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On Complexity, Schwarz-Christoffel Map and Integrable Structures

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Abstract

In the last few years the interest to complex systems, entanglement and their behavior is gradually growing. From one hand side it is dictated by recent development in revealing quantum properties of dynamical systems and quantum computing and on the other side by the remarkable progress in holographic correspondence. In this short note we will focus on some properties of the so-called Krylov complexity. Krylov complexity is an efficient way to quantitatively describe the growth of operators in a theory with respect to a special basis. The latter is generated by the successively nested commutators between the Hamiltonian and the operator. Finally, we consider links between Krylov complexity, Painlevé equations and Toda equations.

1 Introduction

The emergence of qualitative change of behavior in quantum systems is an old issue. In order to understand why and how it happens one must find the minimal amount of information which completely characterizes the system. This issues are important especially in the context of holographic correspondence. The notion of complexity, adapted to particular cases, attracted recently a lot of attention. Informally, the complexity $C_f(x)$ measures *information content*, degrees of *redundancy*, degrees of *structures*, of x, that is

$$C_f(x) = \min_p \{|p| : f(p) = x\},$$

for some “computer” or algorithm f.

Due to its universality there are many concepts and methods about how to precisely define and measure complexity. One approach is by making use of geometry to try to metricize complexity: closeness arises from a metric on the space of unitaries and distance to a fixed reference serves as a complexity measure. Consider for instance, unitary operators U arising from iterating generators $G(s)$ taken from some elementary set of Hermitian operators $\{G(s)\}$. Then, this approach reduces to geodesics on circuit space (see for instance [1, 2]). As an example, one can define complexity for states $\mathcal{C}$ as the minimal length between states driven by generators $G(s)$

$$\mathcal{C}(|\Psi(s_i)\rangle, |\Psi(s_f)\rangle) = \min_{G(s)} \ell(|\Psi(\sigma)\rangle).$$

Another approach is based on the optimization scheme frequently called Krylov chain. In the last few years many papers on the subject appeared [3]- [11]. It has been proved as efficient method for calculating complexity in holographic correspondence and studying behavior of the systems on both sides of correspondence. It was shown that important aspects of the time evolution of complex systems (states/operators) is encoded in the so-called Lanczos coefficients b_n. The latter can be used as indicators for an emerging chaos in the system. For instance, for the asymptotic behavior of b_n one has

$$b_n \sim n^\delta, \quad \delta \geq 1 \text{ - chaotic}, \quad 0 < \delta < 1 \text{ - integrable}$$

If the system exhibits a tendency towards chaos, namely $\delta \geq 1$, one can proceed obtaining the Lyapunov exponents and other characteristics of the system.

Various aspects of Krylov complexity applied to different (holographic) systems can be found for instance in [4]- [9] and references therein, list of which being far from complete.

In this paper we are dealing with operator growth and so called Krylov complexity establishing relations to integrable structures. To develop an effective way for calculations one has first to construct the Krylov basis. Once it is constructed, a natural object to consider is the operator growth, thus Krylov complexity. The next section provides a concise overview of these issues. In the third section we present Schwarz-Christoffel map and its relation to Heun and then to Painlevé equations. Here we suggest Schwarz-Christoffel map as a weight under which the Krylov basis is orthogonalized. This suggests a nontrivial relation to the theory of Painlvé equations. In a subsection we give a simple relation of the Hankel determinant made of time/parameter-dependent moments to Toda equations. To the best of our knowledge, our approach is different from those known in the literature and offers new directions for development. In the Conclusions we provide short comments on the results and discuss future directions.

2 Operator growth and Krylov basis

In the Schrödinger picture unitary evolution mixes the initial state $|\psi\rangle$ with other quantum states as time evolves

$$|\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle = \sum_{n=0}^{\infty} \frac{(-i\mathcal{H}t)^n}{n!}|\psi\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\psi_n\rangle. \tag{2.1}$$

Thus, solving for the time evolution amounts to understanding the states $|\psi_n\rangle \equiv \mathcal{H}^n|\psi\rangle$.

Using operator-state correspondence one can conclude that operators also evolve correspondingly, see for instance [4]. Assume that we have a Heisenberg operator $\mathcal{O}(t)$ that can be formally expanded in a series of *nested commutators* with the Hamiltonian

$$\mathcal{O}(t) = e^{i\mathcal{H}t}\,\mathcal{O}(0)\,e^{-i\mathcal{H}t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\tilde{\mathcal{O}}_n, \tag{2.2}$$

where

$$\tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [\mathcal{H}, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [\mathcal{H}, [\mathcal{H}, \mathcal{O}]] \ldots \tag{2.3}$$

As time progresses, a simple operator $\mathcal{O}(0)$ "grows" in the space of operators of the theory becoming more and more "complex".

In view of Eq. (2.3) it is convenient to introduce an operator called *Liouvillian*:

$$\mathcal{L} := [\mathcal{H}, *] \quad \Longrightarrow \quad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \quad \Longrightarrow \quad \mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0). \tag{2.4}$$

To proceed, we need to construct Krylov space and Krylov basis. To this end consider the general picture of the action of an operator A as the map

$$Ax = b, \qquad A: V \to V^{\#}, \qquad x \in V, \ b \in V^{\#}, \tag{2.5}$$

into the equation in V

$$\tau Ax = \tau b, \qquad \tau A: V \to V, \qquad x \in V, \ \tau b \in V. \tag{2.6}$$

We want to pick the "best" solution x_k from the Krylov space $\mathcal{K}_k(A, b)$ defined as

$$\mathcal{K}_k(A, b) := \text{span}\{b, Ab, A^2 b, \ldots, A^k b\}. \tag{2.7}$$

"Best" means that the remainder is as small as possible over $\mathcal{K}_k(A, b)$, i.e., x_k solves the least squares problem

$$\min_{z \in \mathcal{K}_k(A,b)} ||AZ - b||, \tag{2.8}$$

in the Euclidean norm $|| \cdot ||$.

Let us adapt the above scheme to our problems where the operator A is taken to be the Liouvillian, or super-operator $\mathcal{L}$.

It is natural to expand $\mathcal{O}(t)$ over the states $|\mathcal{O}_n(0)) = \mathcal{O}_n|0\rangle$, however these states may not be orthogonal (and the set $\{|\mathcal{O}_n(0))\}$ may not define a basis). Thus, one has first to orthogonalize the set of states $\{|\mathcal{O}_n(0))\}$ and make them a basis for expansion of $\mathcal{O}(t)$.

The algorithm of orthogonalization (Arnoldi iteration) provides construction of Krylov basis as follows:

1. Set $b_0 \equiv 0$ and $|\mathcal{O}_{-1}) \equiv 0$.
2. Define $|\mathcal{O})_0 = \frac{1}{\sqrt{(\mathcal{O}|\mathcal{O})}}|\mathcal{O})$.
3. For $n = 1$:
   - $|A_1) = \mathcal{L}|\mathcal{O}_0)$.
   - $b_1 = ||A_1||$.
   - If $b_1 \neq 0$ define$|\mathcal{O}_1) = \frac{1}{b_1}|A_1)$.
4. For $n > 1$:
   - $|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$.
   - $b_n = ||A_n|| \equiv \sqrt{(A_n|A_n)}$.
   - If $b_n = 0$ stop the procedure; if not, define $|\mathcal{O}_n) = \frac{1}{b_n}|A_n)$ and go for $n+1$ to step 4.

Now, we can decompose $\mathcal{O}(t)$ in the Krylov basis elements

$$|\mathcal{O}(t)) = \sum_{n=0}^{K-1} \phi_n(t)|\mathcal{O}_n). \tag{2.9}$$

The Heisenberg equation for $\phi_n(t)$ reads

$$-i\dot{\phi}_n = \sum_{m=1}^{K-1} L_{nm}\phi_m(t) = b_{n+1}\phi_{n+1}(t) + b_n\phi_{n-1}(t), \qquad \phi_n(0) = \delta_{n0}.$$

Finally, we are in a position to define *Krylov Complexity and K(rylov)-entropy (Shannon)* as

$$\mathcal{K}(t) = \sum n|\phi_n(t)|^2, \qquad S(t) = \sum |\phi_n(t)|^2 \log|\phi_n(t)|^2. \tag{2.10}$$

The Krylov subspace is a subspace spanned by $\{\mathcal{L}^n|\hat{\mathcal{O}})\}$. Let us formulate the problem of operator growth or Krylov complexity in the Krylov subspace. The chain of states obtained by acting repeatedly with Liouvillian $\mathcal{L}$ we denote as

$$|\hat{\mathcal{O}}_0) := |\hat{\mathcal{O}})\,, \qquad \mathcal{L}|\hat{\mathcal{O}}_n) = \sum_{i=0}^{n+1} h_{i,n}|\hat{\mathcal{O}}_i) \quad (n \geq 0). \tag{2.11}$$

Using the scalar product we define the matrix

$$L_{mn} = (\hat{\mathcal{O}}_m|\mathcal{L}|\hat{\mathcal{O}}_n) = \begin{pmatrix} h_{0,0} & h_{0,1} & h_{0,2} & h_{0,3} & \cdots \\ h_{1,0} & h_{1,1} & h_{1,2} & h_{1,3} & \cdots \\ 0 & h_{2,1} & h_{2,2} & h_{2,3} & \cdots \\ 0 & 0 & h_{3,2} & h_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.12}$$

We have also chosen an appropriate normalization such that $(\hat{\mathcal{O}}_0|\hat{\mathcal{O}}_0) = 1$.

If $(\hat{\mathcal{O}}_m|\mathcal{L}|\hat{\mathcal{O}}_n)$ is a Hermitian matrix, then Eq. (2.11) significantly simplifies as

$$\begin{aligned} &|\hat{\mathcal{O}}_{-1}) := 0\,, \quad |\hat{\mathcal{O}}_0) := |\hat{\mathcal{O}}), \\ &\mathcal{L}|\hat{\mathcal{O}}_n) = a_n|\hat{\mathcal{O}}_n) + b_n|\hat{\mathcal{O}}_{n-1}) + b_{n+1}|\hat{\mathcal{O}}_{n+1}) \quad (n \geq 0), \end{aligned} \tag{2.13}$$

while (2.12) becomes tri-diagonal

$$L_{mn} = (\hat{\mathcal{O}}_m|\mathcal{L}|\hat{\mathcal{O}}_n) = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.. \tag{2.14}$$

The quantity L_{mn} is called Jacobi operator and the entries are the Lanczos coefficients. The orthogonalization procedure results in tri-diagonal relations known for orthogonal polynomials.

The tridiagonal form of the Liouvillian $\mathcal{L}$ in Krylov basis can be written as

$$\mathcal{L} = \sum_{n=0}^{K-1} b_{n+1}\,[\,|\mathcal{O}_n)(\mathcal{O}_{n+1}| + |\mathcal{O}_{n+1})(\mathcal{O}_n|\,] \tag{2.15}$$

The conclusion one can draw is that the Krylov elements consists of orthogonal polynomials $p_n = k_n x^n + \dots$ satisfying recurrent relations:

$$
\begin{aligned}
xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n > 0),\\
xp_0(x) &= a_0 p_1(x) + b_0 p_0(x),
\end{aligned}
\tag{2.16}
$$

with a_n, b_n, c_n real constants and $a_n c_{n+1} > 0$. Also $a_n = \frac{k_n}{k_{n+1}}$, $\frac{c_{n+1}}{h_{n+1}} = \frac{a_n}{h_n}$.

We conclude this overview section noting that to solve the operator growth and complexity problems one has to determine the measure $d\mu$ under which the basis is orthogonal. It's worth also to note that equivalent information is contained in the moment matrix $\mathfrak{M}$ defined by

$$
\mathfrak{M} = \begin{pmatrix} \int x^0 d\mu & \int x d\mu & \cdots & \int x^n d\mu \\ \int x d\mu & \int x^2 d\mu & \cdots & \int x^{n+1} d\mu \\ \cdot & \cdot & \cdots & \\ \int x^n d\mu & \int x^{n+1} d\mu & \cdots & \int x^{2n} d\mu \end{pmatrix} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & \cdot & \cdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}. \tag{2.17}
$$

3 Schwarz-Christoffel map and Painlevé

In this section we will suggest the interpretation of Schwarz-Christoffel map as an orthogonality measure mentioned above.

3.1 Shcwarz-Christoffel map

Let us remind first what is Schwarz-Christoffel map. Consider a polygonal curve Γ having a simply connected interior P. Thus, the Riemann Mapping Theorem states that the upper half plane is conformally equivalent to the interior domain determined by any polygon. In other words, there exists a function S that conformally maps the upper half plane $\mathbb{H}$ onto P. The Schwarz-Christoffel theorem actually realizes such a maps providing explicit formulas.

In order to make Christoffel-Schwarz mapping concrete, take a polygon Γ with vertices $w_1, \dots, w_n$ and interior angles $\theta_1, \dots, \theta_n$ in counterclockwise order. Then Christoffel-Schwarz map in differential form is defined as

$$
\frac{df(w)}{dw} = \gamma \prod_{i=1}^{n} (w - w_i)^{\theta_i - 1}, \tag{3.1}
$$

where w_i are called pre-vertices (on the line), and z_i - the pre-images of the vertices (vertices of the polygon, $z_i = f(w_i)$). The Schwarzian differential equation can be easily obtained

$$
\{f(w), w\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \sum_{i=1}^{n} \left[\frac{1 - \theta_i^2}{2(w - w_i)^2} + \frac{2\beta_i}{w - w_i}\right],
$$

where n is the number of vertices and θ_i are the interior angles at each vertex z_i. It is well known that the solution of the Schwarzian differential equation is given by $z = f(w) = \tilde{y}_1/\tilde{y}_2$, where $\tilde{y}_i$ are the two independent solutions of

$$
\tilde{y}''(w) + \sum_{i=1}^{n} \left[\frac{1 - \theta_i^2}{4(w - w_i)^2} + \frac{\beta_i}{w - w_i}\right] \tilde{y}(w) = 0. \tag{3.2}
$$

There are also some algebraic constraints on the accessory parameters:

$$\sum_i \beta_i = \sum_i (w_i\beta_i + 1 - \theta_i^2) = \sum_i (2w_i\beta_i^2 + w_i(1-\theta_i^2)) = 0. \tag{3.3}$$

Applying change of variables $\tilde{y}(w) = w^{-\theta_0/2}(w-1)^{-\theta_1/2}(w-t)^{-\theta_t/2}y(w)$ one finds the Heun equation in *canonical form*

$$y''(w) + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_t}{w-t} + \frac{1-\theta_1}{w-1}\right) y'(w) + \left(\frac{\kappa_-\kappa_+}{w(w-1)} - \frac{t(t-1)K_0}{w(w-1)(w-t)}\right) y(w) = 0. \tag{3.4}$$

At this point one can follow Slavyanov [12] to make a link from the Heun class differential equation and Painlevé one. To this end one has to make a Legendre transformation to obtain the Lagrangian

$$\mathcal{L}(q,\dot{q},t) = \frac{f(t)}{4P_0(q,t)}\left(\dot{q} - \frac{P_1(q,t)}{f(t)}\right)^2 - \frac{P_2(q,t)}{f(t)}. \tag{3.5}$$

Then the equation of motion for the Lagrangian (3.5) is

$$\ddot{q} = \frac{1}{2}\frac{\partial}{\partial q}\big(\ln\, P_0(q,t)\big)\dot{q}^2 - \left(\frac{\partial}{\partial t}(\ln f(t)) - \frac{\partial}{\partial t}(\ln P_0(q,t))\right)\dot{q} + \frac{P_0(q,t)}{f^2(t)}\left(\frac{\partial}{\partial q}\frac{P_1^2(q,t)}{2P_0(q,t)} + f(t)\frac{\partial}{\partial t}\frac{P_1(q,t)}{P_0(q,t)} - 2\frac{\partial P_2(q,t)}{\partial q}\right). \tag{3.6}$$

The conclusion drawn in [12] is that *any Painlevé equation can be obtained as classical equation of motion of Heun class*!

On the other hand, our interpretation of Schwarz-Christoffel map as measure is based on the following arguments. Without loss of generality, one can assume that the measure $d\mu$ is given by

$$d\mu(x) = w_0(x) = e^{-v(x)}dx, \qquad \int p_n(x)p_m(x)w_0(x)dx = k_n\delta_{nm}.$$

As an example, let us use the recurrent relations (2.16) to obtain ordinary differential equation (ODE) for the polynomials forming the basis. To this end let us go back to tridiagonal recurrence (2.16) and differentiate with respect to x

$$P_n'(x) = -B_n(x)P_n(x) + A_n(x)P_{n-1}(x),$$

where (the measure is $\omega(x) = e^{-v(x)}$):

$$A_n(x) := a_n \int_{-\infty}^{\infty} \frac{v'(x)-v'(y)}{x-y}P_n^2(y)\omega(y)dy,$$
$$B_n(x) := a_n \int_{-\infty}^{\infty} \frac{v'(x)-v'(y)}{x-y}P_n(y)P_{n-1}(y)\omega(y)dy. \tag{3.7}$$

From here it is easy to define a ladder operator corresponding to "annihilation"

$$\left(\frac{d}{dx} + B_n(x)\right)P_n(x) = A_n(x)P_{n-1}(x), \tag{3.8}$$

and the congugate operator (creation operator)

$$\left(-\frac{d}{dx}+B_n(x)+v'(x)\right)P_{n-1}(x)=A_{n-1}(x)\frac{a_n}{a_{n-1}}P_n(x).$$

Putting all these together we end up with a second order ODE

$$\left(-\frac{d}{dx}+B_n+v'\right)\left[\frac{1}{A_n}\left(\frac{d}{dx}+B_n\right)P_n\right]=A_{n-1}\frac{a_n}{a_{n-1}}P_n(x)$$

The obtained ODE can be brought to the form

$$P_n''(x)+S(x)P_n'(x)+Q(x)P_n(x)=0. \tag{3.9}$$

which is Schrödinger type equation

$$P_n''(x)+V(x)P_n(x)=0. \tag{3.10}$$

Comparing with (3.4) we conclude that Schwarz-Christoffel map can serve as a legitimate measure for orthogonal polynomials.

Now consider a polygon with two finite vertices w_1 and w_2 and interior angles θ_1 and θ_2 respectively. Without loss of generality, take $z_1 < z_2$ and consider (conformal function)

$$f_0(z)=(z-z_1)^{\theta_1-1}(z-z_2)^{\theta_2-1}.$$

For the argument of $f'(z)$ we have

$$\arg f'(z)=\begin{cases}0 & \textit{if} \quad z_1<z_2<z,\\ (\theta_2-1)\pi & \textit{if} \quad z_1<z<z_2,\\ (\theta_2-1)\pi+(\theta_1-1)\pi & \textit{if} \quad z_1<z_2<z.\end{cases}$$

It is well known that the weight under which Jacobi polynomials are orthogonal is exactly $(1-x)^{\alpha}(1+x)^{\beta}$! Thus, the measure defined by Schwarz-Christoffel map with $n=2$ defines the measure for the Jacobi polynomials

$$d\mu_0(x)=(1-x)^{\alpha}(1+x)^{\beta} \tag{3.11}$$

However, the ODE for Jacobi polynomials is also of (degenerate) Heun class and therefore can be mapped to a certain Painlevé equation.

3.2 Moments, Schur and relation to Toda

Let us consider a generalization of the measure, which, in the case of Jacobi polynomials, is $w_0(x)=\exp(-v(x))$ with $v(x)=-\alpha\ln(1-x)-\beta\ln(1+x)$. Instead of weight $w_0(x)$ we can consider associate measure $d\mu(x)=w(x)dx$ where $w(x)=w_0(x)e^{-f(x)}$ ($f(x)$ univalent in general) characterized by

- chosen for fixed vector space V and fixed operator acting on it,
- changes in parameters of $f(x)$ produces a new basis

In the case of Jacobi polynomials the simplest deformation is chosing function $f(x)$ to be just a linear term $-x\lambda$. Thus, we consider the measure $w(x,\lambda)=w_0(x)e^{-\lambda x}$ which has associate moments

$$\mu_k(\lambda)=\int x^k w(x,\lambda)dx. \tag{3.12}$$

Thus, the moment matrix can be written as

$$\mathfrak{M}=\left((-1)^{i+j}\partial_\lambda^{i+j}\mathfrak{M}\right)_{ij}=\begin{pmatrix}\mu_0(\lambda) & \mu_1(\lambda) & \cdots & \mu_n(\lambda)\\ \mu_1(\lambda) & \mu_2(\lambda) & \cdots & \mu_{n+1}(\lambda)\\ \cdot & \cdot & \cdots & \\ \mu_n(\lambda) & \mu_{n+1}(\lambda) & \cdots & \mu_{2n}(\lambda)\end{pmatrix}. \tag{3.13}$$

From (3.12) it is clear that differentiating wrt λ one can obtain other moments

$$-\partial_\lambda\mu_n=\mu_{n+1}, \qquad (-1)^k\partial_\lambda^k\mu_n=\mu_{n+k}$$

However, insted of using ∂_λ^k to produce $\mu_n\to\mu_{n+k}$ we define another weight

$$w(x,\lambda)\to w(x,\lambda_i)=w_0(x)\,e^{\sum_i\lambda_i x^i}, \tag{3.14}$$

so that

$$\partial_{\lambda_k}\mu_n=\mu_{n+k}, \qquad [\partial_{\lambda_k},\partial_{\lambda_l}]=0.$$

This deformation of the weight defines a link to representation theory. Indeed, we can expand the exponent in Schur polynomials as

$$e^{\sum_i\lambda_i x^i}=\sum_k S_k[\lambda_i]x^k.$$

Thus, we arrived at a new relation between the moments in terms of Schur polynomials

$$\mu_n(\lambda_i)=\sum_k S_k[\lambda_i]\mu_{n+k}(0). \tag{3.15}$$

This is the second main result in this paper.

It is well known that time-dependent Hankel determinants satisfy Toda equation. Here we suggest another simple derivation based on determinant of moment matrix, which is in fact Hankel determinant. Let us introduce the following notation for (sub)determinant of the moment matrix (3.13)

$$D_m=\det(\mathfrak{M})_{m\times m}. \tag{3.16}$$

Then, we can use Jacobi identity for determinant ($D\equiv D_{n+1}$)

$$D\binom{n}{n}D\binom{n+1}{n+1}-D\binom{n}{n+1}D\binom{n+1}{n}=D\binom{n\;n+1}{n\;n+1}D,$$

where $D\binom{n}{m}$ denotes the determinant with removed n-th row and m-th column. Let us denote by $\tau_{n+1}=D$, $\tau_n=D\binom{n+1}{n+1}$. Having that D_n is Hankel determinant, differentiating wrt λ will produce rearranging of the rows and columns due to vanishing determinants of matrices with equal rows. Then, denoting $\dot\tau_n=D\binom{n}{n+1}=D\binom{n+1}{n}$ (the second equality is obvious), we arrive at the equation

$$\tau_n\ddot\tau_n-\dot\tau_n^2=\tau_{n+1}\tau_{n-1}. \tag{3.17}$$

This is nothing but Toda (molecule) equation!

4 Conclusions

In this paper we establish some relations between Krylov complexity/operator growth and Painlevé equations as well as Toda equations. Before making comments on our results let us mention that some particular relations have been considered in the literature in different context. For instance, the relation between Jacobi orthogonal polynomials and Painlevé equations (in the so called sigma form) have been considered in [13]. Their analysis is based on the ladder operators and differential equations for orthogonal polynomials. Here we provide a short summary of their result. The starting point are generalized Jacobi polynomials P_n [13] which are orthogonal with respect to the measure

$$w(x,t) = (1-x)^{\alpha}(1+x)^{\beta}e^{-tx} = e^{-tx+\alpha\ln(1-x)+\beta\ln(1+x)} = e^{-v(x,t)}. \tag{4.1}$$

Using ladder operators it is straightforward to obtain the differential equation for P_n which reads

$$P_n''(x) - \left(\frac{A'(x)}{A(x)} + v'(x)\right) P_n'(x) + \left(B_n'(x) - B_n(x)\frac{A_n'}{A_n} + \sum_{l=1}^{n-1} A_l\right) P_n = 0.$$

Here the quatities $A_n(x)$, $B_n(x)$ are obtaned as in (3.7)

$$A_n(x) = -\frac{R_n(t)}{z-1} + \frac{t+R_n(t)}{z+1}, \qquad B_n(x) = -\frac{r_n(t)}{z-1} + \frac{r_n(t)-n}{z+1},$$

while for $R_n(x)$ and $r_n(x)$ one obtains

$$R_n(t) = \alpha\int \frac{P_n^2(x)}{1-x} w(x,t)dx, \qquad r_n(t) = \alpha\int \frac{P_n(x)P_{n-1}(x)}{1-x} w(x,t)dx.$$

The authors introduce the function

$$\sigma(t) = \frac{t}{2}\frac{\dot{D}_n(t)}{D_n(t)} - \frac{t}{2}n + n(n+\beta),$$

where β is a constant and $D_n(t)$ to be the Hankel determinant of moments

$$D_n(t) = \det(\mu_{i+j}(t))_{i,j=0}^{n-1}.$$

Then, after long calculations it was shown that $\sigma(t)$ satisfies σ-form of Painlevé V equation

$$(t\sigma''(t))^2 = [\sigma - t\sigma' + (2n+\alpha+\beta)\sigma']^2 + 4[\sigma - n(n+\beta) - t\sigma'][(\sigma')^2 - \alpha\sigma'].$$

In another paper [14] using the same approach it was shown that generalizing the measure as

$$w(x,t) = (1-x)^{\alpha}(1+h)^{\beta}[a + B\theta(x-t)]$$

one finds another Painlevé equation. In this case $\sigma(t)$ defined as

$$\sigma(t) = H_n(t) + d_1 t + d_0, \qquad H_n(t) = t(t-1)\frac{d}{dt}\ln\, D_n(t),$$

where $D_n(t)$ is again Hankel determinant of moments matrix. Then, after long calculations one finds that $\sigma(t)$ satisfies σ-form of Painleve VI.

In this paper we define Schwarz-Christoffel mapping function as a weight of the measure under which our Krylov basis is orthogonal. Utilizing the relation between Schwarz-Christoffel map, Heun class equations and Painlevé equations we established a link between Krylov complexity/operator growth and Painlevé equations. The relations look much easier than known ones and provides further options for investigations. For instance, it would be interesting to make our approach more detailed and apply to concrete systems. Another issue to what relations lead confluent Heun class equations which could be accompanied with Painlevé correspondence.

In the last Section we have shown that Schur polynomials appeared as a link between moments at given time t and the initial ones at $t = 0$. Another line of developments is to investigate the role of Schur polynomials in the type of problems we sketched above.

All these issues are currently under investigation and we hope to report on them in the near future.

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References

[1] Michael A. Nielsen, “A geometric approach to quantum circuit lower bounds”, arXiv:quant-ph/0502070.

[2] R. Jefferson and R. C. Myers, JHEP **10** (2017), 107, [arXiv:1707.08570 [hep-th]].

[3] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi and E. Altman, Phys. Rev. X **9** (2019) no.4, 041017, [arXiv:1812.08657 [cond-mat.stat-mech]].

[4] E. Rabinovici, A. Sánchez-Garrido, R. Shir and J. Sonner, JHEP **06** (2021), 062, [arXiv:2009.01862 [hep-th]].

[5] P. Caputa, J. M. Magan and D. Patramanis, Phys. Rev. Res. **4** (2022) no.1, 013041, [arXiv:2109.03824 [hep-th]].

[6] M. Nozaki, S. Ryu and T. Takayanagi, JHEP **10** (2012), 193, [arXiv:1208.3469 [hep-th]].

[7] J. Erdmenger, S. K. Jian and Z. Y. Xian, [arXiv:2303.12151 [hep-th]].

[8] K. Adhikari, S. Choudhury and A. Roy, Nucl. Phys. B **993** (2023), 116263, [arXiv:2204.02250 [hep-th]].

[9] Z. Y. Fan, JHEP **08** (2022), 232, [arXiv:2206.00855 [hep-th]].

[10] M. Alishahiha and S. Banerjee, [arXiv:2212.10583 [hep-th]].

[11] C. Lv, R. Zhang and Q. Zhou, [arXiv:2303.07343 [quant-ph]].

[12] Slavyanov, S.Y., Stesik, O.L. Theor Math Phys 186, 118-125 (2016).

[13] Estelle Basor et al., Journal of Physics A: Mathematical and Theoretical, 43, 015204 (2010).

[14] Dan Dai, Lun Zhang, Journal of Physics A: Mathematical and Theoretical, 43, 055207 (2010).