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From BRST to light-cone description of higher spin gauge fields*

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Abstract

In this short note we show, at the level of action principles, how the light-cone action of higher spin gauge fields can easily be obtained from the BRST formulation through the elimination of quartets. We analyze how the algebra of cohomology classes is affected by such a reduction. By applying the reduction to the Poincaré generators, we give an alternative way of analyzing the physical spectrum of the Fronsdal type actions, with or without trace condition.

1 Introduction

The title of the lectures given by G. Barnich during the workshop was "Higher spin gauge fields: Basics". The discussion was restricted to free theories in 4 dimensions with integer spin. The following material was covered[1]:

1. Representations of the Poincaré group [3], [4]
2. Mass-shell field representations [5]
3. Variational principles [6], [7], [8]
4. BRST formulation [9], [10], [11]
5. Connection to string theory [9], [10], [11], [13], [12], [14], [15]
6. Trivial pairs and auxiliary fields [16], [17]
7. Connection to the light-cone formulation [18]
8. Connection to Vassiliev's formulation [19], [17]

In order to bridge between these lectures and the lectures by L. Brink "Light-cone frame formulation of field theories and string theories, a non-BRST formulation", we will elaborate on item 7 above and show, as a pedagogical exercice, how the light-cone action can be explicitly reached from the BRST formulation.

More precisely, after reviewing the BRST formulation of the Fronsdal action, we recall that in BRST language, the reduction to the light-cone gauge corresponds to the elimination of quartets [20]. The free light-cone action for higher spin gauge fields [18] is then recovered directly from

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[1]See also the reviews [1, 2] and the references cited therein.

the reduced BRST operator. We develop in general terms the reduction of the Lie algebra of BRST cohomology classes under the elimination of trivial pairs. As an application, we recover the expression of the Poincaré generators [18] in the light cone gauge. Finally, we use these expressions in 4 dimensions to give an alternative proof of the fact that the Fronsdal type action at level s describes

(i) massless particles of helicities $-s, -s+2, \ldots, s-2, s$ if no trace constraint is required,

(ii) massless particles of helicity $\pm s$ if the trace constraint is imposed [8, 21, 22].

2 BRST formulation of the Fronsdal Lagrangian

We will mostly follow the notations and conventions of [17], to which we refer for further details.

2.1 Degrees of freedom, constraints and BRST charge

The variables are $x^\mu, p_\mu, a^\mu, a^{\dagger\mu}$, where $\mu = 0, \ldots, d-1$. Classically, x^μ and p_μ correspond to coordinates on $T^*\mathbb{R}^{d-1}$ and $a^\mu, a^{\dagger\mu}$ correspond to internal degrees of freedom. After quantization, they satisfy the canonical commutation relations

$$[p_\nu, x^\mu] = -\imath\delta^\mu_\nu, \qquad [a^\mu, a^{\nu\dagger}] = \eta^{\mu\nu}, \tag{2.1}$$

where $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$. We assume that x^μ, p_μ are Hermitian, $(x^\mu)^\dagger = x^\mu$, $p^\dagger_\mu = p_\mu$, while a^μ and $a^{\dagger\mu}$ are interchanged by Hermitian conjugation.

The constraints of the system are

$$\mathcal{L} \equiv \eta^{\mu\nu} p_\mu p_\nu = 0, \quad \mathcal{S} \equiv p_\mu a^\mu = 0, \quad \mathcal{S}^\dagger \equiv p_\mu a^{\dagger\mu} = 0. \tag{2.2}$$

For the ghost pairs $(\theta, \mathcal{P})$, $(c^\dagger, b)$, and $(c, b^\dagger)$ corresponding to each of these constraints, we take the canonical commutation relations in the form[2]

$$[\mathcal{P}, \theta] = -\imath, \qquad [c, b^\dagger] = 1, \qquad [b, c^\dagger] = -1. \tag{2.3}$$

The ghost-number assignments are

$$\mathrm{gh}(\theta) = \mathrm{gh}(c) = \mathrm{gh}(c^\dagger) = 1, \qquad \mathrm{gh}(\mathcal{P}) = \mathrm{gh}(b) = \mathrm{gh}(b^\dagger) = -1. \tag{2.4}$$

The hermitian BRST operator is then given by

$$\Omega = \theta\mathcal{L} + c^\dagger\mathcal{S} + \mathcal{S}^\dagger c - \imath\mathcal{P}c^\dagger c, \tag{2.5}$$

and satisfies $\Omega^2 = 0$.

2.2 Representation space

Contrary to reference [17], we now choose the momentum instead of the coordinate representation for the operators (x^μ, p_ν) and also[3] the holomorphic instead of the Fock representation for the oscillators $(a_\mu, a^\dagger_\mu), (c, b^\dagger), (b, c^\dagger)$, while the pair $(\theta, \mathcal{P})$ continues to be quantized in the coordinate representation.

[2]We use the "super" convention that $(ab)^\dagger = (-1)^{|a||b|} b^\dagger a^\dagger$.

[3]This step will allows us to write rather compact formulas below, but is not really necessary for the analysis.

The "Hilbert space" $\mathcal{H}$ consists of the "wave functions" $\psi(p, \theta, \alpha^*, \beta^* \gamma^*) = \langle p\, \theta \alpha^* \beta^* \gamma^*, \psi \rangle$. On these functions, the operators p_μ, θ, $a^{\mu\dagger}$, $b^\dagger$, $c^\dagger$ act as multiplication by p_μ, θ, $\alpha^{\mu *}$, β^*, γ^*, while the operators x^μ, $\mathcal{P}$, a^μ, c, b act as $\imath \frac{\partial}{\partial p_\mu}$, $-\imath \frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial \alpha^*_\mu}$, $\frac{\partial}{\partial \beta^*}$, $-\frac{\partial}{\partial \gamma^*}$. The inner product is defined by

$$\langle \psi, \chi \rangle = \int d^d p\, d\theta (\prod_{\mu=0}^{d-1} \frac{d\alpha^{\mu *} d\alpha^\mu}{2\pi \imath}) d\beta^* d\gamma d\gamma^* d\beta$$
$$\exp\left(-\alpha^{\mu *}\alpha_\mu - \beta^* \gamma + \gamma^* \beta\right) [\psi(p, \theta, \alpha^*, \beta^* \gamma^*)]^* \chi(p, \theta, \alpha^*, \beta^*, \gamma^*). \quad (2.6)$$

The string field becomes

$$\Psi = \int d^d p\, d\theta (\prod_{\mu=0}^{d-1} \frac{d\alpha^*_\mu d\alpha_\mu}{2\pi \imath}) d\beta^* d\gamma d\gamma^* d\beta$$
$$\exp\left(-\alpha^{\mu *}\alpha_\mu - \beta^* \gamma + \gamma^* \beta\right) |p\, \theta \alpha \beta \gamma\rangle \varphi(p, \theta, \alpha^*, \beta^*, \gamma^*). \quad (2.7)$$

The ghost numbers of the component fields of $\varphi(p, \theta, \alpha^*, \beta^*, \gamma^*)$ are defined to be 1/2 minus the ghost number (2.4) of the corresponding state in the first quantized theory. The ghost-number-zero component of the string field can be parameterized as

$$\Psi^{(0)} = \Phi - \imath \theta b^\dagger C + c^\dagger b^\dagger D, \quad (2.8)$$

where

$$\begin{gathered} \langle p\theta\alpha^*\beta^*\gamma^*, \Phi\rangle = \Phi(p, \alpha^*), \qquad \langle p\theta\alpha^*\beta^*\gamma^*, C\rangle = C(p, \alpha^*), \\ \langle p\theta\alpha^*\beta^*\gamma^*, D\rangle = D(p, \alpha^*)\,. \end{gathered} \quad (2.9)$$

2.3 Physical and master actions

The action

$$\boldsymbol{S}[\Psi] = -\tfrac{1}{2} \int d^d p\, \langle \Psi, \Omega \Psi \rangle \quad (2.10)$$

is the solution of the Batalin-Vilkovisky master action associated with the physical action

$$\boldsymbol{S}^{\mathrm{ph}}[\Psi^{(0)}] = -\tfrac{1}{2} \int d^d p\, \langle \Psi^{(0)}, \Omega \Psi^{(0)} \rangle. \quad (2.11)$$

The trace and level operators

$$\mathcal{T} = \eta_{\mu\nu} a^\mu a^\nu + 2cb\,, \qquad N_s = a^\dagger_\mu a^\mu - c^\dagger b + b^\dagger c - s \quad (2.12)$$

commute with the BRST charge Ω and satisfy $[N_s, \mathcal{T}] = -2\mathcal{T}$ so that they can be consistently imposed as restrictions on the Hilbert space and on the string field: $\tilde{\mathcal{H}}_s = \mathrm{Ker}\mathcal{T} \cap \mathrm{Ker} N_s$, $N_s \tilde{\Psi} = 0 = \mathcal{T}\tilde{\Psi}$. Assuming appropriate reality conditions, the restricted action

$$\boldsymbol{S}^{\mathrm{ph}}[\tilde{\Psi}^{(0)}] = -\tfrac{1}{2} \int d^d p\, \langle \tilde{\Psi}^{(0)}, \Omega \tilde{\Psi}^{(0)} \rangle \quad (2.13)$$

is the gauge invariant Fronsdal action [8] (written with additional auxiliary fields contained in $C(p, \alpha^*)$), while

$$\boldsymbol{S}[\tilde{\Psi}] = -\tfrac{1}{2} \int d^d p\, \langle \tilde{\Psi}, \Omega \tilde{\Psi} \rangle \quad (2.14)$$

is the associated proper solution of the Batalin-Vilkovisky master equation.

3 Reduction to the light-cone formulation

As an alternative to the analysis in [8], [21] or [22], we will analyze the physical spectrum of actions (2.11) and (2.13) by going to the light-cone formulation. In order to do so, we use the grading introduced in [20] to relate the BRST to the light-cone formulation in the context of string theory. This grading allows one to identify quartets for the full BRST charge on the level of the lowest part in the expansion.

3.1 Unconstrained reduction

Here and in the following, indices are lowered and raised with $\eta_{\mu\nu} = diag(-1, 1 \ldots, 1)$. The light-cone operators are defined through $p^{\pm} = \frac{1}{\sqrt{2}}(p^0 \pm p^{d-1})$, $a^{\pm} = \frac{1}{\sqrt{2}}(a^0 \pm a^{d-1})$, with $[a^-, a^{+\dagger}] = -1 = [a^+, a^{-\dagger}]$ with transverse momenta and oscillators $p^i, a^i, a^{i\dagger}$, $i = 1, \ldots, d-2$, unchanged. In the following, we denote the transverse momenta and oscillators collectively by a superscript T. The BRST operator becomes

$$\begin{aligned}\Omega = \theta(p_i p^i - 2p^- p^+) + c^\dagger(-p^- a^+ - p^+ a^- + p_i a^i) + \\ + (-p^- a^{+\dagger} - p^+ a^{-\dagger} + p_i a^{i\dagger})c - \imath \mathcal{P} c^\dagger c. \quad (3.1)\end{aligned}$$

The grading is defined through the eigenvalues of the operator

$$G = 2a^{-\dagger} a^+ - 2a^{+\dagger} a^- - b^\dagger c - c^\dagger b, \quad (3.2)$$

so that

$$\begin{aligned} &|a^+| = 2, \qquad |a^{+\dagger}| = 2, \qquad |a^{-\dagger}| = -2, \qquad |a^-| = -2, \\ &|c| = 1, \qquad |c^\dagger| = 1, \qquad |b^\dagger| = -1, \qquad |b| = -1, \end{aligned} \quad (3.3)$$

with all other operators having grading zero. The operator G acts on the wave functions $\phi(p, \theta, \alpha^*, \beta^*, \gamma^*)$ and the associated string fields as

$$G = -2\alpha^{-*}\frac{\partial}{\partial \alpha^{-*}} + 2\alpha^{+*}\frac{\partial}{\partial \alpha^{+*}} - \beta^*\frac{\partial}{\partial \beta^*} + \gamma^*\frac{\partial}{\partial \gamma^*}. \quad (3.4)$$

The Hilbert space $\mathcal{H}$ can be decomposed according to this grading and, as the wave functions are polynomial in the holomorphic variables, each wave function has a component with lowest degree. The BRST operator decomposes as $\Omega = \sum_{i=-1}^{3} \Omega_i$, with

$$\begin{aligned} &\Omega_{-1} = -p^+(c^\dagger a^- + a^{-\dagger} c), \qquad \Omega_0 = \theta(-2p^- p^+ + p_i p^i), \\ &\Omega_1 = c^\dagger p_i a^i + p_i a^{i\dagger} c, \qquad \Omega_2 = -\imath \mathcal{P} c^\dagger c, \qquad \Omega_3 = -p^-(c^\dagger a^+ + a^{+\dagger} c). \end{aligned} \quad (3.5)$$

If we assume $p^+ \neq 0$, which is a standard assumption in the light-cone approach, states depending non trivially on $\alpha^{+*}, \alpha^{-*}, \beta^*, \gamma^*$ form quartets for the differential Ω_{-1}. Indeed, we have

$$\begin{aligned} &K_1 = -\frac{1}{p^+}(a^{+\dagger} b - b^\dagger a^+), \qquad [K_1, \Omega_{-1}] = N_q, \\ &N_q = -a^{+\dagger} a^- - a^{-\dagger} a^+ - c^\dagger b + b^\dagger c, \end{aligned} \quad (3.6)$$

where the operator N_q acts on states and the associated string fields as the operator that counts the unphysical variables,

$$N_q = \alpha^{-*}\frac{\partial}{\partial \alpha^{-*}} + \alpha^{+*}\frac{\partial}{\partial \alpha^{+*}} + \beta^*\frac{\partial}{\partial \beta^*} + \gamma^*\frac{\partial}{\partial \gamma^*}. \quad (3.7)$$

The Hilbert space $\mathcal{H}$ can accordingly be decomposed as $\mathcal{H} = \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$ with $\mathcal{G} = \mathrm{Im}\,\Omega_{-1}$, $\mathcal{F} = \mathrm{Im}\,K_1$ and $\mathcal{E} \subset \mathrm{Ker}\,\Omega_{-1}$ corresponding to the states $\psi(p, \theta, \alpha^{*T})$ that are eigenvectors of N_q with eigenvalue zero.

Proposition **3.6** of [17] then implies that the full BRST charge is invertible between $\mathcal{F}$ and $\mathcal{G}$: if $\overset{\mathcal{FG}}{\Theta} \equiv (\overset{\mathcal{GF}}{\Omega})^{-1}$ denotes the inverse of the BRST charge $\overset{\mathcal{GF}}{\Omega}$ between $\mathcal{F}$ and $\mathcal{G}$ and $\rho = (\overset{\mathcal{GF}}{\Omega}_{-1})^{-1}$, we have

$$\overset{\mathcal{FG}}{\Theta} = \sum_{n\geqslant 0} (-1)^n \rho[(\sum_{i\geqslant 0} \overset{\mathcal{GF}}{\Omega}_i)\rho]^n. \tag{3.8}$$

Note that the inverse exists because the degree of each wave function is bounded from below. Furthermore, because the cohomology of Ω_{-1} is concentrated in degree 0, the system $(\Omega, \mathcal{H})$ can be reduced to the system $(\Omega_0, \mathcal{E})$. According to proposition **3.4** of [17], the associated field theories are related through the elimination of generalized auxiliary fields[4]. In addition, the elimination of the quartets is compatible with the inner product: after elimination of the generalized auxiliary fields the inner product on $\mathcal{H}$ reduces to an inner product on $\mathcal{E}$ that makes $\overset{\mathcal{EE}}{\Omega_0}$ formally self-adjoint. This means that the elimination can be done on the level of the corresponding master equations, and corresponds to the elimination of generalized auxiliary fields in the sense of [23]. Explicitly, the reduced master action obtained from (2.10) reads

$$\begin{aligned} \boldsymbol{S}[\Psi^{\mathcal{E}}] &= -\frac{1}{2} \int d^dp \, \langle \Psi^{\mathcal{E}}, \Omega_0 \Psi^{\mathcal{E}} \rangle \; = \\ &= \; -\frac{1}{2} \int d^dp \left(\prod_{i=1}^{d-2} \frac{d\alpha_i^* d\alpha_i}{2\pi\imath} \right) \exp{(-\alpha^{*i}\alpha_i)} [\varphi(p, \alpha^{*T})]^* (-2p^-p^+ + p_ip^i) \varphi(p, \alpha^{*T}). \end{aligned} \tag{3.9}$$

As it depends only on ghost number 0 fields, it coincides with the physical action and there is no gauge invariance left.

3.2 Imposing the level and trace constraints

Because both Ω_{-1} and K_1 commute with N_s the elimination can be done consistently also in $\mathrm{Ker}\,N_s$.

The reduction can also be done in $\mathrm{Ker}\,\mathcal{T}$. To see this we use an expansion with respect to the eigenvalues of N_q. The trace operator decomposes as $\mathcal{T} = \mathcal{T}^{-2} + \mathcal{T}^0$, with $\mathcal{T}^{-2} = -2a^-a^+ + 2cb$, $\mathcal{T}^0 = a^ia_i \equiv \mathcal{T}^T$, while each wave function decomposes as $\psi = \psi^0 + \dots \psi^M$. Suppose $\mathcal{T}\psi = 0$. If $\Omega_{-1}\psi = 0$, we have, for $M \neq 0$, $\psi^M = \Omega_{-1}(\frac{1}{M}K_1\psi^M)$. Because $[\mathcal{T}, K_1] = -\frac{4}{p^+}a^+b$, the term $K_1\psi$ is double $\mathcal{T}$ traceless. In fact one can adjust $K_1\psi$ to make it $\mathcal{T}$ traceless using the expansion in homogeneity in all oscilators. Namely, for each monomial $\psi_{(d)}$ of definite homogeneity d in all the oscillators, $\phi_{(d)} \equiv K_1\psi_{(d)} - k(-2\alpha^{*+}\alpha^{*-} + 2\gamma^*\beta^* + \alpha^{*i}\alpha^*_i)\mathcal{T}K_1\psi_{(d)}$ is $\mathcal{T}$ traceless for an appropriate choice of k. Since Ω_{-1} commutes with both $\mathcal{T}$ and $-2\alpha^{*+}\alpha^{*-} + 2\gamma^*\beta^* + \alpha^{*i}\alpha^*_i$, it follows, for $M \neq 0$, that $\Omega_{-1}(\frac{1}{M}\phi_{(d)}) = \psi^M_{(d)} + \dots$, where the dots denote terms of order strictly lower than M. Hence, by summing over all monomials, $\psi - \Omega_{-1}(\frac{1}{M}\phi)$ is $\mathcal{T}$ traceless, Ω_{-1} closed and its expansion stops at the latest at $M - 1$. By induction we conclude that $\mathcal{T}\psi = 0 = \Omega_{-1}\psi$ implies $\psi = \psi^0 + \Omega_{-1}\chi$, with $\mathcal{T}\chi = 0$ and $\mathcal{T}^T\psi^0 = \mathcal{T}\psi^0 = 0$.

The same reasoning as in the previous subsection then shows that the Fronsdal master action

[4]Because we are not concerned with issues concerning locality of the associated field theories in our discussion, the coordinates x^μ and the momenta p_μ are considered on the same foot with the other degrees of freedom. The discussion of [17] is then simplified because only the Hilbert space $\mathcal{H}$ is involved in the discussion.

(2.14) can be reduced to the master and physical action

$$
\begin{aligned}
\boldsymbol{S}[\tilde{\Psi}^{\mathcal{E}}] = -\tfrac{1}{2}\int d^dp\,\langle\tilde{\Psi}^{\mathcal{E}},\Omega_0\tilde{\Psi}^{\mathcal{E}}\rangle \ &= \\
= \ -\tfrac{1}{2}\int d^dp \left(\prod_{l=1}^{d-2}\frac{d\alpha_l^* d\alpha_l}{2\pi\imath}\right) &\exp\left(-\alpha^{*i}\alpha_i\right)[\tilde{\varphi}(p,\alpha^{*T})]^*(-2p^-p^+ + p_ip^i)\tilde{\varphi}(p,\alpha^{*T})
\end{aligned}
\tag{3.10}
$$

with $\mathcal{T}^T\tilde{\Psi}^{\mathcal{E}} = 0 = \mathcal{T}^T\varphi(p,\alpha^{*T})$ and also $N_s\tilde{\Psi}^{\mathcal{E}} = 0 = N_s\varphi(p,\alpha^{*T})$.

3.3 Reduction of observables

By observables, we mean in this context BRST cohomology classes of operators, i.e., equivalence classes of operators A which commute with the BRST charge Ω, up to such operators which are in the image of the adjoint action of Ω,

$$
[\Omega, A] = 0, \quad A \sim A + [\Omega, B], \tag{3.11}
$$

As discussed for instance in [13, 16, 17], if one restricts to antihermitian operators in ghost number zero, these cohomology classes describe equivalence classes of global symmetries of the action (2.11), with

$$
\delta_A\Psi^{(0)} = A\Psi^{(0)}. \tag{3.12}
$$

In order for such observables to be well defined for the constrained action (2.13), they have to commute with the trace and the level constraint,

$$
[N_s, A] = 0 = [\mathcal{T}, A]. \tag{3.13}
$$

In the following discussion we have in mind in particular the observables that describe the Poincaré transformations. There are several ways to reduce such observables to the light-cone formulation. One possibility, discussed in [18], is to work out the compensating gauge transformation needed to stay in the light-cone gauge when performing the covariant Poincaré transformations. Another possibility, followed in [11], is to use the quantum Dirac bracket. Alternatively, one could use a Dirac-type antibracket for representatives $\langle\Psi, A\Psi\rangle$ of antifield BRST cohomology classes in ghost number -1, which describe the global symmetry on the level of the master action (2.10). This approach has the advantage to extend to the non linear and non Lagrangian setting and will be discussed elsewhere. Here we will follow a first quantized BRST approach.

Let $(h_A) \equiv (e_\alpha, f_a, g_a)$ denote a basis for $\mathcal{H} = \mathcal{E}\oplus\mathcal{F}\oplus\mathcal{G}$. Assuming only existence of $(\overset{\mathcal{GF}}{\Omega})^{-1} \equiv \overset{\mathcal{FG}}{\Theta}$, we can consider the following invertible change of basis:

$$
\tilde{e}_\alpha = e_\alpha - f_b(\overset{\mathcal{FE}}{R})^b_\alpha, \qquad \overset{\mathcal{FE}}{R} = \overset{\mathcal{FG}}{\Theta}\,\overset{\mathcal{GE}}{\Omega} \tag{3.14}
$$

$$
\tilde{f}_a = f_b(\overset{\mathcal{FG}}{\Theta})^b_a, \tag{3.15}
$$

$$
\tilde{g}_a = e_\beta(\overset{\mathcal{EG}}{L})^\beta_a + f_b(\overset{\mathcal{FF}}{\Omega}\,(\overset{\mathcal{FG}}{\Theta})^b_a + g_a, \qquad \overset{\mathcal{EG}}{L} = \overset{\mathcal{EF}}{\Omega}\,\overset{\mathcal{FG}}{\Theta}. \tag{3.16}
$$

In other words, $\tilde{h} = hC$ where

$$
C = \begin{pmatrix} \overset{\mathcal{EE}}{\delta} & 0 & \overset{\mathcal{EG}}{L} \\ -\overset{\mathcal{EE}}{R} & \overset{\mathcal{FG}}{\Theta} & \overset{\mathcal{FF}}{\Omega}\,\overset{\mathcal{FG}}{\Theta} \\ 0 & 0 & \overset{\mathcal{GG}}{\delta} \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} \overset{\mathcal{EE}}{\delta} & 0 & -\overset{\mathcal{EG}}{L} \\ \overset{\mathcal{GE}}{\Omega} & \overset{\mathcal{GF}}{\Omega} & \overset{\mathcal{GG}}{\Omega} \\ 0 & 0 & \overset{\mathcal{GG}}{\delta} \end{pmatrix}. \tag{3.17}
$$

with δ denoting the identity matrix. The new basis is chosen in such a way that the expression for Ω simplifies to

$$\hat{\Omega} = C^{-1}\Omega C = \begin{pmatrix} \tilde{\Omega} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \overset{\mathcal{GF}}{\delta} & 0 \end{pmatrix}, \tag{3.18}$$

with

$$\tilde{\Omega} = \overset{\mathcal{EE}}{\Omega} - \overset{\mathcal{EG}}{L}\,\overset{\mathcal{GE}}{\Omega} - \overset{\mathcal{EF}}{\Omega}\,\overset{\mathcal{FE}}{R} + \overset{\mathcal{EG}}{L}\,\overset{\mathcal{GF}}{\Omega}\,\overset{\mathcal{FE}}{R} = \overset{\mathcal{EE}}{\Omega} - \overset{\mathcal{EF}}{\Omega}\,\overset{\mathcal{FG}}{\Theta}\,\overset{\mathcal{GE}}{\Omega}. \tag{3.19}$$

If $\hat{A} = C^{-1}AC$, and A is of parity $|A|$, $[\Omega, A] = 0$ implies

$$\hat{A} = \begin{pmatrix} \tilde{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + [\hat{\Omega}, A_E], \tag{3.20}$$

where

$$\tilde{A} = \overset{\mathcal{EE}}{A} - \overset{\mathcal{EG}}{L}\,\overset{\mathcal{GE}}{A} - \overset{\mathcal{EF}}{A}\,\overset{\mathcal{FE}}{R} + \overset{\mathcal{EG}}{L}\,\overset{\mathcal{GF}}{A}\,\overset{\mathcal{FE}}{R}, \tag{3.21}$$

$$A_E = \begin{pmatrix} 0 & 0 & (-1)^{|A|}(\overset{\mathcal{EF}}{A} - \overset{\mathcal{EG}}{L}\,\overset{\mathcal{GF}}{A})\,\overset{\mathcal{FG}}{\Theta} \\ (\overset{\mathcal{GE}}{A} - \overset{\mathcal{GF}}{A}\,\overset{\mathcal{FE}}{R}) & \overset{\mathcal{GF}}{A}\,\overset{\mathcal{FG}}{\Theta} & (-1)^{|A|}\overset{\mathcal{GH}}{\Omega}\,\overset{\mathcal{HF}}{A}\,\overset{\mathcal{FG}}{\Theta} \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.22}$$

In particular,

$$\Omega_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \overset{\mathcal{FF}}{\delta} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

Equation (3.20) implies that the map g from linear operators on $\mathcal{H}$ to linear operators on $\mathcal{E}$ defined through $g(A) = \tilde{A}$ induces a well defined map $g^{\#}$ from the cohomology of $[\Omega, \cdot]$ to the cohomology of $[\tilde{\Omega}, \cdot]$. It also follows directly from (3.20) that $g^{\#}([A]) = [\tilde{A}]$ is an isomorphism which preserves the Lie algebra of observables: $g^{\#}([[A], [B]] = [[\tilde{A}], [\tilde{B}]]$.

3.4 Lorentz generators in the light-cone

Let $\xi_\mu = \delta_\mu + \omega_{\mu\nu}x^\nu$, with $\omega_{\mu\nu}$ antisymmetric. The Poincaré generators are then obtained from the antihermitian generating operators

$$L(\omega, a) = -\imath\big(\xi^\mu p_\mu + a^\dagger_\mu [p^\mu, \xi^\nu] a_\nu\big), \tag{3.24}$$

satisfying

$$[\Omega, L] = 0, \quad [L(\omega_1, \delta_1), L(\omega_2, \delta_2)] = L([\omega_1, \omega_2], \omega_1\delta_2 - \omega_2\delta_1) \tag{3.25}$$

through

$$J^{\mu\nu} \equiv -2\imath\frac{\partial L}{\partial \omega_{\mu\nu}} = \big(x^\mu p^\nu - x^\nu p^\mu - \imath(a^{\mu\dagger}a^\nu - a^{\nu\dagger}a^\mu)\big), \tag{3.26}$$

$$P^\mu \equiv \imath\frac{\partial L}{\partial \delta_\mu} = p^\mu. \tag{3.27}$$

Now we apply the reduction formula (3.21) to the Poincaré generators in the light-cone basis, $P^+, P^-, P^i, J^{ij}, J^{+i}, J^{-i}, J^{+-}$. First we note that the translations and the orbital parts of the Lorentz transformations carry vanishing degree and map $\mathcal{E}$ to $\mathcal{E}$. Moreover all the terms in (3.21) besides the first one vanish for these operators because R and L carry strictly positive degree.

The same reasoning applies to the spin part $S^{ij} = -\imath({a^i}^\dagger a^j - {a^j}^\dagger a^i)$ of J^{ij}, which are unchanged in the reduction. The spin operator $S^{+-} = -\imath({a^+}^\dagger a^- - {a^-}^\dagger a^+)$ also carries vanishing degree, but since its restriction to $\mathcal{E}$ vanishes, $\tilde{S}^{+-} = 0$. The spin operator $S^{+i} = -\imath({a^+}^\dagger a^i - {a^i}^\dagger a^+)$ carries degree $+2$ and therefore $\tilde{S}^{+i} = 0$. Finally, the spin operator $S^{-i} = -\imath({a^-}^\dagger a^i - {a^i}^\dagger a^-)$ has degree -2: therefore the first term in (3.21) vanishes. Furthermore, the fourth term also vanishes. Indeed, it contains both R and L operators whose expansion contains terms with degree greater or equal to 2. This is so because Θ is of strictly positive degree while the only possible contributions to $\overset{\mathcal{GE}}{\Omega}$ and $\overset{\mathcal{EF}}{\Omega}$ are coming from Ω_1 given in (3.5). Explicitly, one thus has

$$\tilde{S}^{-i} = -\overset{\mathcal{EF}}{S^{-i}}\,\rho\,\overset{\mathcal{GE}}{\Omega}_1 - \overset{\mathcal{EF}}{\Omega}_1\,\rho\,\overset{\mathcal{GE}}{S^{-i}}\,. \tag{3.28}$$

Choosing $\mathcal{G} = \Omega_{-1}\mathcal{H}$, $\mathcal{F} = \sum_{M>0}\frac{1}{M}K_1\Omega_{-1}\mathcal{H}^M$, where the expansion is with respect to the eigenvalues of N_q, we have $\rho = K_1$ and, for an arbitrary state $\psi^{\mathcal{E}} \in \mathcal{E}$,

$$\rho\overset{\mathcal{GE}}{\Omega}_1\psi^{\mathcal{E}} = \rho c^\dagger p_j a^j\,\psi^{\mathcal{E}} = K_1 c^\dagger p_j a^j\,\psi^{\mathcal{E}} = \frac{1}{p^+}{a^+}^\dagger p_j a^j\,\psi^{\mathcal{E}} \tag{3.29}$$

$$\rho\overset{\mathcal{GE}}{S^{-i}}\psi^{\mathcal{E}} = -\imath\rho {a^-}^\dagger a^i\,\psi^{\mathcal{E}} = -\imath K_1 {a^-}^\dagger a^i\,\psi^{\mathcal{E}} = \frac{\imath}{p^+}b^\dagger a^i\,\psi^{\mathcal{E}}\,. \tag{3.30}$$

Summing up, one finds

$$\tilde{S}^{-i} = \frac{\imath}{p^+}({a^i}^\dagger p^j a_j - p^j a_j^\dagger a^i). \tag{3.31}$$

3.5 Physical spectrum

We now analyze the particle content of the actions (2.11) and (2.13) in 4 dimensions.

The Pauli-Lubanski vector is given by

$$\tilde{W}_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{J}^{\nu\rho}P^\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{S}^{\nu\rho}P^\sigma, \qquad \epsilon_{0123} = -1. \tag{3.32}$$

Using

$$\tilde{S}^{-1} = \frac{\imath p_2}{p^+}({a^1}^\dagger a^2 - {a^2}^\dagger a^1)\,, \qquad \tilde{S}^{-2} = -\frac{\imath p_1}{p^+}({a^1}^\dagger a^2 - {a^2}^\dagger a^1)\,, \tag{3.33}$$

and $\epsilon_{+-12} = 1$, $\eta_{-+} = -1$ we get

$$\tilde{W}_+ = \tilde{S}^{12}P^- + \tilde{S}^{-1}P^2 - \tilde{S}^{-2}P^1 = -\imath({a^1}^\dagger a^2 - {a^2}^\dagger a^1)(p_+ - \frac{p_\mu p^\mu}{p^+}), \tag{3.34}$$

$$\tilde{W}_1 = \tilde{S}^{-2}P^+ = -\imath({a^1}^\dagger a^2 - {a^2}^\dagger a^1)p_1 \tag{3.35}$$

$$\tilde{W}_2 = -\tilde{S}^{-1}P^+ = -\imath({a^1}^\dagger a^2 - {a^2}^\dagger a^1)p_2 \tag{3.36}$$

$$\tilde{W}_- = -\tilde{S}^{12}P^+ = -\imath({a^1}^\dagger a^2 - {a^2}^\dagger a^1)p_- \tag{3.37}$$

Let us define

$$\alpha^* = \frac{1}{\sqrt{2}}(\alpha^{1*} - \imath\alpha^{2*}), \qquad \bar{\alpha}^* = \frac{1}{\sqrt{2}}(\alpha^{1*} + \imath\alpha^{2*})\,. \tag{3.38}$$

Then the operator $-\imath({a^1}^\dagger a^2 - {a^2}^\dagger a^1)$ acts on a state $\phi(\alpha^*, \bar{\alpha}^*)$ as $\bar{\alpha}^*\frac{\partial}{\partial\bar{\alpha}^*} - \alpha^*\frac{\partial}{\partial\alpha^*}$. Reduced fields $\varphi(p, \alpha^*, \bar{\alpha}^*)$ of level s, $N_s\varphi(p, \alpha^*, \bar{\alpha}^*) = 0$ can be decomposed as

$$\varphi(p,\alpha^*,\bar{\alpha}^*) = \sum_{k=0}^{s} (\bar{\alpha}^*)^k (\alpha^*)^{s-k} \varphi_{-s+2k}(p).$$

If these fields are on-shell, $p_\mu p^\mu \varphi_{-s+2k}(p) = 0$, the action of the Pauli-Lubanski vector is given by

$$\tilde{W}_\mu \varphi_{-s+2k}(p) = (-s+2k) P_\mu \varphi_{-s+2k}(p). \tag{3.39}$$

hence, when restricted to level s, action (2.11) describes massless particles of helicities $-s, -s+2, \ldots, s-2, s$. In addition, because $\mathcal{T}^T$ acts as $\frac{\partial}{\partial\alpha^*}\frac{\partial}{\partial\bar{\alpha}^*}$, only the fields $\varphi_s(p), \varphi_{-s}(p)$ satisfy the trace condition, so that action (2.13) describes massless particles of helicities $\pm s$.

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