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Some remarks on hamiltonian reduction of third derivative extension of MCS model

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Abstract

A Hamiltonian reduction of a higher derivative model described by a Lagrangian action containing three terms, the topological mass term, Maxwell term and a third derivative extension of the Chern-Simons term is achieved.

1 Introduction

The addition of topological mass term to the Maxwell term (MCS model) leads to topologically massive electrodynamics a first-class theory with a single massive degree of freedom, described by a second order derivative action [1–4]. An interesting model can be built up in $D = 3$ by adding to the MCS model a third order derivative extension that involve the Chern-Simons (TCS) term [5]

$$S = \int d^3x \left(c\varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho - \frac{a}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2b} \varepsilon_{\mu\nu\rho} F^{\mu\lambda} \partial^\nu F^\rho{}_\lambda \right). \tag{1}$$

2 Hamiltonian reduction of MCSTCS model

In this paper after the canonical analysis of the MCSTCS model, the hamiltonian of the model is expressed in term of a reduced set of variables by solving the constraints [6, 7]. The canonical analysis of the MCSTCS model will be done by a variant [8–11] of the Ostrogradsky method [12,13]. This approach is done by going through the third derivative order MCSTCS model to an equivalent first order one by introducing some new fields B_μ as

$$B_\mu = \partial_0 A_\mu, \tag{2}$$

and enforce the Lagrangian constraints

$$B_\mu - \partial_0 A_\mu = 0, \tag{3}$$

by Lagrange multiplier ξ^μ

$$\mathcal{L} = c\varepsilon_{0ij} A^0 \partial^i A^j + c\varepsilon_{i0i} A^i B^j + c\varepsilon_{ij0} A^i \partial^j A^0 - \frac{a}{4} F_{ij} F^{ij}$$

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$$
-\frac{a}{2}\left(B_i-\partial_i A_0\right)\left(B^i-\partial^i A^0\right)-\frac{1}{2b}\varepsilon_{0ij}\left(B^k-\partial^k A^0\right)\partial^i F^j{}_k
$$
$$
-\frac{1}{2b}\varepsilon_{i0j}\left(\partial^i A^0-B^i\right)\left(\partial^j B_0-\partial_0 B^j\right)-\frac{1}{2b}\varepsilon_{i0j}F^{ik}\left(\partial^j B_k-\partial_k B^j\right)
$$
$$
-\frac{1}{2b}\varepsilon_{ij0}F^{ik}\partial^j\left(B_k-\partial_k A_0\right)+\xi^\mu\left(B_\mu-\partial_0 A_\mu\right) \tag{4}
$$

and then canonical analysis is performed using Dirac's constrained algorithm [14, 15].

Performing the canonical analysis of the model described by the Lagrangian (4) we are left with a system subject to the constraints

$$
\chi_i \equiv \pi_i+\frac{1}{2b}\varepsilon_{0ij}\left(B^j-\partial^j A^0\right)\approx 0, \tag{5}
$$
$$
G^{(1)} \equiv \partial_i p^i+c\varepsilon_{0ij}\partial^i A^j\approx 0, \tag{6}
$$
$$
G^{(2)} \equiv -p_0+\partial_i\pi^i\approx 0, \tag{7}
$$
$$
G^{(2)} \equiv \pi^0\approx 0, \tag{8}
$$

where we denote by $\{p^\mu,\pi^i\}$ the canonical momenta conjugate to the fields $\{A_\mu,B_i\}$. The canonical hamiltonian is given by

$$
H = \int d^2x\left[-c\varepsilon_{0ij}A^0\partial^i A^j-c\varepsilon_{i0i}A^iB^j-c\varepsilon_{ij0}A^i\partial^j A^0+\frac{a}{4}F_{ij}F^{ij}\right.
$$
$$
+\frac{a}{2}\left(B_i-\partial_i A_0\right)\left(B^i-\partial^i A^0\right)+\frac{1}{2b}\varepsilon_{0ij}\left(B^k-\partial^k A^0\right)\partial^i F^j{}_k \tag{9}
$$
$$
+\frac{1}{2b}\varepsilon_{i0j}\left(\partial^i A^0-B^i\right)\partial^j B_0+\frac{1}{2b}\varepsilon_{i0j}F^{ik}\left(\partial^j B_k-\partial_k B^j\right)
$$
$$
\left.+\frac{1}{2b}\varepsilon_{ij0}F^{ik}\partial^j\left(B_k-\partial_k A_0\right)+p^iB_i\right]. \tag{10}
$$

Examining the structure of the matrix, we find that the constraints $\chi_i\approx 0$ represent an independent subset of second-class constraints, while $\left\{G^{(1)},G^{(2)},G^{(3)}\right\}\approx 0$ represent an independent subset of first-class constraints. The number of physical degrees of freedom of the equivalent first order system is equal to

$$
\begin{aligned}
\mathcal{N}_O &= (12\text{ canonical variables}-2\text{ scc}-2\times 3\text{ fcc})/2\\
&= 2.
\end{aligned} \tag{11}
$$

We perform a total gauge-fixing imposing the canonical gauge conditions

$$
-\partial^i A_i\approx 0,\quad -A_0\approx 0,\quad B_0\approx 0. \tag{12}
$$

We remove all constraints by the Dirac bracket technique. Nonzero Dirac brackets between the variables $\{A_i,B_i,p^i\}$ are

$$
\left[B_i(x),B_j(y)\right]^*_{x_0=y_0} = -b\varepsilon_{0ij}\delta^2(\mathbf{x}-\mathbf{y}), \tag{13}
$$
$$
\left[A_i(x),p^j(y)\right]^*_{x_0=y_0} = \left(\delta_i^j-\frac{\partial_i\partial^j}{\partial_k\partial^k}\right)\delta^2(\mathbf{x}-\mathbf{y}), \tag{14}
$$
$$
\left[p_i(x),p_j(y)\right]^*_{x_0=y_0} = -c\varepsilon_{0ij}\delta^2(\mathbf{x}-\mathbf{y}). \tag{15}
$$

while the canonical Hamiltonian takes the form

$$
H = \int d^2x\left[-c\varepsilon_{i0j}A^iB^j+\frac{a}{4}F_{ij}F^{ij}+\frac{a}{2}B_iB^i\right.
$$

$$+\frac{1}{2b}\varepsilon_{0ij}B^k\partial^i F^j{}_k - \frac{1}{2b}\varepsilon_{i0j}F^{ik}\partial_k B^j + p^i B_i\Bigg]. \tag{16}$$

The fields $\{A_i, p^i\}$ are not really independent variables on the phase space because they are subject to an additional constraint. The algebra (13)-(15) is solved in terms of the free fields and canonical conjugate momenta $\{\alpha, \pi_\alpha, \varphi, \pi_\varphi\}$

$$\begin{aligned} A_i &= -\varepsilon_{0ij}\hat{\partial}^j\alpha, & (17)\\ p_i &= \varepsilon_{0ij}\hat{\partial}^j\pi_\alpha - c\hat{\partial}_i\alpha, & (18)\\ B_i &= -\varepsilon_{0ij}\hat{\partial}^j\pi_\varphi + b\hat{\partial}_i\varphi. & (19)\end{aligned}$$

where $\hat{\partial}_i \equiv \frac{\partial_i}{\sqrt{-\partial^2}}$. In terms of the new fields/momenta pairs the Hamiltonian reduce to

$$H = \int d^2x \left[\frac{a}{2}\alpha\partial_i\partial^i\alpha + \alpha\partial_i\partial^i\varphi - \frac{ab^2}{2}\varphi^2 + 2cb\alpha\varphi - \frac{a}{2}\pi_\alpha^2 - \frac{1}{2}\pi_\beta^2 + \frac{1}{2}\pi_\varphi^2\right]. \tag{20}$$

Passing to the Lagrangian formulation we obtain

$$S = \int d^3x \left[-\frac{a}{2}\alpha\Box\alpha - \alpha\left(\Box + 2cb\right)\varphi + \frac{ab^2}{2}\varphi^2\right]. \tag{21}$$

Setting $c = 0$ in (21) we get

$$S_{(c=0)} = \int d^3x \left[-\frac{a}{2}\alpha\Box\alpha - \alpha\Box\varphi + \frac{ab^2}{2}\varphi^2\right] \tag{22}$$

that can be written in diagonal form

$$S_{(c=0)} = \int d^3x \left[-\frac{1}{2a}\overline{\alpha}\Box\overline{\alpha} + \frac{1}{2a}\varphi\left(\Box + 4a^2b^2\right)\varphi\right], \tag{23}$$

where

$$\overline{\alpha} = a\alpha + \varphi, \tag{24}$$

result in agreement with those obtained in [5].

For $a = 0$ (21) reduce to

$$S_{(a=0)} = \int d^3x \left[-\alpha\left(\Box + 2cb\right)\varphi\right]. \tag{25}$$

that can be diagonalized

$$S_{(a=0)} = \frac{1}{2}\int d^3x \left[-\overline{\alpha}\left(\Box + 2cb\right)\overline{\alpha} + \overline{\varphi}\left(\Box + 2cb\right)\overline{\varphi}\right] \tag{26}$$

in terms of

$$\overline{\alpha} = \frac{1}{\sqrt{2}}\left(\alpha + \varphi\right), \qquad \overline{\varphi} = \frac{1}{\sqrt{2}}\left(\alpha - \varphi\right), \tag{27}$$

to represent two massive degrees of freedom with a relative ghost sign.

Keeping all three terms ($a \neq 0$ and $c \neq 0$), action (21) can be written in diagonal form for $b = 2c$

$$S = \frac{1}{2}\int d^3x \left\{-\widetilde{\alpha}\left[\Box + \frac{4c^2\left(\sqrt{a^2+4} - a\right)}{\sqrt{a^2+4} + a}\right]\widetilde{\alpha} + \widetilde{\varphi}\left[\Box + \frac{4c^2\left(\sqrt{a^2+4} + a\right)}{\sqrt{a^2+4} - a}\right]\widetilde{\varphi}\right\}, \tag{28}$$

where the concrete form of the $\widetilde{\alpha}$ and $\widetilde{\varphi}$ is not important for our purpose. We find that the MCSTCS is free of tachyons (the c^2 terms signs are all positive) and based on the relative sign of the two terms from (28) we have a massive ghost.

3 Conclusions

In this paper, we have achieved a Hamiltonian reduction of a higher derivative model described by a Lagrangian action containing three terms, the topological mass term, Maxwell term and a third derivative extension of the Chern-Simons term. The MCSTCS model is free of tachyons, but is plagued by ghost. Similarly, the CSTCS model describes two massive degrees of freedom with a relative ghost sign. The MTCS model describes a pair of excitations, one is massless and the other a massive ghost [5].

References

[1] P. K. Townsend, K. Pilch and P. van Nieuwenhuizen, Phys. Lett. B 136, 38 (1984).

[2] S. Deser and R. Jackiw, Phys. Lett. B 139, 371 (1984).

[3] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (NY) 140, 372 (1982).

[4] R. Banerjee, H. J. Rothe and K. D. Rothe, Phys. Rev. D 55, 6339 (1997).

[5] S. Deser and R. Jackiw, Phys. Lett. B 451, 73 (1999).

[6] R. Banerjee and S. Kumar, Phys. Rev. D 63, 125008 (2001).

[7] S. C. Sararu, M. T. Udristioiu, AIP Conf. Proc. 1916, 020009 (2017);

[8] J. M. Pons, Lett. Math. Phys. 17, 181 (1989).

[9] R. Banerjee, P. Mukherjee and B. Paul, JHEP 1108, 085 (2011).

[10] S. C. Sararu, Eur. Phys. J. C 75, 526 (2015).

[11] S. C. Sararu, M. T. Udristioiu, Mod. Phys. Lett. A 31, 1650205 (2016)

[12] M. Ostrogradsky, Mem. Ac. St. Petersbourg VI 4, 385 (1850).

[13] D. M. Gitman, S. L. Lyakhovich and I. V. Tyutin, Sov. Phys. J. 28, 554 (1985).

[14] P. A. M. Dirac, Can. J. Math. 2, 129 (1950).

[15] P. A. M. Dirac, Lectures on Quantum Mechanics (Academic Press, 1967).