A non-linear action of a real scalar field on two-dimensional flat spacetime, without derivative interaction term, of the form*

$$S[\phi] = \frac{1}{2}\langle \phi \,|\, \Box \,|\, \phi \rangle + \int d^2x\, V(\phi)\,, \qquad V(\phi) = \mathcal{O}(\phi^2)\,,$$

is invariant under an infinite number of local infinitesimal rigid symmetry transformations, independent of the coordinate x^μ, if and only if

$$V(\phi) = \pm\left(\frac{m}{\alpha}\right)^2 (\,\cosh(\alpha\,\phi) - 1)\,, \qquad m \in \mathbb{R}\,,\ \alpha \in \mathbb{R}_0\,,$$

i.e. either it corresponds to a free massless scalar field ($m = 0$), a free massive scalar field ($m \neq 0$, $\alpha \to 0$) or sine-Gordon theory ($m \neq 0$, $\alpha \neq 0$).

Moreover, via linearisation, there is a one-to-one correspondence between:

- *The non-trivial coordinate-independent local symmetry transformations of the sine-Gordon Lagrangian,*
- *The Lie algebra of coordinate-independent off-shell symmetries of a free real scalar field quotiented by the ideal of on-shell-trivial symmetry generators,*
- *A proper Abelian Lie subalgebra of the on-shell higher-spin algebra of the Minkowski plane,*
- *The space of harmonic odd polynomials in the momenta $\mathrm{P}_\mu = -i\,\partial_\mu$. These differential operators T may be represented by real traceless symmetric constant tensors λ:*

$$\mathrm{T} = i\,\lambda^{\mu_1\cdots\mu_{2q+1}}\partial_{\mu_1}\ldots\partial_{\mu_{2q+1}} \;+\; \text{on-shell-trivial}\,.$$

Proof: The first part of the theorem is a straightforward consequence of the results of [11] in the case when $V(\phi)$ is at least quadratic in ϕ. The second part is easily proven by computing all coordinate-independent symmetries of a free real scalar field via the method of [8] and comparing with the symmetry transformations of [11]. In both cases, the Noether correspondence between non-trivial conserved currents and non-trivial symmetries is performed via the Hamiltonian formulation of a two-dimensional scalar field in the light-cone coordinates. □

5 A gauge principle for higher-spins ?

The analogy with lower-spins suggests to guess the full non-Abelian gauge theory by making use of the "gauge principle." Moreover, this point of view actually provides a concrete motivation for using the higher-spin algebras in the interaction problem.

The idea is to consider some "matter" system described by a quadratic action (3) with some algebra of rigid symmetries. The rigid symmetries U of this matter sector are by definition in the "fundamental" representation of the algebra of off-shell symmetries of the action (3). Connections are usually introduced in order to "gauge" these rigid symmetries by allowing U to be a smooth function on $\mathbb{R}^n$ taking values in the group of off-shell symmetries of the action (3). In order to construct a covariant derivative $D = \partial + \Gamma$, one introduces a connection defined as a covariant vector field Γ_μ taking values in the Lie algebra of off-shell symmetries and transforming as

$$| \psi \rangle \longrightarrow \mathrm{U} \, | \psi \rangle \,, \quad \Gamma \longrightarrow \mathrm{U} \, D \, \mathrm{U}^{-1} \,, \tag{8}$$

in such a way that

$$D \, | \psi \rangle \longrightarrow \mathrm{U} \, D \, | \psi \rangle \,.$$

The minimal coupling is the replacement of all partial derivatives ∂ in the kinetic operator $\mathrm{K}(\partial)$ by covariant derivatives D which ensures that the quadratic action $\langle \psi \mid \mathrm{K}(D) \mid \psi \rangle$ is preserved by gauge symmetries (8). The connection transforms in the adjoint representation of the rigid symmetries while the matter field transforms in the fundamental.

The introduction of a connection requires the introduction of some new dynamical fields: the "gauge" sector. In Yang-Mills gauge theories, the rigid symmetry is internal and the connection is itself the spin-1 gauge field h_μ. For spacetime symmetries, the relation between the connection and the gauge field is more complicated. For instance, in Einstein gravity the Levi-Civita connection is expressed in terms of the first derivative of the metric via the torsionlessness and metricity constraints. In general, the spin-s tensor field propagating on a constant-curvature spacetime is expected to be the perturbation of some background field

$$g_{\mu_1 \ldots \mu_s} = \overset{(0)}{g}_{\mu_1 \ldots \mu_s} + \varepsilon \, h_{\mu_1 \ldots \mu_s} \,,$$

so that the deformed gauge symmetries would be of the form

$$\delta_\xi g_{\mu_1 \mu_2 \ldots \mu_s} = \varepsilon \, (D\xi)_{\mu_1 \mu_2 \ldots \mu_s} \,, \tag{9}$$

where the covariant derivative $D = \nabla + \mathcal{O}(\varepsilon)$ starts as the covariant derivative with respect to the Levi–Civita connection for the space-time metric plus non-minimal corrections. Thus the background connection is identified with the Levi-Civita connection for the background metric, and the linearization of (9) reproduces (1). Furthermore, the reducibility parameters of (1) exactly correspond to the gauge symmetries (9) leaving the background geometry invariant. In the present case, this group of rigid symmetries contains the isometry group $\mathfrak{g}$ of the constant-curvature spacetime. The classical theory of (in)homogeneous pseudo-orthogonal groups tells us that completely symmetric tensor fields which are invariant under $\mathfrak{g}$ are constructed from products of the background metric:

$$\overset{(0)}{g}_{(\mu_1 \mu_2} \cdots \overset{(0)}{g}_{\mu_{2m-1} \mu_{2m})} \,.$$

Thus only even-spin symmetric tensor fields can be perturbations of a non-vanishing higher-spin background in a constant-curvature spacetime. The first-order deformation of the gauge symmetries (1) following from (9) would be of the schematic form

$$\overset{(1)}{\delta}_\xi \, h_{\mu_1 \mu_2 \ldots \mu_s} = (\, \overset{(1)}{\Gamma} \cdot \xi \,)_{\mu_1 \mu_2 \ldots \mu_s} \,, \tag{10}$$

where $\overset{(1)}{\Gamma}$ stands for the linearized connection (including the linearized Levi-Civita connection) and the dot stands for the action on the gauge parameter ξ.

The conclusion is that there are two complementary but distinct ways of using rigid symmetries of the free theory in order to guess the proper gauge symmetry principle of higher-spin gauge theories. On the one hand, in the free (or integrable) "matter" sector, rigid symmetries may be gauged by the introduction of a connection and the minimal coupling leaves the action quadratic in the matter fields. The idea of using a massive scalar field as matter sector and an infinite tower of massless symmetric tensor fields as interacting gauge sector is in agreement with the isomorphism between the off-shell higher-spin algebra and of the reducibility parameters. If tensor fields are used as free matter sector, then the symmetry algebra would be larger. The structure of the universal enveloping algebra points towards a larger infinite tower of gauge fields including mixed-symmetry tensors. On the other hand, in the free "gauge" sector, rigid symmetries linked with reducibility parameters may arise from the linearization (10) of the gauge symmetries of some non-linear action. Thus the complete knowledge of the rigid symmetries and the reducibility parameters of free higher-spin gauge theories would indicate what can be the linearized connection.

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