Reduction of couplings: from Finiteness to Fuzzy extra dimensions

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Abstract

Finite Unified Theories (FUTs) are N=1 supersymmetric Grand Unified Theories, which can be made all-loop finite, both in the dimensionless (gauge and Yukawa couplings) and dimensionful (soft supersymmetry breaking terms) sectors. This remarkable property, based on the reduction of couplings at the quantum level, provides a drastic reduction in the number of free parameters, which in turn leads to an accurate prediction of the top quark mass in the dimensionless sector, and predictions for the Higgs boson mass and the supersymmetric spectrum in the dimensionful sector. Here we examine the predictions of two such FUTs. Next we consider gauge theories defined in higher dimensions, where the extra dimensions form a fuzzy space (a finite matrix manifold). We emphasize some striking features emerging such as (i) the appearance of non-abelian gauge theories in four dimensions starting from an abelian gauge theory in higher dimensions, (ii) the fact that the spontaneous symmetry breaking of the theory takes place entirely in the extra dimensions and (iii) the renormalizability of the theory both in higher as well as in four dimensions. This scheme represents so far an excellent example in which classical reduction of couplings takes place. However since it leads to renormalizable theories, has the ingredients to become a framework for quantum reduction too.

1 Introduction

The theoretical efforts to establish a deeper understanding of Nature have led to very interesting frameworks such as String theories and Non-commutative Geometry both of which aim to describe physics at the Planck scale. Looking for the origin of the idea that coordinates might not commute we might have to go back to the days of Heisenberg. In the recent years the birth of such speculations can be found in refs. [1, 2]. In the spirit of Non-commutative Geometry also particle models with non-commutative gauge theory were explored [3] (see also [4]), [5, 6]. On the other hand the present intensive research has been triggered by the natural realization of non-commutativity of space in the string theory context of D-branes in the presence of a constant background antisymmetric field [7]. After the work of Seiberg and Witten [8], where a map (SW map) between non-commutative and commutative gauge theories has been described, there has been a lot of activity also in the construction of non-commutative phenomenological Lagrangians, for example various non-commutative standard model like Lagrangians have been proposed [9, 10][1]. In particular in ref. [10], following the SW map methods developed in refs. [11], a non-commutative standard model with $SU(3) \times SU(2) \times U(1)$ gauge group has been presented. These non-commutative models represent interesting generalizations of the SM and hint at possible new physics. However they do not address the usual problem of the SM, the presence of a plethora of free parameters mostly related to the ad hoc introduction of the Higgs and Yukawa sectors in the theory. At this stage it is worth recalling that various schemes, with the Coset Space Dimensional Reduction (CSDR) [14–17] being pioneer, were suggesting that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally after CSDR to Yukawa couplings in four dimensions.

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[1]These SM actions are mainly considered as effective actions because they are not renormalizable. The effective action interpretation is consistent with the SM in [10] being anomaly free [12]. Non-commutative phenomenology has been discussed in [13].

In the successes of the CSDR scheme certainly should be added the possibility to obtain chiral theories in four dimensions [18–21] as well as softly broken supersymmetric or non-supersymmetric theories starting from a supersymmetric gauge theory defined in higher dimensions [22].

The original plan of this paper was to present an overview covering the following subjects:
a) Quantum Reduction of Couplings and Finite Unified Theories
b) Classical Reduction of Couplings and Coset Space Dimensional Reduction
c) Renormalizable Unified Theories from Fuzzy Higher Dimensions [23]
The aim was to present an unified description of our current attempts to reduce the free parameters of the Standard Model by using Finite Unification and extra dimensions, but due to space limitations we will cover only the first and the third subjects.

Finite Unified Theories are $N = 1$ supersymmetric Grand Unified Theories (GUTs) which can be made finite even to all-loop orders, including the soft supersymmetry breaking sector. The method to construct GUTs with reduced independent parameters [25,26] consists of searching for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. Of particular interest is the possibility to find RGI relations among couplings that guarantee finitenes to all-orders in perturbation theory [27, 28]. In order to achieve the latter it is enough to study the uniqueness of the solutions to the one-loop finiteness conditions [27, 28]. The constructed *finite unified* $N = 1$ supersymmetric SU(5) GUTs, using the above tools, predicted correctly from the dimensionless sector (Gauge-Yukawa unification), among others, the top quark mass [29]. The search for RGI relations and finiteness has been extended to the soft supersymmetry breaking sector (SSB) of these theories [30, 31], which involves parameters of dimension one and two. Eventually, the full theories can be made all-loop finite and their predictive power is extended to the Higgs sector and the supersymmetric spectrum (s-spectrum).

2   Reduction of Couplings and Finiteness in $N = 1$ SUSY Gauge Theories

Here let us review the main points and ideas concerning the *reduction of couplings* and *finiteness* in $N = 1$ supersymmetric theories. A RGI relation among couplings g_i, $\Phi(g_1, \cdots, g_N) = 0$, has to satisfy the partial differential equation $\mu \, d\Phi/d\mu = \sum_{i=1}^{N} \beta_i \, \partial\Phi/\partial g_i = 0$, where β_i is the β-function of g_i. There exist $(N-1)$ independent Φ's, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs) [26], $\beta_g \, (dg_i/dg) = \beta_i$, $i = 1, \cdots, N$, where g and β_g are the primary coupling and its β-function. Using all the $(N-1)\,\Phi$'s to impose RGI relations, one can in principle express all the couplings in terms of a single coupling g. The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, whose uniqueness can be investigated at the one-loop level.

Finiteness can be understood by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W = \frac{1}{2} \, m^{ij} \, \Phi_i \, \Phi_j + \frac{1}{6} \, C^{ijk} \, \Phi_i \, \Phi_j \, \Phi_k \ , \tag{1}$$

where m^{ij} (the mass terms) and C^{ijk} (the Yukawa couplings) are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G.

The one-loop β-function of the gauge coupling g is given by

$$\beta_g^{(1)} \;=\; \frac{dg}{dt} = \frac{g^3}{16\pi^2} \big[\sum_i l(R_i) - 3 \, C_2(G) \big] \ , \tag{2}$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C^{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_i^j of the matter fields Φ_i as:

$$\beta_C^{ijk} = \frac{d}{dt} \, C^{ijk} = C^{ijp} \sum_{n=1} \frac{1}{(16\pi^2)^n} \, \gamma_p^{k(n)} + (k \leftrightarrow i) + (k \leftrightarrow j) \ . \tag{3}$$

At one-loop level γ_i^j is given by

$$\gamma_i^{j(1)} = \frac{1}{2} C_{ipq} C^{jpq} - 2\, g^2\, C_2(R_i) \delta_i^j \;, \tag{4}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C_{ijk}^*$.

All the one-loop β-functions of the theory vanish if the β-function of the gauge coupling $\beta_g^{(1)}$, and the anomalous dimensions $\gamma_i^{j(1)}$, vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G) \,, \; \frac{1}{2} C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i) \;, \tag{5}$$

where $l(R_i)$ is the Dynkin index of R_i, and $C_2(G)$ is the quadratic Casimir invariant of the adjoint representation of G.

A very interesting result is that the conditions (5) are necessary and sufficient for finiteness at the two-loop level [32, 33].

The one- and two-loop finiteness conditions (5) restrict considerably the possible choices of the irreducible representations R_i for a given group G as well as the Yukawa couplings in the superpotential (1). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a $U(1)$ gauge group is incompatible with the condition (5), due to $C_2[U(1)] = 0$. This leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding low-energy, effective theory.

The finiteness conditions impose relations between gauge and Yukawa couplings. Therefore, we have to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs) to all orders. The all-loop order finiteness theorem of ref. [27] is based on: (a) the structure of the supercurrent in $N = 1$ SYM and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [27]. Alternatively, similar results can be obtained [28, 34] using an analysis of the all-loop NSVZ gauge beta-function [35].

3   Soft supersymmetry breaking and finiteness

The above described method of reducing the dimensionless couplings has been extended [30, 31] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. Recently very interesting progress has been made [36–44] concerning the renormalization properties of the SSB parameters, based conceptually and technically on the work of ref. [38]. In this work the powerful supergraph method [41] for studying supersymmetric theories has been applied to the softly broken ones by using the "spurion" external space-time independent superfields [42]. In the latter method a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire "vacuum expectation values". Based on this method the relations among the soft term renormalization and that of an unbroken supersymmetric theory have been derived. In particular the β-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. The key point in the strategy of refs. [36]- [44] in solving the set of coupled differential equations so as to be able to express all parameters in a RGI way, was to transform the partial differential operators involved to total derivative operators [36]. It is indeed possible to do this on the RGI surface which is defined by the solution of the reduction equations. In addition it was found that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule at one-loop [40]. This result was generalized to two-loops for finite theories [44], and then to all-loops for general Gauge-Yukawa and Finite Unified Theories [37].

In order to obtain a feeling of some of the above results, consider the superpotential given by (1)

along with the Lagrangian for SSB terms

$$-\mathcal{L}_{\mathrm{SB}} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j$$
$$+ \frac{1}{2} (m^2)^j_i \phi^{*\,i} \phi_j + \frac{1}{2} M \lambda\lambda + \mathrm{H.c.}, \tag{6}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i, λ are the gauginos and M their unified mass. Since only finite theories are considered here, it is assumed that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. It is also assumed that the reduction equations admit power series solutions of the form

$$C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n} . \tag{7}$$

According to the finiteness theorem [27], the theory is then finite to all-orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma^{j(1)}_i$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved by [33]

$$h^{ijk} = -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)} g + O(g^5) . \tag{8}$$

An additional constraint in the SSB sector up to two-loops [44], concerns the soft scalar masses as follows

$$\frac{(m_i^2 + m_j^2 + m_k^2)}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2} \Delta^{(2)} + O(g^4) \tag{9}$$

for i, j, k with $\rho^{ijk}_{(0)} \neq 0$, where $\Delta^{(2)}$ is the two-loop correction

$$\Delta^{(2)} = -2 \sum_l [(m_l^2/MM^\dagger) - (1/3)] \, T(R_l), \tag{10}$$

which vanishes for the universal choice [33], i.e. when all the soft scalar masses are the same at the unification point.

If we know higher-loop β-functions explicitly, we can follow the same procedure and find higher-loop RGI relations among SSB terms. However, the β-functions of the soft scalar masses are explicitly known only up to two loops. In order to obtain higher-loop results, we need something else instead of knowledge of explicit β-functions, e.g. some relations among β-functions.

The recent progress made using the spurion technique [41, 42] leads to the following all-loop relations among SSB β-functions, [36–44]

$$\beta_M = 2\mathcal{O}\left(\frac{\beta_g}{g}\right) , \tag{11}$$

$$\beta_h^{ijk} = \gamma^i_{\ l} h^{ljk} + \gamma^j_{\ l} h^{ilk} + \gamma^k_{\ l} h^{ijl}$$
$$-2\gamma^i_{1\,l} C^{ljk} - 2\gamma^j_{1\,l} C^{ilk} - 2\gamma^k_{1\,l} C^{ijl} , \tag{12}$$

$$(\beta_{m^2})^i_{\ j} = \left[\Delta + X\frac{\partial}{\partial g}\right] \gamma^i_{\ j} , \tag{13}$$

$$\mathcal{O} = \left(Mg^2\frac{\partial}{\partial g^2} - h^{lmn}\frac{\partial}{\partial C^{lmn}}\right) , \tag{14}$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} \tag{15}$$

$$+ \tilde{C}_{lmn}\frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn}\frac{\partial}{\partial C^{lmn}} , \tag{16}$$

where $(\gamma_1)^i_{\ j} = \mathcal{O}\gamma^i_{\ j}$, $C_{lmn} = (C^{lmn})^*$, and

$$\tilde{C}^{ijk} = (m^2)^i_{\ l} C^{ljk} + (m^2)^j_{\ l} C^{ilk} + (m^2)^k_{\ l} C^{ijl} . \tag{17}$$

It was also found [43] that the relation

$$h^{ijk} = -M(C^{ijk})' \equiv -M \frac{dC^{ijk}(g)}{d\ln g} \ , \tag{18}$$

among couplings is all-loop RGI. Furthermore, using the all-loop gauge β-function of Novikov *et al.* [35] given by

$$\beta_g^{\mathrm{NSVZ}} = \frac{g^3}{16\pi^2} \left[\frac{\sum_l T(R_l)(1 - \gamma_l/2) - 3C(G)}{1 - g^2 C(G)/8\pi^2} \right] \ , \tag{19}$$

it was found the all-loop RGI sum rule [37],

$$
\begin{aligned}
m_i^2 + m_j^2 + m_k^2 = |M|^2 \Big\{ & \frac{1}{1 - g^2 C(G)/(8\pi^2)} \frac{d\ln C^{ijk}}{d\ln g} \\
& + \frac{1}{2} \frac{d^2 \ln C^{ijk}}{d(\ln g)^2} \Big\} + \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2} \frac{d\ln C^{ijk}}{d\ln g} \ .
\end{aligned}
\tag{20}
$$

In addition the exact-β-function for m^2 in the NSVZ scheme has been obtained [37] for the first time and is given by

$$
\begin{aligned}
\beta_{m_i^2}^{\mathrm{NSVZ}} = \Bigg[& |M|^2 \Big\{ \frac{1}{1 - g^2 C(G)/(8\pi^2)} \frac{d}{d\ln g} + \frac{1}{2} \frac{d^2}{d(\ln g)^2} \Big\} \\
& + \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2} \frac{d}{d\ln g} \Bigg] \gamma_i^{\mathrm{NSVZ}} \ .
\end{aligned}
\tag{21}
$$

4 Finite Unified Theories

In this section we examine two concrete $SU(5)$ finite models, where the reduction of couplings in the dimensionless and dimensionful sector has been achieved. For other interesting Finite Unified Theories based on cross group structure see ref. [45]. A predictive Gauge-Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.

2. Three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i$, $\mathbf{10}_i$ ($i = 1, 2, 3$), which obviously should not couple to the adjoint $\mathbf{24}$.

3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model. The model of ref. [29], which will be labeled **A**, and a slight variation of this model (labeled **B**), which can also be obtained from the class of the models suggested by Kazakov *et al.* [36] with a modification to suppress non-diagonal anomalous dimensions[2].

The superpotential which describes the two models takes the form [29, 44]

$$
\begin{aligned}
W = \sum_{i=1}^{3} [\ & \frac{1}{2} g_i^u \, \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \, \mathbf{10}_i \overline{\mathbf{5}}_i \, \overline{H}_i \ ] \\
& + g_{23}^u \, \mathbf{10}_2 \mathbf{10}_3 H_4 + g_{23}^d \, \mathbf{10}_2 \overline{\mathbf{5}}_3 \, \overline{H}_4 + g_{32}^d \, \mathbf{10}_3 \overline{\mathbf{5}}_2 \, \overline{H}_4 \\
& + \sum_{a=1}^{4} g_a^f \, H_a \, \mathbf{24} \, \overline{H}_a + \frac{g^\lambda}{3} \, (\mathbf{24})^3 \ ,
\end{aligned}
\tag{22}
$$

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[2]An extension to three families, and the generation of quark mixing angles and masses in Finite Unified Theories has been addressed in [46], where several realistic examples are given. These extensions are not considered here.

where H_a and $\overline{H}_a$ $(a = 1, \ldots, 4)$ stand for the Higgs quintets and anti-quintets.

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ for the models $\{\mathbf{A}, \mathbf{B}\}$ are:

$$
\begin{aligned}
(g_1^u)^2 &= \{\frac{8}{5}, \frac{8}{5}\}g^2 \ , \ (g_1^d)^2 = \{\frac{6}{5}, \frac{6}{5}\}g^2 \ , \\
(g_2^u)^2 &= (g_3^u)^2 = \{\frac{8}{5}, \frac{4}{5}\}g^2 \ , \\
(g_2^d)^2 &= (g_3^d)^2 = \{\frac{6}{5}, \frac{3}{5}\}g^2 \ , \\
(g_{23}^u)^2 &= \{0, \frac{4}{5}\}g^2 \ , \ (g_{23}^d)^2 = (g_{32}^d)^2 = \{0, \frac{3}{5}\}g^2 \ , \\
(g^\lambda)^2 &= \frac{15}{7}g^2 \ , \ (g_2^f)^2 = (g_3^f)^2 = \{0, \frac{1}{2}\}g^2 \ , \\
(g_1^f)^2 &= 0 \ , \ (g_4^f)^2 = \{1, 0\}g^2 \ .
\end{aligned}
\tag{23}
$$

According to the theorem of ref. [27] these models are finite to all orders. After the reduction of couplings the symmetry of W is enhanced [29, 44].

The main difference of the models $\mathbf{A}$ and $\mathbf{B}$ is that three pairs of Higgs quintets and anti-quintets couple to the $\mathbf{24}$ for $\mathbf{B}$ so that it is not necessary to mix them with H_4 and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of $SU(5)$.

In the dimensionful sector, the sum rule gives us the following boundary conditions at the GUT scale [44]:

$$
m_{H_u}^2 + 2m_{\mathbf{10}}^2 = m_{H_d}^2 + m_{\overline{\mathbf{5}}}^2 + m_{\mathbf{10}}^2 = M^2 \ \ \text{for} \ \ \mathbf{A} \ ;
\tag{24}
$$

$$
m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2 \ , \ m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3} \ ,
$$

$$
m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3} \ \ \ \text{for} \ \ \mathbf{B},
\tag{25}
$$

where we use as free parameters $m_{\overline{\mathbf{5}}} \equiv m_{\overline{\mathbf{5}}_3}$ and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for the model $\mathbf{A}$, and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for $\mathbf{B}$, in addition to M.

5   Predictions of Low Energy Parameters

Since the gauge symmetry is spontaneously broken below M_{GUT}, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (23), the $h = -MC$ relation, and the soft scalar-mass sum rule (9) at M_{GUT}, as applied in the two models. Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones with the relevant boundary conditions. Below M_{GUT} their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale M_s (which we define as the average of the stop masses) and therefore below that scale the effective theory is just the SM.

The predictions for the top quark mass M_t are ~ 183 and ~ 173 GeV in models $\mathbf{A}$ and $\mathbf{B}$ respectively, as can be seen if fig.1. Comparing these predictions with the most recent experimental value $M_t^{exp} = (172.72.9)$ GeV [47], and recalling that the theoretical values for M_t may suffer from a correction of $\sim 4\%$ [48], we see that clearly model $\mathbf{B}$ is preferred. In addition the value of $\tan\beta$ is found to be $\tan\beta \sim 54$ and ~ 48 for models $\mathbf{A}$ and $\mathbf{B}$ respectively.

In the SSB sector, besides the constraints imposed by finiteness there are further restrictions imposed by phenomenology. In the case where all the soft scalar masses are universal at the unfication scale, there is no region of M below $O(few\ TeV)$ in which $m_{\tilde{\tau}} > m_{\chi^0}$ is satisfied (where $m_{\tilde{\tau}}$ is the lightest $\tilde{\tau}$ mass, and m_{χ^0} the lightest neutralino mass, which is the lightest supersymmetric particle). But once the universality condition is relaxed this problem can be solved naturally (thanks to the sum rule). More specifically, using the sum rule (9) and imposing the conditions a) successful radiative
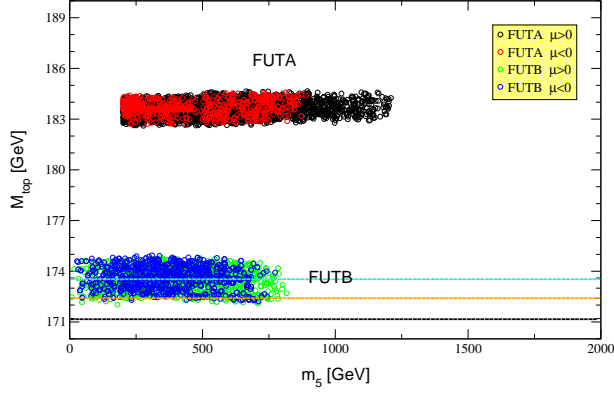
Figure 1: The physical top mass M_{top} as function of $m_\mathbf{5}$ for different values of M for models **FUTA** and **FUTB**, for $\mu < 0$ and $\mu > 0$.
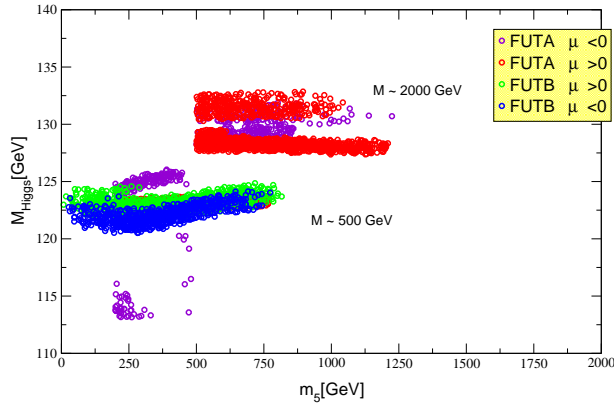

Figure 2: The lightest Higgs mass, m_H, as function of $m_\mathbf{5}$ for different values of M for both models.

electroweak symmetry breaking, b) $m_{\tilde{\tau}}^2 > 0$ and c) $m_{\tilde{\tau}} > m_{\chi^0}$, a comfortable parameter space for both models (although model **B** requires large $M \sim 1$ TeV) is found.

As additional constraints, we consider the following observables: the anomalous magnetic moment of the muon, $(g-2)_\mu$, rare b decays $BR(b \to s\gamma)$ and $BR(B_s \to \mu^+\mu^-)$, as well as the density of cold dark matter in the Universe, assuming it consists mainly of neutralinos.

For the branching ratio $BR(b \to s\gamma)$ [49], we take the present experimental value estimated by the Heavy Flavour Averaging Group (HFAG) is [50]

$$\mathrm{BR}(b \to s\gamma) = (3.54^{+0.30}_{-0.28}) \times 10^{-4}, \tag{26}$$

where the error includes an uncertainty due to the decay spectrum, as well as the statistical error. In the case of the anomalous magnetic moment of the muon $a_\mu \equiv (g-2)_\mu$, we compare our different models with

$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{theo}} = (25.29.2) \times 10^{-10}. \tag{27}$$

For the branching ratio $\mathrm{BR}(B_s \to \mu^+\mu^-)$, the SM prediction is $(3.40.5) \times 10^{-9}$ [51], and the present experimental upper limit from the Fermilab Tevatron collider is 3.4×10^{-7} at the 95% C.L. [52], providing the possibility for the MSSM to dominate the SM contribution.

The lightest supersymmetric particle (LSP) is an excellent candidate for cold dark matter (CDM) [53], with a density that falls naturally within the range

$$0.094 < \Omega_{\mathrm{CDM}}h^2 < 0.129 \tag{28}$$

favoured by a joint analysis of WMAP and other astrophysical and cosmological data [54].
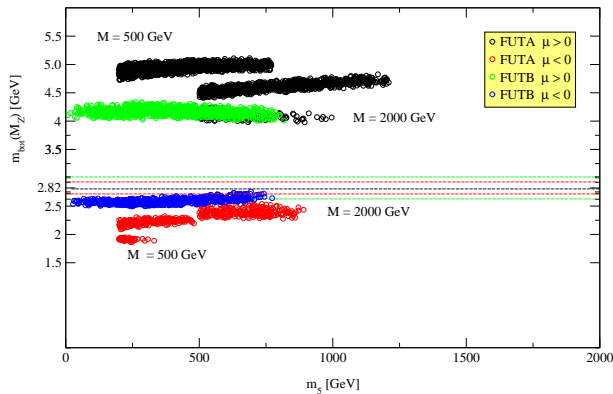
Figure 3: The running bottom mass $m_{bot}(M_Z)$ as function of m_5 for different values of M for models **FUTA** and **FUTB**, for $\mu < 0$ and $\mu > 0$.

In fig.2 we show the **FUTA** and **FUTB** results concerning M_h, for different values of M, for the cases where $\mu < 0$ and $\mu > 0$, the LSP is a neutralino χ^0 and the constraints imposed by the cold dark matter density Eq. (28), are satisfied.

The results for $\mu > 0$ and $\mu < 0$ are different for **FUTA**: with $\mu < 0$ the spectrum starts with an LSP around 750 GeV, whereas for $\mu > 0$ the spectrum starts around 500 GeV. The main difference, though, is in the value of the running bottom mass $m_{bot}(m_{bot})$, where we have included the corrections coming from bottom squark-gluino loops and top squark-chargino loops [55].

We give the predictions for the running bottom quark mass evaluated at M_Z, $m_{bot}(M_Z)$, to avoid the large QCD uncertainties at the pole mass. The value of $mb(M_Z)$ depends strongly on the sign of μ, due to the above mentioned susy radiative corrections. As can be seen from fig.3, both for models **A** and **B** the values for $\mu > 0$ are above the central experimental value, with $m_{bot} \sim 4.0 - 5.0$ GeV . For $\mu < 0$ on the other hand, model **B** has a clear overlap with the experimental allowed values, $m_{bot} \sim 2.5 - 2.8$ whereas for model **A**, $m_{bot} \sim 2.0 - 2.6$, there is only a small region of allowed parameter space at three sigma level, and only for large values of M.

In the case of **FUTB** the spectrum starts around $300 \sim 400$ GeV, and the $m_{bot} \sim 4 - 4.3$ GeV for $\mu < 0$ and $m_{bot} \sim 4.8 - 5.1$ GeV for $\mu > 0$.

The Higgs mass prediction of the two models is, although the details of each of the models differ, in the following range

$$m_h = \quad \sim 112 - 132 \quad GeV, \tag{29}$$

where the uncertainty comes from variations of the gaugino mass M and the soft scalar masses, and from finite (i.e. not logarithmically divergent) corrections in changing renormalization scheme. The one-loop radiative corrections have been included [57] for m_h, but not for the rest of the spectrum. In making the analysis, the value of M was varied from $200 - 2000$ GeV. We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5% of the FUT boundary conditions. This small variation does not give a noticeable effect in the results at low energies. The requirement $m_h > 114.4$ GeV [58] (neglecting the theoretical uncertainties) excludes the possibility of $M = 200$ GeV for FUTA, as seen also from the graph.

A more detailed numerical analysis, where the results of our program and of the known programs FeynHiggs [59] and Suspect [60] are combined, is currently in progress [61].

6   Unified Theories from Fuzzy Higher Dimensions

Coset Space Dimensional Reduction (CSDR) [14–17] is a unification scheme for obtaining realistic particle models from gauge theories on higher D-dimensional spaces M^D. It suggests that a unification of the gauge and Higgs sectors of the Standard Model can be achieved in higher than four dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally, after CSDR, to Yukawa couplings in four dimensions. We present a study of the CSDR in the non-commutative

context which sets the rules for constructing new particle models that might be phenomenologically interesting. One could study CSDR with the whole parent space M^D being non-commutative or with just non-commutative Minkowski space or non-commutative internal space. We specialize here to this last situation and therefore eventually we obtain Lorentz covariant theories on commutative Minkowski space. We further specialize to fuzzy non-commutativity, i.e. to matrix type non-commutativity. Thus, following [23], we consider non-commutative spaces like those studied in refs. [2,5,6] and implementing the CSDR principle on these spaces we obtain we obtain the rules for constructing new particle models.

7 The Fuzzy sphere

The fuzzy sphere [2, 62] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2 (for example spanned by the spherical harmonics) is truncated at a given frequency and thus becomes finite dimensional. The truncation has to be consistent with the associativity of the algebra and this can be nicely achieved relaxing the commutativity property of the algebra. The fuzzy sphere is the "space" described by this non-commutative algebra. The algebra itself is that of $N \times N$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbf{R}^3$ modulo the relation $\sum_{\hat{a}=1}^{3} x_{\hat{a}} x_{\hat{a}} = r^2$. The fuzzy sphere S_F^2 at fuzziness level $N-1$ is the non-commutative manifold whose coordinate functions $iX_{\hat{a}}$ are $N \times N$ hermitian matrices proportional to the generators of the N-dimensional representation of $SU(2)$. They satisfy the condition $\sum_{\hat{a}=1}^{3} X_{\hat{a}} X_{\hat{a}} = \alpha r^2$ and the commutation relations

$$[X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} X_{\hat{c}} \ , \tag{30}$$

where $C_{\hat{a}\hat{b}\hat{c}} = \varepsilon_{\hat{a}\hat{b}\hat{c}}/r$ while the proportionality factor α goes as N^2 for N large. Indeed it can be proven that for $N \to \infty$ one obtains the usual commutative sphere.

On the fuzzy sphere there is a natural $SU(2)$ covariant differential calculus. This calculus is three-dimensional and the derivations $e_{\hat{a}}$ along $X_{\hat{a}}$ of a function f are given by $e_{\hat{a}}(f) = [X_{\hat{a}}, f]$. Accordingly the action of the Lie derivatives on functions is given by

$$\mathcal{L}_{\hat{a}} f = [X_{\hat{a}}, f] \ ; \tag{31}$$

these Lie derivatives satisfy the Leibniz rule and the $SU(2)$ Lie algebra relation

$$[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} \mathcal{L}_{\hat{c}}. \tag{32}$$

In the $N \to \infty$ limit the derivations $e_{\hat{a}}$ become $e_{\hat{a}} = C_{\hat{a}\hat{b}\hat{c}} x^{\hat{b}} \partial^{\hat{c}}$ and only in this commutative limit the tangent space becomes two-dimensional. The exterior derivative is given by

$$df = [X_{\hat{a}}, f] \theta^{\hat{a}} \tag{33}$$

with $\theta^{\hat{a}}$ the one-forms dual to the vector fields $e_{\hat{a}}$, $< e_{\hat{a}}, \theta^{\hat{b}} > = \delta_{\hat{a}}^{\hat{b}}$. The space of one-forms is generated by the $\theta^{\hat{a}}$'s in the sense that for any one-form $\omega = \sum_i f_i dh_i \, t_i$ we can always write $\omega = \sum_{\hat{a}=1}^{3} \omega_{\hat{a}} \theta^{\hat{a}}$ with given functions $\omega_{\hat{a}}$ depending on the functions f_i, h_i and t_i. The action of the Lie derivatives $\mathcal{L}_{\hat{a}}$ on the one-forms $\theta^{\hat{b}}$ explicitly reads

$$\mathcal{L}_{\hat{a}}(\theta^{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}} \theta^{\hat{c}} \ . \tag{34}$$

On a general one-form $\omega = \omega_{\hat{a}} \theta^{\hat{a}}$ we have $\mathcal{L}_{\hat{b}} \omega = \mathcal{L}_{\hat{b}}(\omega_{\hat{a}} \theta^{\hat{a}}) = [X_{\hat{b}}, \omega_{\hat{a}}] \theta^{\hat{a}} - \omega_{\hat{a}} C^{\hat{a}}_{\hat{b}\hat{c}} \theta^{\hat{c}}$ and therefore

$$(\mathcal{L}_{\hat{b}} \omega)_{\hat{a}} = [X_{\hat{b}}, \omega_{\hat{a}}] - \omega_{\hat{c}} C^{\hat{c}}_{\hat{b}\hat{a}} \ ; \tag{35}$$

this formula will be fundamental for formulating the CSDR principle on fuzzy cosets.

The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S_F^2$, is easily obtained from that on M^4 and on S_F^2. For example a one-form A defined on $M^4 \times S_F^2$ is written as

$$A = A_\mu dx^\mu + A_{\hat{a}} \theta^{\hat{a}} \tag{36}$$

with $A_\mu = A_\mu(x^\mu, X_{\hat{a}})$ and $A_{\hat{a}} = A_{\hat{a}}(x^\mu, X_{\hat{a}})$.

One can also introduce spinors on the fuzzy sphere and study the Lie derivative on these spinors. Although here we have sketched the differential geometry on the fuzzy sphere, one can study other (higher-dimensional) fuzzy spaces (e.g. fuzzy CP^M) and with similar techniques their differential geometry.

8 Actions in higher dimensions seen as four-dimensional actions (Expansion in Kaluza-Klein modes)

First we consider on $M^4 \times (S/R)_F$ a non-commutative gauge theory with gauge group $G = U(P)$ and examine its four-dimensional interpretation. $(S/R)_F$ is a fuzzy coset, for example the fuzzy sphere S_F^2. The action is

$$\mathcal{A}_{YM} = \frac{1}{4g^2} \int d^4x \, kTr \, tr_G \, F_{MN} F^{MN}, \tag{37}$$

where kTr denotes integration over the fuzzy coset $(S/R)_F$ described by $N \times N$ matrices; here the parameter k is related to the size of the fuzzy coset space. For example for the fuzzy sphere we have $r^2 = \sqrt{N^2 - 1}\pi k$ [2]. In the $N \to \infty$ limit kTr becomes the usual integral on the coset space. For finite N, Tr is a good integral because it has the cyclic property $Tr(f_1 \dots f_{p-1} f_p) = Tr(f_p f_1 \dots f_{p-1})$. It is also invariant under the action of the group S, that is infinitesimally given by the Lie derivative. It is also invariant under the action of the group S, that is infinitesimally given by the Lie derivative. In the action (37) tr_G is the gauge group G trace. The higher-dimensional field strength F_{MN}, decomposed in four-dimensional space-time and extra-dimensional components, reads as follows $(F_{\mu\nu}, F_{\mu\hat{b}}, F_{\hat{a}\hat{b}})$; explicitly the various components of the field strength are given by

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{38} \\
F_{\mu\hat{a}} &= \partial_\mu A_{\hat{a}} - [X_{\hat{a}}, A_\mu] + [A_\mu, A_{\hat{a}}], \\
F_{\hat{a}\hat{b}} &= [X_{\hat{a}}, A_{\hat{b}}] - [X_{\hat{b}}, A_{\hat{a}}] + [A_{\hat{a}}, A_{\hat{b}}] - C^{\hat{c}}_{\hat{a}\hat{b}} A_{\hat{c}}. \tag{39}
\end{aligned}
$$

Under an infinitesimal G gauge transformation $\lambda = \lambda(x^\mu, X^{\hat{a}})$ we have

$$\delta A_{\hat{a}} = -[X_{\hat{a}}, \lambda] + [\lambda, A_{\hat{a}}] \ , \tag{40}$$

thus F_{MN} is covariant under local G gauge transformations: $F_{MN} \to F_{MN} + [\lambda, F_{MN}]$. This is an infinitesimal abelian $U(1)$ gauge transformation if λ is just an antihermitian function of the coordinates $x^\mu, X^{\hat{a}}$ while it is an infinitesimal non-abelian $U(P)$ gauge transformation if λ is valued in $\mathrm{Lie}(U(P))$, the Lie algebra of hermitian $P \times P$ matrices. In the following we will always assume $\mathrm{Lie}(U(P))$ elements to commute with the coordinates $X^{\hat{a}}$. In fuzzy/non-commutative gauge theory and in Fuzzy-CSDR a fundamental role is played by the covariant coordinate,

$$\varphi_{\hat{a}} \equiv X_{\hat{a}} + A_{\hat{a}} \ . \tag{41}$$

This field transforms indeed covariantly under a gauge transformation, $\delta(\varphi_{\hat{a}}) = [\lambda, \varphi_{\hat{a}}]$. In terms of φ the field strength in the non-commutative directions reads,

$$
\begin{aligned}
F_{\mu\hat{a}} &= \partial_\mu \varphi_{\hat{a}} + [A_\mu, \varphi_{\hat{a}}] = D_\mu \varphi_{\hat{a}}, \tag{42} \\
F_{\hat{a}\hat{b}} &= [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}_{\hat{a}\hat{b}} \varphi_{\hat{c}} \ ; \tag{43}
\end{aligned}
$$

and using these expressions the action reads

$$\mathcal{A}_{YM} = \int d^4x \, Tr \, tr_G \left(\frac{k}{4g^2} F_{\mu\nu}^2 + \frac{k}{2g^2} (D_\mu \varphi_{\hat{a}})^2 - V(\varphi) \right), \tag{44}$$

where the potential term $V(\varphi)$ is the $F_{\hat{a}\hat{b}}$ kinetic term (in our conventions $F_{\hat{a}\hat{b}}$ is antihermitian so that $V(\varphi)$ is hermitian and non-negative)

$$
\begin{aligned}
V(\varphi) &= -\frac{k}{4g^2} Tr\, tr_G \sum_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} \\
&= -\frac{k}{4g^2} Tr\, tr_G \left([\varphi_{\hat{a}}, \varphi_{\hat{b}}][\varphi^{\hat{a}}, \varphi^{\hat{b}}] - 4C_{\hat{a}\hat{b}\hat{c}}\varphi^{\hat{a}}\varphi^{\hat{b}}\varphi^{\hat{c}} + 2r^{-2}\varphi^2 \right).
\end{aligned} \tag{45}
$$

The action (44) is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^\mu, X^{\hat{a}})$ can indeed be interpreted just as an M^4 gauge transformation. We write

$$
\lambda(x^\mu, X^{\hat{a}}) = \lambda^\alpha(x^\mu, X^{\hat{a}})\mathcal{T}^\alpha = \lambda^{h,\alpha}(x^\mu)T^h\mathcal{T}^\alpha , \tag{46}
$$

where $\mathcal{T}^\alpha$ are hermitian generators of $U(P)$, $\lambda^\alpha(x^\mu, X^{\hat{a}})$ are $n \times n$ antihermitian matrices and thus are expressible as $\lambda(x^\mu)^{\alpha,h}T^h$, where T^h are antihermitian generators of $U(n)$. The fields $\lambda(x^\mu)^{\alpha,h}$, with $h = 1, \ldots n^2$, are the Kaluza-Klein modes of $\lambda(x^\mu, X^{\hat{a}})^\alpha$. We now consider on equal footing the indices h and α and interpret the fields on the r.h.s. of (46) as one field valued in the tensor product Lie algebra $\text{Lie}(U(n)) \otimes \text{Lie}(U(P))$. This Lie algebra is indeed $\text{Lie}(U(nP))$ (the $(nP)^2$ generators $T^h\mathcal{T}^\alpha$ being $nP \times nP$ antihermitian matrices that are linear independent). Similarly we rewrite the gauge field A_ν as

$$
A_\nu(x^\mu, X^{\hat{a}}) = A_\nu^\alpha(x^\mu, X^{\hat{a}})\mathcal{T}^\alpha = A_\nu^{h,\alpha}(x^\mu)T^h\mathcal{T}^\alpha, \tag{47}
$$

and interpret it as a $\text{Lie}(U(nP))$ valued gauge field on M^4, and similarly for $\varphi_{\hat{a}}$. Finally $Tr\, tr_G$ is the trace over $U(nP)$ matrices in the fundamental representation.

Up to now we have just performed an ordinary fuzzy dimensional reduction. Indeed in the commutative case the expression (44) corresponds to rewriting the initial lagrangian on $M^4 \times S^2$ using spherical harmonics on S^2. Here the space of functions is finite dimensional and therefore the infinite tower of modes reduces to the finite sum given by Tr.

9 Non-trivial Dimensional reduction in the case of Fuzzy Extra Dimensions

Next we reduce the number of gauge fields and scalars in the action (44) by applying the Coset Space Dimensional Reduction (CSDR) scheme. Since $SU(2)$ acts on the fuzzy sphere $(SU(2)/U(1))_F$, and generally the group S acts on the fuzzy coset $(S/R)_F$, we can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be invariant under the infinitesimal $SU(2)$, respectively S, action up to an infinitesimal gauge transformation

$$
\mathcal{L}_{\hat{b}}\phi = \delta_{W_{\hat{b}}}\phi = W_{\hat{b}}\phi, \tag{48}
$$

$$
\mathcal{L}_{\hat{b}}A = \delta_{W_{\hat{b}}}A = -DW_{\hat{b}}, \tag{49}
$$

where A is the one-form gauge potential $A = A_\mu dx^\mu + A_{\hat{a}}\theta^{\hat{a}}$, and $W_{\hat{b}}$ depends only on the coset coordinates $X^{\hat{a}}$ and (like A_μ, A_a) is antihermitian. We thus write $W_{\hat{b}} = W_{\hat{b}}^\alpha \mathcal{T}^\alpha$, $\alpha = 1, 2 \ldots P^2$, where $\mathcal{T}^i$ are hermitian generators of $U(P)$ and $(W_b^i)^\dagger = -W_b^i$, here $\dagger$ is hermitian conjugation on the $X^{\hat{a}}$'s.

In terms of the covariant coordinate $\varphi_{\hat{d}} = X_{\hat{d}} + A_{\hat{d}}$ and of

$$
\omega_{\hat{a}} \equiv X_{\hat{a}} - W_{\hat{a}} , \tag{50}
$$

the CSDR constraints assume a particularly simple form, namely

$$
[\omega_{\hat{b}}, A_\mu] = 0, \tag{51}
$$

$$
C_{\hat{b}\hat{d}\hat{e}}\varphi^{\hat{e}} = [\omega_{\hat{b}}, \varphi_{\hat{d}}]. \tag{52}
$$

In addition we have a consistency condition following from the relation $[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\mathcal{L}_{\hat{c}}$:

$$
[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}}\omega_c, \tag{53}
$$

where $\omega_{\hat{a}}$ transforms as $\omega_{\hat{a}} \to \omega_{\hat{a}}' = g\omega_{\hat{a}}g^{-1}$. One proceeds in a similar way for the spinor fields [23].

9.1 Solving the CSDR constraints for the fuzzy sphere

We consider $(S/R)_F = S_F^2$, i.e. the fuzzy sphere, and to be definite at fuzziness level $N-1$ ($N \times N$ matrices). We study here the basic example where the gauge group is $G = U(1)$. In this case the $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}})$ appearing in the consistency condition (53) are $N \times N$ antihermitian matrices and therefore can be interpreted as elements of $\text{Lie}(U(N))$. On the other hand the $\omega_{\hat{a}}$ satisfy the commutation relations (53) of $\text{Lie}(SU(2))$. Therefore in order to satisfy the consistency condition (53) we have to embed $\text{Lie}(SU(2))$ in $\text{Lie}(U(N))$. Let T^h with $h = 1, \ldots, N^2$ be the generators of $\text{Lie}(U(N))$ in the fundamental representation, we can always use the convention $h = (\hat{a}, u)$ with $\hat{a} = 1, 2, 3$ and $u = 4, 5, \ldots, N^2$ where the $T^{\hat{a}}$ satisfy the $SU(2)$ Lie algebra,

$$[T^{\hat{a}}, T^{\hat{b}}] = C^{\hat{a}\hat{b}}_{\ \ \hat{c}} T^{\hat{c}} \ . \tag{54}$$

Then we define an embedding by identifying

$$\omega_{\hat{a}} = T_{\hat{a}}. \tag{55}$$

The constraint (51), $[\omega_{\hat{b}}, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centralizer of the image of $SU(2)$ in $U(N)$, i.e.

$$K = C_{U(N)}(SU((2))) = SU(N-2) \times U(1) \times U(1) \ ,$$

where the last $U(1)$ is the $U(1)$ of $U(N) \simeq SU(N) \times U(1)$. The functions $A_\mu(x, X)$ are arbitrary functions of x but the X dependence is such that $A_\mu(x, X)$ is $\text{Lie}(K)$ valued instead of $\text{Lie}(U(N))$, i.e. eventually we have a four-dimensional gauge potential $A_\mu(x)$ with values in $\text{Lie}(K)$. Concerning the constraint (52), it is satisfied by choosing

$$\varphi_{\hat{a}} = r\varphi(x)\omega_{\hat{a}} \ , \tag{56}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ which is a singlet under the four-dimensional gauge group K.

The choice (55) defines one of the possible embedding of $\text{Lie}(SU(2))$ in $\text{Lie}(U(N))$. For example we could also embed $\text{Lie}(SU(2))$ in $\text{Lie}(U(N))$ using the irreducible N-dimensional rep. of $SU(2)$, i.e. we could identify $\omega_{\hat{a}} = X_{\hat{a}}$. The constraint (51) in this case implies that the four-dimensional gauge group is $U(1)$ so that $A_\mu(x)$ is $U(1)$ valued. The constraint (52) leads again to the scalar singlet $\varphi(x)$.

In general, we start with a $U(1)$ gauge theory on $M^4 \times S_F^2$. We solve the CSDR constraint (53) by embedding $SU(2)$ in $U(N)$. There exist p_N embeddings, where p_N is the number of ways one can partition the integer N into a set of non-increasing positive integers [2]. Then the constraint (51) gives the surviving four-dimensional gauge group. The constraint (52) gives the surviving four-dimensional scalars and eq. (56) is always a solution but in general not the only one. By setting $\phi_{\hat{a}} = \omega_{\hat{a}}$ we obtain always a minimum of the potential. This minimum is given by the chosen embedding of $SU(2)$ in $U(N)$.

10 Discussion on the Fuzzy-CSDR

Non-commutative Geometry has been regarded as a promising framework for obtaining finite quantum field theories and for regularizing quantum field theories. In general quantization of field theories on non-commutative spaces has turned out to be much more difficult and with less attractive ultraviolet features than expected.

The Fuzzy-CSDR has different features from the ordinary CSDR leading therefore to new four-dimensional particle models. It may well be that Fuzzy-CSDR provides more realistic four-dimensional theories.

A major difference between fuzzy and ordinary SCDR is that in the fuzzy case one always embeds S in the gauge group G instead of embedding just R in G. This is due to the fact that the differential

calculus on the fuzzy coset space is based on $dimS$ derivations instead of the restricted $dimS - dimR$ used in the ordinary one. As a result the four-dimensional gauge group $H = C_G(R)$ appearing in the ordinary CSDR after the geometrical breaking and before the spontaneous symmetry breaking due to the four-dimensional Higgs fields does not appear in the Fuzzy-CSDR. In Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR constraints. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Correspondingly in the Yukawa sector of the theory we have the results of the spontaneous symmetry breaking, i.e. massive fermions and Yukawa interactions among fermions and the physical Higgs field. Having massive fermions in the final theory is a generic feature of CSDR when S is embedded in G [23]. We see that if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a $U(1)$ in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions.

The final point that we would like to stress here is the question of the renormalizability of the gauge theory defined on $M^4 \times (S/R)_F$ [24]. First we notice that the theory exhibits certain features so similar to a higher-dimensional gauge theory defined on $M^4 \times S/R$ that naturally it could be considered as a higher-dimensional theory too. For instance the isometries of the spaces $M^4 \times S/R$ and $M^4 \times (S/R)_F$ are the same. It does not matter if the compact space is fuzzy or not. For example in the case of the fuzzy sphere, i.e. $M^4 \times S^2_F$, the isometries are $SO(3,1) \times SO(3)$ as in the case of the continuous space, $M^4 \times S^2$. Similarly the coupling of a gauge theory defined on $M^4 \times S/R$ and on $M^4 \times (S/R)_F$ are both dimensionful and have exactly the same dimensionality. On the other hand the first theory is clearly non-renormalizable, while the latter is renormalizable (in the sense that divergencies can be removed by a finite number of counterterms). So from this point of view one finds a partial justification of the old hopes for considering quantum field theories on non-commutative structures. If this observation can lead to finite theories too, it remains as an open question.

Acknowledgments

References

[1] A. Connes, "Non-commutative Geometry", Academic Press 1994.

[2] J. Madore, "An Introduction to Non-commutative Differential Geometry and its Physical Applications", London Mathematical Society Lecture Note Series 257, Cambridge University Press 1999.

[3] A. Connes and J. Lott, Nucl. Phys. Proc. Suppl. **18B**, 29 (1991).

[4] C. P. Martin, J. M. Gracia-Bondia and J. C. Varilly, Phys. Rept. **294**, 363 (1998).

[5] M. Dubois-Violette, J. Madore and R. Kerner, Phys. Lett. B **217**, 485 (1989); Class. Quant. Grav. **6**, 1709 (1989); M. Dubois-Violette, R. Kerner and J. Madore, J. Math. Phys. **31**, 323 (1990).

[6] J. Madore, Phys. Lett. B **305**, 84 (1993); Wroclaw 1992 Proceedings, Spinors, twistors, Clifford algebras and quantum deformations, 285-298.

[7] A. Connes, M. R. Douglas and A. Schwarz, JHEP **9802**, 003 (1998).

[8] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999).

[9] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, hep-th/0107055.

[10] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002); P. Aschieri, B. Jurco, P. Schupp and J. Wess, Nucl. Phys. B **651**, 45 (2003).

[11] B. Jurco, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **17**, 521 (2000);B. Jurco, P. Schupp and J. Wess, Nucl. Phys. B **604**, 148 (2001);B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **21**, 383 (2001); G. Barnich, F. Brandt and M. Grigoriev, JHEP **0208**, 023 (2002).

[12] F. Brandt, C. P. Martin and F. R. Ruiz, JHEP **0307** (2003) 068.

[13] C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. B **518**, 201 (2001); W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, Eur. Phys. J. C **29**, 441 (2003); I. Hinchliffe, N. Kersting and Y. L. Ma, Int. J. Mod. Phys. A **19** (2004) 179; P. Schupp, J. Trampetic, J. Wess and G. Raffelt, Eur. Phys. J. C **36** (2004) 405.

[14] P. Forgacs and N. S. Manton, Commun. Math. Phys. **72**, 15 (1980).

[15] D. Kapetanakis and G. Zoupanos, Phys. Rept. **219**, 1 (1992).

[16] Y. A. Kubyshin, I. P. Volobuev, J. M. Mourao and G. Rudolph, "Dimensional Reduction Of Gauge Theories, Spontaneous Compactification And Model Building," *Lecture notes in Physics*, **Vol. 349**, Springer Verlag, Heidelberg 1989.

[17] F. A. Bais, K. J. Barnes, P. Forgacs and G. Zoupanos, Proc. *"HEP1985"*, Bari, Italy, p.60; P. Forgacs, D. Lust and G. Zoupanos, Proc. *"2nd Hellenic School on EPP"*, Corfu, Greece, Sep 1-20, 1985, p.86; G. Zoupanos, K. Farakos, D. Kapetanakis, G. Koutsoumbas,Proc. *"23 Rencontres de Moriond"*, Les Arcs, France, 1988, p.559; ibid, Proc. *"ICHEP 1988"*, Munich, Germany, p.1137; G. Zoupanos, Proc. *"Int. Warsaw Meeting on EPP"* 27-31 May, 1991, Kazimierz, Poland, p.446.

[18] G. Chapline and N. S. Manton, Nucl. Phys. **B 184**, 391 (1981); F. A. Bais, K. J. Barnes, P. Forgacs and G. Zoupanos, Nucl. Phys. **B 263**, 557 (1986); K. Farakos, G. Koutsoumbas, M. Surridge and G. Zoupanos, Nucl. Phys. **B 291**, 128 (1987); ibid., Phys. Lett. **B 191**, 135 (1987); Y. A. Kubyshin, J. M. Mourao, I. P. Volobujev, Int. J. Mod. Phys. **A 4**, 151 (1989).

[19] N. S. Manton, Nucl. Phys. **B 193**, 502 (1981); G. Chapline and R. Slansky, Nucl. Phys. B **209**, 461 (1982).

[20] C. Wetterich, Nucl. Phys. **B 222**, 20 (1983); L. Palla, Z. Phys. **C 24**, 195 (1984); K. Pilch and A. N. Schellekens, J. Math. Phys. **25**, 3455 (1984); P. Forgacs, Z. Horvath and L. Palla, Z. Phys. **C 30**, 261 (1986); K. J. Barnes, P. Forgacs, M. Surridge and G. Zoupanos, Z. Phys. **C 33**, 427 (1987).

[21] K. Farakos, D. Kapetanakis, G. Koutsoumbas, G. Zoupanos, Phys. Lett. **B 211**, 322 (1988); B. E. Hanlon and G. C. Joshi, Phys. Lett. **B 298**, 312 (1993).

[22] P. Manousselis and G. Zoupanos, JHEP 0411,025 (2004); ibid, JHEP **0203**, 002 (2002); ibid., Phys. Lett. B **518**, 171 (2001); ibid., Phys. Lett. B **504**, 122 (2001).

[23] P. Aschieri, J. Madore, P. Manousselis, G. Zoupanos, JHEP **0404** 034 (2004); ibid, *Fortsch.Phys.* **52** 718 (2004).

[24] P. Aschieri, J. Madore, P. Manousselis, G. Zoupanos, 3rd Summer School in Modern Mathematical Physics, Zlatibor, Serbia, and Montenegro, hep-th/0503039.

[25] Kubo, J.,Mondragón, M. and Zoupanos, G. (1994) *Nucl. Phys.* **B424** 291.

[26] Zimmermann, W., (1985) *Com. Math. Phys.* **97** 211; Oehme, R. and Zimmermann, W. (1985) *Com. Math. Phys.* **97** 569; Ma,E. (1978) *Phys.Rev* **D 17** 623; ibid (1985) **D31** 1143.

[27] Lucchesi, C., Piguet, O. and Sibold, K. (1988) *Helv. Phys. Acta* **61** 321; Piguet, O. and Sibold, K. (1986) *Intr. J. Mod. Phys.* **A1** 913; (1986) *Phys. Lett.* **B177** 373; see also Lucchesi, C. and Zoupanos, G. (1997) *Fortsch. Phys.* **45** 129.

[28] A. Ermushev, D. Kazakov and O. Tarasov (1987) *Nucl. Phys.* **281** 72; D. Kazakov, (1987) *Mod. Phys. Lett.* **A9** 663.

[29] D. Kapetanakis, M.Mondragón and G. Zoupanos (1993) *Zeit. f. Phys.* **C60** 181; Mondragón, M. and Zoupanos, G. (1995) *Nucl. Phys. B* (Proc. Suppl.) **37C** 98.

[30] Kubo, J.,Mondragón, M. and Zoupanos, G. (1996) *Phys. Lett.* **B389** 523.

[31] Jack, I. and Jones, D.R.T. (1995) *Phys. Lett.* **B349** 294.

[32] D.R.T. Jones, L. Mezincescu and Y.-P. Yao, *Phys. Lett.* **B148** 317 (1984).

[33] Jack, I. and Jones, D.R.T. (1994) *Phys. Lett.* **B333** 372.

[34] R. Leigh and M. Strassler, (1995) *Nucl. Phys.* **B447** 95.

[35] Novikov, V., Shifman, M., Vainstein, A., and Zakharov, V. (1983) *Nucl. Phys.***B229** 381; (1986)*Phys. Lett.* **B166** 329; Shifman, M. (1996) *Int.J. Mod. Phys.***A11** 5761.

[36] Avdeev, L.V., Kazakov, D.I. and Kondrashuk, I.N. (1998) *Nucl. Phys.* **B510** 289; Kazakov, D.I. (1999) *Phys.Lett.* **B449** 201.

[37] T. Kobayashi, J. Kubo and G. Zoupanos, (1998) Phys. Lett. **B427** 291; T. Kobayashi et al, in proc. of "Supersymmetry, Supergravity and Superstrings", pp. 242-268, Seoul, 1999; T.Kobayashi et. al (2001) *Surveys High Energy Phys.* **16** 87-129.

[38] Yamada, Y. (1994) *Phys.Rev.* **D50** 3537.

[39] J. Hisano and M. Shifman, *Phys. Rev.* **D56** (1997) 5475.

[40] Kawamura, T., Kobayashi, T. and Kubo, J. (1997) *Phys. Lett.* **B405** 64.

[41] Delbourgo, R. (1975) *Nuovo Cim* **25A** 646; Salam, A. and Strathdee, J. (1975) *Nucl. Phys.* **B86** 142; Fujikawa, K. and Lang, W. (1975) *Nucl. Phys.* **B88** 61; Grisaru, M.T., Rocek, M. and Siegel, W. (1979) *Nucl. Phys.* **B59** 429.

[42] L.Girardello and M.Grisaru (1982)*Nucl. Phys.* **B194** 65.

[43] I. Jack, D.R.T. Jones and A. Pickering, *Phys. Lett.* **B426** (1998) 73.

[44] T. Kobayashi, J. Kubo, M.Mondragón and G. Zoupanos, (1998) *Nucl. Phys.* **B511** 45; Acta Phys. Polon. **B30** (1999) 2013; M. Mondragó n and G. Zoupanos, Acta Phys. Polon. B **34** (2003) 5459.

[45] E. Ma, M. Mondragó n and G. Zoupanos, JHEP **0412** (2004) 026 [arXiv:hep-ph/0407236].

[46] K. S. Babu, T. Enkhbat and I. Gogoladze (2003) Phys. Lett. B **555** 238.

[47] K. Sato, EPS05, Lisbon, Portugal, to be published in the proceedings.

[48] For an extended discussion and a complete list of references see: Kubo, J.,Mondragón, M. and Zoupanos, G. (1997) *Acta Phys. Polon.* **B27** 3911.

[49] (1996); P. Gambino and M. Misiak, *Nucl. Phys.* **B 611** (2001) 338, R. Barate et al. [ALEPH Collaboration], *Phys. Lett.* **B 429** (1998) 169; S. Chen et al. [CLEO Collaboration], *Phys. Rev. Lett.* **87** (2001) 251807, K. Abe et al. [Belle Collaboration], *Phys. Lett.* **B 511** (2001) 151, B. Aubert et al. [BABAR Collaboration]; hep-ex/0207074.

[50] R. Barate et al. [ALEPH Collaboration], *Phys. Lett.* **B 429** (1998) 169; S. Chen et al. [CLEO Collaboration], *Phys. Rev. Lett.* **87** (2001) 251807; P. Koppenburg et al. [Belle Collaboration], *Phys. Rev. Lett.* **93** (2004) 061803; B. Aubert et al. [BABAR Collaboration], hep-ex/0207074.

[51] G. Buchalla and A. Buras, *Nucl. Phys.* **B 400** (1993) 225; M. Misiak and J. Urban, *Phys. Lett.* **B 451** (1999) 161; A. Buras, *Phys. Lett.* **B 566** (2003) 115.

[52] M. Herndon [CDF and D0 Collaborations], FERMILAB-CONF-04-391-E, to appear in the proceedings of 32nd International Conference on High-Energy Physics (ICHEP 04), Beijing, China, 16-22 Aug 2004.

[53] H. Goldberg, *Phys. Rev. Lett.* **50** (1983) 1419; J. Ellis, J. Hagelin, D. Nanopoulos, K. Olive and M. Srednicki, *Nucl. Phys.* **B 238** (1984) 453.

[54] C. Bennett et al., *Astrophys. J. Suppl.* **148** (2003) 1; D. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. **148** (2003) 175.

[55] L. J. Hall, R. Rattazzi and U. Sarid, Phys. Rev. D **50**, (1994) 7048 ; R. Hempfling, Z. Phys. C **63**, (1994) 309; M. Carena, M. Olechowski, S. Pokorski and C. E. M. Wagner, Nucl. Phys. B **426** (1994) 269.

[56] K. Hagiwara et al., *Phys. Rev. D66*, 010001 (2002).

[57] Gladyshev, A.V., Kazakov, D.I., de Boer, W., Burkart, G. and Ehret, R. (1997) *Nucl. Phys.* **B498** 3; Carena, M. et. al., (1995) *Phys. Lett.* **B355** 209.

[58] LEP Higgs working group, *Phys. Lett.* **B 565** (2003) 61.

[59] S. Heinemeyer, W. Hollik and G. Weiglein, Comput. Phys. Commun. **124** (2000) 76; Eur. Phys. J. C **9** (1999) 343.

[60] A. Djouadi, J. L. Kneur and G. Moultaka, arXiv:hep-ph/0211331.

[61] Djouadi, A., Heinemeyer, S.,Mondragón, M. and Zoupanos, G., work in progress.

[62] J. Madore, Class. Quant. Grav. **9** (1992) 69.