

New quantization approach to study non-polynomial potentials

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Abstract

An approach of the proper quantization rule to find exact solution of radial Schrödinger equation for non-polynomial potentials, which are quasi exactly solvable, is developed in this work. Using this rule, the determination of the energy spectrum E_n for non-polynomial potentials is somewhat impossible and limited, which provides their energy at ground state only. To overcome this difficulty, we devised in this approach the potential into two potentials: an exact potential $V_e(r)$ and a non-linear extension $V_a(r)$, the potential will be expressed as: $V(r) = V_e(r) + V_a(r)$, then calculate the energy level ϵ_n and the ground state ϵ_0 energy for exactly solvable potential $V_e(r)$ using the proper quantization rule, from which an analytical expression of energy E_n for non-polynomial potential $V(r)$ related with the ground level energy E_0 is found easily. We conclude that the study of non-exactly potential remain to the study of the first exactly solvable potential.

Keywords: *Proper quantization rule, exact solutions, non-polynomial potential, Riccati equation.*

1 Introduction

In quantum mechanics, find an exact solution of Schrödinger equation forms the most important subject of research. This continuous interest is because of the solutions, used in differential field of physics, contain all information about the quantum system. Also, such solutions can be used to test the analytical method. Among different approach used to obtain the exact eigenvalues and the wave function of solvable potentials; we consider the traditional method [1], which reduce the Schrödinger equation via transformations into a well-known hypergeometric, confluent hypergeometric differential equation whose solutions are the special functions, showing as well as the Asymptotic Iteration Method (AIM) [2, 3], the Nikitarov-Uvarov (Nu) method[4],...etc. The algebraic method related to the study of the Hamiltonian, the supersymmetric quantum mechanics (SusyQM) [5, 6, 7], the factorization method[8, 9]. Recently, new methods based on the exact and proper quantization rule [10, 11, 12], and the Supersymmetric (WKB) method were investigated.

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It is well known that the number of potentials exactly solvable is limited, which have given recently a great deal of interest to generate a novel class of potentials, for which the Schrödinger equation is exactly. Construct this new family of solvable potential and generating an extension of this one, is possible by means of a number of technic, such as the algebraic Darboux transformations [13, 14, 15, 16] based on the factorization method, point canonical transformations [17, 18], the supersymmetric quantum mechanics [19]. These indeed provides a powerful approach to a recent research. Recently, in the same context, more study were published. All most of the authors have proposed exact solutions of the non-polynomial extension of solvable potentials, using several methods. Such Quesene [19] have constructed a new exactly solvable extended of radial oscillator and scarf I potentials, Cariñena [20] , Sesma [21], Qiang Dong [22], Nasser Saad *et al.* [23], Y.Grandati [24] both of them have treated the Generalized isotonic oscillator. Agboola *et al.*[25] have analyzed a class of four of non-polynomially potentials, employing Bethe ansatz method. In regard to Qiang Dong *et al.*[26], they exposed in their paper a work concerning Non-polynomially modified oscillator, using a Confluent Hunt equation.

Motivated by this works, we propose in this paper, a study to obtain the exact solution of some non-linear potential related with the radial oscillator, and coulomb potential respectively using another efficient technical; so-called Qiang Dong proper quantization rule [10, 11, 12] based on the quantization rule. This rule present a disadvantage, that it is unable to generate energy spectra of non-polynomial potentials, we propose to calculate the eigenvalues of the well-known solvable potential, then determined the ground state energy E_0 for non-polynomial potential, thus, an analytical form of energy E_n for each potential of the following potentials was deduced by combination of the energy spectrum of the solvable potential, with much more simple calculations and in economical manner:

- Generalized isotonic oscillator [20, 21, 22, 23, 24], $V(r) = \frac{1}{2}\omega^2 r^2 + g \frac{r^2 - a^2}{(r^2 + a^2)^2}$,
- Singular anharmonic potential [21, 27], $V(r) = \frac{1}{2}\omega^2 r^2 + \frac{e}{r^4} + \frac{d}{r^6}$,
- Non polynomially modified oscillator [21, 26, 28, 29, 30], $V(r) = \frac{1}{2}\omega^2 r^2 + \frac{\lambda r^2}{1 + \delta r^2}$.
- Soft-core coulomb potential [25] $V(r) = \frac{G}{r} - \frac{F}{r + \beta}$.

We will present the rest of this paper as follows: section 2: is a brief description of exact and proper quantization rule as a powerful tool in calculating energy levels of exactly solvable systems. In section 3; at first, we apply this method to calculate the energy spectra both of the radial oscillator, and the coulomb potential. The form of ground energies of these non-polynomial potentials are exposed in the section 4. A relation between the energy spectrum of these two types of non-polynomial potentials and their ground energy is found in section 5, then general expressions of energies for each one of the potentials type is exposed and marks the end of this section. Finally, in section 6, we give some concluding remarks .

2 Exact and proper quantization rule

Consider the one-dimensional non-relativistic stationary Schrödinger equation (we employed the natural unit $\hbar = 2m = 1$) expressed as,

$$\frac{d^2\psi(x)}{dx^2} + [E - V(x)]\psi(x) = 0, \quad (1)$$

by making transformation $\phi(x) = \frac{1}{\psi(x)} \frac{d\psi(x)}{dx}$, the Schrödinger equation can be expressed as a nonlinear Riccati equation,

$$\frac{d\phi(x)}{dx} = -[E - V(x)] - \phi^2(x), \quad (2)$$

where $\phi(x)$ solution of the Riccati equation, defined as the logarithmic derivative of wave function $\psi(x)$.

From Eq. (2), $\phi(x)$ decreases monotonically with respect to x between two turning points x_A and x_B , where $E \geq V(x)$. As x increases across a node of wave function, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$ and decreases again.

By study of the above one dimensional Schrödinger equation, Ma and Xu [10, 11] proposed a new exact quantization rule, derived from the semi-classical *WKB* (Wentzel-Kramer-Brillouin) condition. Its integral expression without approximation (so the rule is exact) is,

$$\int_{x_A}^{x_B} k(x) dx = N\pi + \int_{x_A}^{x_B} \frac{\phi(x)k'(x)}{\phi'(x)} dx, \quad (3)$$

where $K(x) = \sqrt{[E - V(x)]}$ is the classical momentum function for the energy E .

x_A and x_B are two classical turning points determined by the resolution of $E = V(x)$.

$N = (n + 1)$ is the number of nodes of $\phi(x)$ in the region $E \geq V(x)$, and n is the quantum number; represent the number of node of wave function $\psi(x)$ between two turning points.

The integral term $\int_{x_A}^{x_B} \frac{\phi(x)k'(x)}{\phi'(x)} dx$ is called the quantum correction. Ma and Xu [10, 11]

were found that this term is independ of the number of nodes of wave function $\psi(x)$ for all exactly solvable potentials, they considered that the quantum correction can be calculated from its ground state as,

$$\int_{x_A}^{x_B} k_n(x) dx = N\pi + \int_{x_A}^{x_B} \frac{\phi_0(x)k'_0(x)}{\phi'_0(x)} dx, \quad (4)$$

with this approach, it should be mentioned that it is necessary to calculate two complicated integrals. In fact, to simplify this improved quantization method; Qiang Dong proposed a proper Quantization rule [12] given by,

$$\int_{x_A}^{x_B} K_n(x) dx - \int_{x_{0A}}^{x_{0B}} K_0(x) dx = n\pi, \quad (5)$$

it consists of calculation of the first integral $\int_{x_A}^{x_B} K_n(x) dx$, then replace the energy levels E_n

in the result by the ground energy expression E_0 to obtain the second integral. Which simplify the calculation of complicated integrals, occurs previously.

Obviously, this new exact quantization rule, was also generalized to 3D Schrödinger equation, and will be written as,

$$\int_{r_A}^{r_B} K_n(r)dr - \int_{r_A}^{r_B} K_0(r)dr = n\pi. \quad (6)$$

3 Energy level for exactly solvable potentials

We propose, in this section, to apply the proper quantization rule in order to find an analytical expression of the energy spectrum with respect of Radial oscillator and coulomb potential which are exactly solvable:

3.1 Radial oscillator

The radial oscillator in three dimensional is given by,

$$V(r) = \frac{1}{2}\omega^2 r^2, \quad (7)$$

the corresponding radial Schrödinger equation can be written as (Throughout the rest of this paper, we take the unit $\hbar = m = 1$),

$$\frac{d^2\psi_n(r)}{dx^2} + [2\epsilon_n + \frac{l(l+1)}{r^2} - \omega^2 r^2]\psi_n(r) = 0, \quad (8)$$

where $\omega \succ 0$ denote the oscillator frequency and $l = -1, 0, 1, \dots$ is the angular momentum quantum angular number. And E_n is the energy eigenvalue.

Making the substitution $r^2 = z$, the turning points z_A and z_B determined by solving $E = V(r)$ are given by,

$$z_A + z_B = \frac{2\epsilon_n}{\omega^2} \quad \text{and} \quad z_A \cdot z_B = \frac{l(l+1)}{\omega^2}, \quad (9)$$

the logarithmic derivative for the ground state has the form: $\phi_0(r) = -ar + \frac{b}{r}$, (where $a \succ 0$ due to the monotonic property). Substituting $\phi_0(x)$ into the nonlinear Riccati equation below,

$$\frac{d\phi_0(r)}{dx} = - \left[2\epsilon_0 + \frac{l(l+1)}{z} - \omega^2 z \right] - \phi_0^2(r), \quad (10)$$

resolving this equation, allows us to determine the ground energy,

$$\epsilon_0 = \omega \left(\frac{3}{2} + l \right), \quad (11)$$

we find that $\phi_0(r) = -\omega r + \frac{(l+1)}{r}$.

Let now calculate the first integral for the momentum $k(r)$ appearing in Eq. (6) as follows,

$$\int_{r_A}^{r_B} K_n(r)dr = \int_{r_A}^{r_B} \sqrt{2(\epsilon_n - V(r))}dr = \frac{\omega}{2} \int_{z_A}^{z_B} \frac{\sqrt{(z_A - z)(z - z_B)}}{z} dz \quad (12)$$

$$= \frac{\omega\pi}{2} \left[\frac{1}{2} \left(\frac{2\epsilon_n}{\omega^2} \right) - \sqrt{\frac{l(l+1)}{\omega^2}} \right], \quad (13)$$

the second integral is deduced by replacing ϵ_n in Eq. (12) with ϵ_0 in Eq. (11) and considering Eq. (6) one has, the following well-known result of the energy spectrum of radial oscillator

$$\epsilon_n = \omega \left(2n + \frac{3}{2} + l \right), \quad (14)$$

from where the energy spectrum is equidistant,

$$\epsilon_n = (2n\omega + \epsilon_0), \quad n = 0, 1, 2, \dots \quad (15)$$

3.2 Coulomb potential

Now, we consider the coulomb potential defined as

$$V(r) = \frac{G}{r}, \quad (16)$$

where $G \succ 0$ constant parameter.

the corresponding radial part of schrödinger equation for a particle in the presence of this potential is:

$$\frac{d^2\psi_n(r)}{dx^2} + [2\epsilon_n + \frac{l(l+1)}{r^2} - \frac{G}{r}]\psi_n(r) = 0, \quad (17)$$

where ϵ_n denotes the energy eigenvalues.

the turning points r_A and r_B are determined by solving $\epsilon_n = V(r)$, where $r_A \prec r_B$

$$r_A = \frac{\epsilon_n}{2G} - \frac{1}{2} \sqrt{\left(-\frac{\epsilon_n}{G}\right)^2 - \frac{4l(l+1)}{2\epsilon_n}}, \quad (18)$$

$$r_B = \frac{\epsilon_n}{2G} + \frac{1}{2} \sqrt{\left(-\frac{\epsilon_n}{G}\right)^2 - \frac{4l(l+1)}{2\epsilon_n}}, \quad (19)$$

the logarithmic derivative ϕ_0 in the Riccati equation (2) take the form $\phi_0 = -a + \frac{b}{r}$ (where $a \succ 0$), substituting this into equation (2), allows us to obtain the explicit form of the ground state energy and the Riccati function

$$\epsilon_0 = -\frac{1}{2} \left(-\frac{G}{l+1} \right)^2, \quad \phi_0 = -\frac{G}{l+1} \quad (20)$$

now, the first momentum quantum integral $\int_{r_A}^{r_B} K_n(r) dr$ can be take the simplify form and calculated as:

$$\int_{r_A}^{r_B} K_n(r) dr = \int_{r_A}^{r_B} \sqrt{2(\epsilon_n - V(r))} dr = \sqrt{2\epsilon_n} \int_{r_A}^{r_B} \frac{\sqrt{(r-r_A)(r_B-r)}}{r} dr, \quad (21)$$

thus

$$\int_{r_A}^{r_B} K_n(r) dr = \frac{\pi}{2} \sqrt{-\frac{2G^2}{\epsilon_n}} - \pi \sqrt{l(l+1)}, \quad (22)$$

replacing ϵ_n in equation (22) by ϵ_0 given in equation (20) yields the second integral as

$$\int_{r_A}^{r_B} K_0(r) dr = \frac{\pi}{2} \sqrt{-\frac{2G^2}{\epsilon_0}} - \pi \sqrt{l(l+1)}, \quad (23)$$

and now, by considering the Quantization rule in equation (6), we obtain immediately an analytical expression of the well-known eigenvalues for coulomb potential as following

$$\epsilon_n = \frac{-G^2}{2(n+l+1)^2}, \quad (24)$$

4 Ground state energies for non-polynomial potentials

Now, let proceed to find a solution of nonlinear oscillator potentials at the ground state via the Proper quantization rule. Considering non-polynomial oscillator as follows:

4.1 Generalized isotonic oscillator

The quantum system described by the generalized isotonic oscillator, have attracted more attention, and given in three dimensional by,

$$V(r) = \frac{1}{2}\omega^2 r^2 + g \frac{r^2 - a^2}{(r^2 + a^2)^2}, \quad \text{and } g, a \succ 0, \quad (25)$$

this Non-polynomial potential converges to the isotonic and the harmonic oscillator when a tends to zero and to infinity respectively if g remains constant. It is found that this potential is placed between the isotonic and the harmonic oscillator.

Considering, the Schrödinger radial equation for this potential which take the form,

$$\frac{d^2\psi_n(r)}{dx^2} + [2E_n + \frac{l(l+1)}{r^2} - \omega^2 r^2 - 2g \frac{r^2 - a^2}{(r^2 + a^2)^2}] \psi_n(r) = 0, \quad (26)$$

this later, can be transformed to the nonlinear Riccati equation for the ground level,

$$\frac{d\phi_0(r)}{dx} = -[2E_0 + \frac{l(l+1)}{r^2} - \omega^2 r^2 - 2g \frac{r^2 - a^2}{(r^2 + a^2)^2}] - \phi_0^2(r) = 0, \quad (27)$$

taken the wave function for the ground state as $\phi_0(r) = -ar + \frac{b}{r} + \frac{2cr}{(r^2 + a^2)}$, with $a \succ 0$, substituting it into Riccati equation Eq. (27), one can find the ground state energy as,

$$E_0 = \omega \left(\frac{5}{2} + l - \sqrt{1 + 4g} \right), \quad (28)$$

indeed, the Riccati function becomes $\phi_0(r) = -\omega r + \frac{(l+1)}{r} - \frac{(1 - \sqrt{1 + 4g})r}{(r^2 + a^2)}$.

4.2 Singular anharmonic potential

The singular anharmonic potential is represented in three dimensional by the following formula,

$$V(r) = \frac{1}{2}\omega^2 r^2 + \frac{e}{r^4} + \frac{d}{r^6}, \quad (29)$$

where $e, d \succ 0$ constants parameter.

It's radial Schrödinger equation can be written as,

$$\frac{d^2\psi_n(r)}{dx^2} + [2E_n + \frac{l(l+1)}{r^2} - \omega^2 r^2 - 2\left(\frac{e}{r^4} + \frac{d}{r^6}\right)]\psi_n(r) = 0, \quad (30)$$

and can be given as the following nonlinear Riccati equation,

$$\frac{d\phi_0(r)}{dx} = -[2E_0 + \frac{l(l+1)}{r^2} - \omega^2 r^2 - 2\left(\frac{e}{r^4} + \frac{d}{r^6}\right)] - \phi_0^2(r) = 0, \quad (31)$$

after taking $\phi_0(r) = -ar + \frac{b}{r} + \frac{c}{r^3}$, with $a \succ 0$, and substituting it into Riccati equation, one gets the energy of the ground state,

$$E_0 = \omega \left(2 + \frac{e}{\sqrt{2d}}\right), \quad (32)$$

and $\phi_0(r) = -\omega r + \frac{\left(\frac{3}{2} + \frac{e}{\sqrt{2d}}\right)}{r} + \frac{\sqrt{2d}}{r^3}$.

4.3 Non polynomially modified oscillator

A great interest in several areas was focused to study this potential, particularly in laser theory. N. Besiss and G. Bessis [28], Flessas[29]. And recently, N. Saad et *all* [30], D. Agboola and *all* [25], Qiang Dong and *all* [26] were analyzed this potential. The non-polynomial modified oscillator have the following expression,

$$V(r) = \frac{1}{2}\omega^2 r^2 + \frac{\lambda r^2}{1 + \delta r^2}, \quad \text{with } \omega, \delta \succ 0, \quad -\infty \prec x \prec +\infty, \quad (33)$$

which can be reduced to the harmonic radial oscillator when the parameter λ converge to zero. The corresponding radial Schrödinger equation is given as,

$$\frac{d^2\psi_n(r)}{dx^2} + [2E_n + \frac{l(l+1)}{r^2} - \omega^2 r^2 - \frac{2\lambda r^2}{1 + \delta r^2}]\psi_n(r) = 0, \quad (34)$$

can be transformed to the Riccati equation at the ground state like,

$$\frac{d\phi_0(r)}{dx} = -[2E_0 + \frac{l(l+1)}{r^2} - \omega^2 r^2 - \frac{2\lambda r^2}{1 + \delta r^2}] - \phi_0^2(r) = 0, \quad (35)$$

if taking $\phi_0(r) = -ar + \frac{b}{r} + \frac{cr}{1 + \delta r^2}$, where $a \succ 0$, substituting it into the above Riccati equation, we find the ground eigenvalue energy,

$$E_0 = \frac{\lambda}{\delta} + \omega \left(\frac{7}{2} + l\right), \quad (36)$$

and the solution of Riccati equation is given by $\phi_0(r) = -\omega r + \frac{(l+1)}{r} + \frac{2\delta r}{1 + \delta r^2}$.

4.4 Soft-core coulomb potential

Consider the Soft-core Coulomb potential [25] given by

$$V(r) = \frac{G}{r} - \frac{Z}{r + \beta}, \quad (37)$$

where $G \neq Z > 0$ and $\beta > 0$ are constant parameters. Such potential is of interest in atomic and molecular physics. The corresponding radial Schrödinger equation is

$$\frac{d^2\psi_n(r)}{dx^2} + [2E_n + \frac{l(l+1)}{r^2} - \frac{G}{r} + \frac{Z}{r+\beta}]\psi_n(r) = 0, \quad (38)$$

where E_n is the energy eigenvalue. The last equation transforms to the bellow non-linear Riccati equation and becomes

$$\frac{d\phi_0(r)}{dx} = -[2E_0 + \frac{l(l+1)}{r^2} - \frac{G}{r} + \frac{Z}{r+\beta}] - \phi_0^2(r) = 0, \quad (39)$$

After substituting the logarithmic derivative $\phi_0(r) = -ar + \frac{b}{r} + \frac{c}{1+\beta}$, where $a > 0$, into the above Riccati equation, one can obtain the explicit form of the energy at the ground state E_0 and the unknown parameters a , b and c as

$$E_0 = -\frac{1}{2} \left(\frac{Z-G}{l+2} \right)^2, \quad \text{and} \quad a = \left(\frac{Z-G}{l+2} \right), \quad b = (l+1), \quad c = 1, \quad (40)$$

5 Energies expressions potentials

Now, we are in the position to determine the energy levels expression for anharmonic potentials. Looking at the Eqs.(28), (32) and (36), one can conclude that the ground level energy of radial oscillator potential appears in the ground state energy expression of non-polynomial potentials, from which, with a simple treatment, allows us to obtain a general analytical expression of energy spectrum for these class of potentials in respect to the ground level energy E_0 for each potential by the given formula,

$$E_n = E_0 + 2n\omega, \quad (41)$$

nevertheless, the energy spectrum of the nonlinear oscillator is that of the radial oscillator plus a constant c , which differs from one potential to another. So the height ΔE is constant,

$$E_n = \epsilon_n + c, \quad \text{with } c \text{ is constant}, \quad (42)$$

replacing the ground state energy E_0 of each non-polynomial potentials into the energy formula Eq. (41), we obtain easily the energy spectra, corresponding to non-polynomial quantum potentials bellows:

Generalized isotonic oscillator,

$$E_n = \omega \left(2n + \frac{5}{2} + l - \sqrt{1+4g} \right). \quad (43)$$

Singular anharmonic potential,

$$E_n = \omega \left(2n + 2 + \frac{e}{\sqrt{2d}} \right). \quad (44)$$

Non polynomially modified oscillator,

$$E_n = \frac{\lambda}{\delta} + \omega \left(2n + \frac{7}{2} + l \right). \quad (45)$$

Likewise, the energy levels E_n and the energy for the ground state E_0 of Soft-core coulomb potential is expressed in terms of the energy spectrum ϵ_n of the coulomb potential and provides by solving the equations (22) and (23), as follows

$$\sqrt{\frac{-2(G-Z)^2}{E_n}} = 2 + \sqrt{\frac{-2G^2}{\epsilon_n}}, \quad (46)$$

then

$$\sqrt{\frac{-2(G-Z)^2}{E_0}} + 2n = 2 + \sqrt{\frac{-2G^2}{\epsilon_n}}, \quad (47)$$

hence

$$\sqrt{\frac{-2(G-Z)^2}{E_n}} = 2n + \sqrt{\frac{-2(G-Z)^2}{E_0}}, \quad (48)$$

using this result it will be easy to obtain the the energy eigenvalues E_n of the Soft-core coulomb potential is given by resolving the equation (48)

$$E_n = -\frac{1}{2} \left(\frac{Z-G}{n+l+2} \right)^2, \quad (49)$$

6 Conclusion

At last, throughout of this work, we have shown that it is very easy to obtain the eigenenergy of the quantum systems described by non-pollynomial potentials, expressed with solvable potential and a nonlinear extension using proper quantization rule. It consists of transforming the Schrödinger equation into the Riccati differential equation; in which, the appropriate solution ϕ_0 function at the ground level was inserted. In a simple resolution of the Riccati equation, one can obtain the ground level energy E_0 for these potentials type. And find that the energy spectrum of non-polynomial quantum potentials can be determined from its ground state energy only; without needing to calculate integral term obtained by combination of energy spectra of solvable potential. The results gated are in agreement with those already obtained.

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