Physics AUC, vol. 30 (part II), 85-95 (2020) **PHYSICS AUC**

Holographic Fisher Information Metric for Models with Non-Relativistic Symmetry

H. Dimov[a,b], M. Radomirov[b], R. C. Rashkov[b,c], and T. Vetsov *[b]

[a] *The Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia*

[b] *Department of Physics, Sofia University, 5 J. Bourchier Blvd., 1164 Sofia, Bulgaria*

[c] *Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstr. 8–10, 1040 Vienna, Austria*

Abstract

In this paper we make a quantitative check of the gauge/gravity correspondence for holographic systems with non-relativistic symmetries. The focus is on the calculation of Fisher information metric on the space of coupling constants on both sides of the duality. We consider the particular duality between non-relativistic dipole field theory and string theory in Schrödinger spacetime. Our study shows exact match between the relevant quantities on both sides of the correspondence, thus suggesting its validity in this case.

1 Introduction

One of the most powerful and revealing discoveries in recent years is the realization of the holographic principle in the context of string theory [1]. At the heart of this idea lies the confidence that there is a unifying framework of all forces of nature at the fundamental level. In this regard, the holographic principle conjectures the existence of a particular duality between quantum and gravitational theories, which greatly extends our understanding of spacetime and matter. Known also as the gauge/gravity correspondence it relates a higher dimensional gravitational (string) theory in the bulk of spacetime to a quantum field theory, living on the lower dimensional asymptotic boundary.

The specific dictionary of the duality relates not only the symmetries in both theories, but also their strong/weak coupling regimes. It turns out that if the quantum system is strongly correlated and thus nonperturbative, its dual gravitational theory is weakly coupled and allows perturbative calculations. The latter can be extended back to the quantum theory by duality. Ever since the original proposal [1] of this conjecture[1] it has been repeatedly shown to hold in many cases.

*Emails: `dimov @theor.jinr.ru` and `h_dimov,radomirov,rash,vetsov@phys.uni-sofia.bg`.
Proceedings of the WORKSHOP ON QUANTUM FIELDS AND NONLINEAR PHENOMENA, Craiova, Romania, 2020.

[1] The AdS/CFT correspondence.

An important consequence of the holographic correspondence is that spacetime can be an emergent phenomenon from the dynamics of the quantum degrees of freedom. In this case, a key concept is the boundary entanglement entropy, which is conjectured to be the dual quantity of a codimension-2 extremal surface[2], referred to as the Ryu-Takayanagi (RT) surface, in the bulk [2, 3]. On the other hand, the volume, enclosed by this surface, is dual to another key concept called the quantum Fisher information metric (QFIM). The latter gives in general a measure for the distance between two quantum states, while its holographic dual Fisher metric (HFIM) encodes information about the shape of the entangling region enclosed by the RT surface [4, 5].

In general, the bulk entanglement entropy, which is interpret as the $1/N$ quantum correction to the boundary entanglement entropy, is hard to obtain directly from a generic bulk state. However, as pointed out in [5], the leading $1/N$ quantum correction to the Fisher information metric is related to the corresponding quantum correction to the boundary entanglement entropy. Due to the fact that one can calculate the holographic Fisher metric in the bulk perturbatively, as shown by [6], we are compelled to study this quantity in both sides of the gauge/gravity correspondence.

Particular cases, where the holographic principle has not been thoroughly checked, are holographic systems with non-relativistic symmetries. Important features of non-relativistic holography have been uncovered in [7], where strong arguments for integrability and quantitative matching between string and gauge theory predictions have been presented. Furthermore, new studies have lead to several interesting applications of non-relativistic holography in condensed matter physics and string theory such as the description of SQCD-like gauge theories in the context of D-brane constructions [8–14], Sachdev-Ye-Kitaev (SYK) model [15], Fermi unitary gas [16], and models with trapped super cooled atoms [17, 18]. One of the main reasons for studying such non-relativistic models from the holographic point of view is the fact that in most cases they appear as strongly coupled system.

In this paper we will quantitatively check the holographic principle by explicitly calculating and comparing the Fisher information metric from both sides of a particular realization of the non-relativistic holographic correspondence. Namely, when there is a duality between a strongly correlated dipole gauge theory and Schrödinger gravitational background. The structure of the paper is the following. In Section 2 we briefly discus the procedure, known as null-Melvin twist, allowing us to generate backgrounds with non-relativistic symmetry, such as the Schrödinger spacetime. In Section 3 we will perform an exact match of the quantum Fisher metric from both sides of the gauge/gravity correspondence, thus proving that the holographic principle holds in Schrödinger spacetime. Finally, in Section 4, we will briefly comment on the result.

2 Generating the Schrödinger background

Well-established fact is that the symmetry of the free Schrödinger equation is the so called Schrödinger group, which consists of spatial translations, rotations, Galilean boosts, dilatations (where time and space dilate with different factors) and one additional special conformal transformation. From group theoretic point of view, the Schrödinger group can be thought of as a non-relativistic analogue of the conformal group. In fact, the Schrödinger

[2]The entanglement entropy for an entangling region in the boundary conformal field theory of an asymptotically AdS spacetime is proposed to be given by a minimal codimension-2 area in the bulk.

group in d dimensions can be embedded into the relativistic conformal group $SO(2, d+2)$ in $(d+1)$ dimensions [18–20].

The holographic construction of spaces with the Schrödinger group being the maximal group of isometries[3] can be achieved by particular TsT (T-duality-shift-T-duality) transformations along some of the null coordinates of the metric. In order to realize the Schrödinger symmetry geometrically, we will consider the maximally symmetric $AdS_5 \times S^5$ space in light-cone coordinates

$$\frac{ds^2}{L^2} = \frac{2dx^+dx^- + dx^i dx_i + dz^2}{z^2} + \left(d\chi + \frac{1}{2}\sin^2\mu\,(d\alpha + \cos\theta\, d\phi)\right)^2 + ds^2_{\mathbb{CP}^2}, \tag{1}$$

which is invariant under the whole conformal group. However, it can consequently be TsT deformed to reduce the symmetry down to the Schrödinger group. This type of deformations are known also as null-Melvin twist. Here, the metric of the round S^5 is written in the form of Hopf fibration, where χ is a coordinate for the Hopf fibers. To generate the null-Melvin twist we perform the following steps:

- Make a T-duality along χ following the standard Buscher rules.
- Make a shift, $x^- \to x^- + \hat{\mu}\hat{\chi}$, with parameter $\hat{\mu}$, where $\hat{\chi}$ is the T-dualized χ.
- Make T-duality back on $\hat{\chi}$.

Following these rules one easily finds [22]

$$ds^2 = L^2\left(-\frac{(\hat{\mu}dx^+)^2}{z^4} + \frac{2dx^+d\hat{x}^- + dx^i dx_i + dz^2}{z^2}\right) + ds^2_{\hat{S}^5}, \tag{2}$$

where the metric on the new 5-sphere is now given by[4]

$$\begin{aligned}\frac{ds^2_{\hat{S}^5}}{L^2} &= d\hat{\chi}^2 + d\mu^2 + \frac{1}{4}\sin^2\mu\left(d\alpha^2 + d\theta^2 + d\phi^2\right)\\ &\quad + \sin^2\mu\, d\hat{\chi}\, d\alpha + \sin^2\mu\cos\theta\, d\hat{\chi}\, d\phi + \frac{\sin^2\mu\cos\theta}{2}\, d\alpha\, d\phi.\end{aligned} \tag{3}$$

The first part of the metric (2) is the desired Schrödinger background. For our considerations it will be useful to write the metric of the Schrödinger spacetime in $(d+3)$-dimensional global coordinates, as written originally by [18, 19],

$$ds^2_{Schr_{d+3}} = L^2\left(-\frac{dt^2}{r^4} + \frac{2d\xi dt + d\vec{x}^{\,2}}{r^2} + \frac{dr^2}{r^2}\right), \tag{4}$$

where $\vec{x}$ is a d-dimensional vector. The boundary of the background (4) is at $r = 0$ and the generator associated with translations along the compact ξ direction can be identified with the mass operator $M = i\partial_\xi$. The latter is not a geometric dimension in the usual sense, because each operator of the boundary theory can be taken to have a fixed discrete momentum conjugate to ξ.

[3] For a detailed group-theoretical perspective on non-relativistic holography see [21].

[4] Additionally, the deformed TsT background acquires also a non-zero B-field, while in the original $AdS_5 \times S^5$ theory there was no initial B-field turned on. Anyway, the B-field is not important for our current investigation.

It is conjectured that the dual quantum field theory of the Schrödinger spacetime is a specific non-relativistic and primarily non-local field theory, called dipole theory. It lives on the $(d+1)$-dimensional boundary at $r=0$, which is coordinazied by $(t, \vec{x})$.

In what follows, we will consider the computation of the quantum Fisher information metric and its dual holographic counterpart, considering only the local sectors of the theories from both sides of the holographic correspondence.

3 Check of the Non-Relativistic Holography

3.1 Exact computation of QFIM

Let us consider a conformal field theory (CFT) living on the $(d+1)$-dimensional boundary of the Schrödinger spacetime, described by the Lagrangian $\mathcal{L}_0$ for an Euclidean time interval[5] $\tau \in (-\infty, 0)$. At $\tau = 0$ we perturb the original theory by some quantum operator $\lambda\mathcal{O}(\tau)$ with a coupling λ. In this way, we have a new theory with $\mathcal{L}_1 = \mathcal{L}_0 + \lambda\mathcal{O}$, for $\tau > 0$. Traditionally, we can compute the quantum Fisher information metric $G_{\lambda\lambda}$, between the ground states of both theories, by the series expansion of the quantum fidelity at temporal infinities,

$$\mathcal{F}(\lambda + \delta\lambda) \equiv |\langle\psi_1(\tau \to \infty)|\psi_0(\tau \to -\infty)\rangle| = 1 - G_{\lambda\lambda}\delta\lambda^2 + \mathcal{O}(\delta\lambda^3), \tag{5}$$

where $|\psi_0\rangle$ and $|\psi_1\rangle$ are the ground states in the first and the second theory, correspondingly. The generic expression for $G_{\lambda\lambda}$ is given by [6, 23]

$$G_{\lambda\lambda} = \frac{1}{2}\int\limits_{V_{\mathbb{R}^d}} d^d x_1 \int\limits_{V_{\mathbb{R}^d}} d^d x_2 \int\limits_{-\infty}^{-\epsilon} d\tau_1 \int\limits_{\epsilon}^{\infty} d\tau_2 \left(\langle\mathcal{O}(\tau_1, x_1)\mathcal{O}(\tau_2, x_2)\rangle - \langle\mathcal{O}(\tau_1, x_1)\rangle\langle\mathcal{O}(\tau_2, x_2)\rangle\right). \tag{6}$$

The presence of the cutoff ϵ around $\tau = 0$ is necessary to address any ultraviolet divergences in case there is a discontinuity when passing from the original to the deformed Lagrangian.

Considering primary operators with conformal dimension Δ and taking into account that for such operators $\langle\mathcal{O}\rangle = 0$, then only the 2-point function, $\langle\mathcal{O}(\tau_1, x_1)\mathcal{O}(\tau_2, x_2)\rangle$, contributes to the QFIM. The computation of the 2-point correlator, between primary operators from the dual gauge field theory in the Schrödinger case, has been performed by [24, 25]. Its explicit form yields

$$\langle\mathcal{O}(\tau_2, \vec{x}_2)\mathcal{O}(\tau_1, \vec{x}_1)\rangle = \frac{i\Delta\left(\frac{M}{2}\right)^{\Delta-1} e^{-i\pi\frac{\Delta}{2}}}{\pi^{d/2}\,\Gamma\left(\Delta - \frac{d}{2} - 1\right)} \frac{\theta(\tau_2 - \tau_1)}{(\tau_2 - \tau_1)^{\Delta}} e^{\frac{iM}{2}\frac{(\vec{x}_2 - \vec{x}_1)^2}{\tau_2 - \tau_1}}, \tag{7}$$

where d is the dimension of $\vec{x}$ space, $\theta(\tau_2 - \tau_1)$ is the unit step function for an Euclidean time interval, M is the quantized momentum along the compact direction ξ with radius $1/M$, and the conformal dimension of the operators is[6]

$$\Delta = 1 + \frac{d}{2} + \sqrt{\left(1 + \frac{d}{2}\right)^2 + m^2 + M^2}. \tag{8}$$

[5]We will refer to t as the real time and $\tau = -it$ as the Wick-rotated time.

[6]The parameter m is the mass of the dual bulk scalar field probing the Schrödinger background.

We will consider $\tau_1 \leq \tau_2$, thus the QFIM for a single marginal deformation yields

$$G_{\lambda\lambda}^{(CFT)} = \frac{1}{2}\int d^d x_1 \int d^d x_2 \int_{-\infty}^{-\epsilon} d\tau_1 \int_{\epsilon}^{\infty} d\tau_2 \left\langle \mathcal{O}(\tau_1, x_1)\, \mathcal{O}(\tau_2, x_2)\right\rangle = C\epsilon^{2+\frac{d}{2}-\Delta}, \tag{9}$$

where the normalization constant is[7]

$$C = \frac{\Delta 2^{d-2\Delta+3} e^{-\frac{i\pi}{4}(d+2\Delta)} M^{\Delta-\frac{d}{2}-1} V_{\mathbb{R}^d}}{(d-2\Delta+4)\,\Gamma\left(\Delta-\frac{d}{2}\right)}. \tag{10}$$

The integrals over τ_1 and τ_2 converge only for

$$2\Delta > d+4, \tag{11}$$

which falls within the scope of some previous conditions for holographic systems [6, 26]. Note that the divergence behavior of the QFIM is determined only by the regulator ϵ.

In the next Section we proceed with the computation of the holographic Fisher information metric in the dual Schrödinger gravitational background, which will show more complex behavior than QFIM. The analysis of the relativistic AdS case was performed in [6].

3.2 Perturbative computation of HFIM

As we already mentioned, the holographic correspondence states a duality between the operators from the boundary gauge theory and a classical scalar field probing the dual gravitational background. Any changes in the boundary gauge theory will produce changes in the profile of the scalar field from the dual gravitational theory.

In order to encode the relevant quantum information onto the dynamics of the massive scalar field $\phi(x)$ on the gravity side, one has to consider the following picture. Initially, we have an undeformed CFT with an Euclidean Lagrangian $\mathcal{L}_0$ and primary operators dual to a probe scalar field $\phi_0(x)$ in the bulk. This theory can exist unrestricted for an Euclidean time interval $\tau \in (-\infty, +\infty)$. On the other hand, we can consider the deformed theory $\mathcal{L}_1 = \mathcal{L}_0 + \lambda\mathcal{O}$ as an independent theory from $\mathcal{L}_0$ within the same time interval $\tau \in (-\infty, +\infty)$. Its operator content is dual to a new profile of the probe scalar field, namely $\phi_1(x)$ in the bulk. However, turning on a deformation $\lambda\mathcal{O}$ at $\tau = 0$ in the original theory $\mathcal{L}_0$, will produce a third theory with $\mathcal{L}_2 = \mathcal{L}_1$, valid only for $\tau > 0$. Its content is dual to a bulk scalar field with a profile $\phi_2(x)$.

As suggested by [6], the first step in finding the HFIM is to rewrite the fidelity (5) in terms of the partition functions of the three different theories, namely

$$\langle\psi_1|\psi_0\rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}}, \tag{12}$$

were Z_0 and Z_1 are the partition functions for the original and the deformed theories in the range $\tau \in (-\infty, +\infty)$, while Z_2 is the partition function for the deformed theory only for $\tau > 0$. The second step is to consider the large N limit on the gravity side, where $Z_k = e^{-I_k}$, $k = 0, 1, 2$, with I_k being the on-shell action of the gravity solution dual to the

[7] With $V_{\mathbb{R}^d}$ we notate the volume of $\mathbb{R}^d$ space.

corresponding field theory configuration. In this case, it is the action of the massive scalar field $\phi_k(x)$, probing the Shrödinger geometry[8] $g_{\mu\nu}(x)$:

$$I_k = -\frac{1}{\kappa^2}\int\limits_{\mathcal{M}} d^D x\sqrt{|g|}\left(\frac{1}{2}(R-2\Lambda) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k - \frac{1}{2}m^2\phi_k^2 + V(\phi_k)\right) + I_{\partial\mathcal{M}}, \tag{13}$$

where $I_{\partial\mathcal{M}}$ is some proper boundary term. Therefore, one finds

$$\langle\psi_1|\psi_0\rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}} = e^{\frac{1}{2}(I_0+I_1)-I_2}. \tag{14}$$

In this picture, the deformation of the CFT by a single primary operator induces an interaction term $\lambda\mathcal{O}$ with a coupling λ in the CFT Lagrangian, thus changing the initial dual bulk gravitational action I_0 by $I_k = I_0 + \delta I_k(\lambda)$ with $\delta I_0 = 0$. Hence, one has

$$\langle\psi_1|\psi_0\rangle = \frac{Z_2}{\sqrt{Z_0 Z_1}} = e^{\frac{1}{2}(I_0+I_1)-I_2} = e^{\frac{1}{2}(I_0+\delta I_0+I_0+\delta I_1)-I_0-\delta I_2} = e^{\frac{1}{2}\delta I_1-\delta I_2}. \tag{15}$$

This reduces the computation of HFIM only to finding the variations δI_1 and δI_2 of the on-shell gravitational action. In particular, when the massive field is turned off ($\phi_0(x) = 0$), one has the initial Schrödinger background geometry with metric $g^{(0)}_{\mu\nu}$, i.e. $Z_0 = \exp\left(-I_0[\phi_0, g^{(0)}_{\mu\nu}]\right)$. On the other hand, the profiles $\phi_{1,2}(x)$ of the scalar field for I_1 and I_2 will in general be spacetime depended and can be calculated by the corresponding bulk-to-boundary propagators. Since we are interested in perturbative solutions in lower powers of λ we can insist on the following transformations of the fields[9]

$$\phi(x) = \phi_0(x) + \lambda\varphi(x), \qquad g_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) + \lambda^2 h_{\mu\nu}(x), \tag{16}$$

where $\varphi(x)$ and $h_{\mu\nu}(x)$ are some perturbations generated after turning on the deformation in the dual CFT. Notice that the metric receives corrections at order λ^2 since the scalar field enters quadratically in Einstein field equations. Hence, the variation of the bulk action δI to lowest orders of λ yields

$$\delta I = I[\phi, g_{\mu\nu}] - I_0[\phi_0, g^{(0)}_{\mu\nu}] = \lambda^2 \int \frac{\delta^2 I}{\delta\phi(x)\delta\phi(y)}\bigg|_{\lambda=0} \varphi(x)\varphi(y) + \mathcal{O}(\lambda^3).$$

Note that the first variations of the action with respect to the metric and the scalar field vanish due to the equations of motion. Now, we can easily compute the second variation of the action with respect to the scalar field to find

$$\delta I_k = \frac{1}{2\kappa^2}\int d^{d+1}x\,\sqrt{|g|}\,\left(g^{(0)\mu\nu}\partial_\mu\phi_k\,\partial_\nu\phi_k + m^2\phi_k^2\right) = \frac{1}{2\kappa^2}\int d^d x\,\sqrt{|\gamma|}\,n_\mu\,g^{(0)\mu\nu}\phi_k\,\partial_\nu\phi_k, \tag{17}$$

with n_μ being the unit normal vector, γ is the determinant of the induced metric on the boundary, and $\phi_k(x)$ are the scalar field configurations dual to the operators of the corresponding deformed and undeformed CFTs, while probing the fixed background $g^{(0)}_{\mu\nu}$.

[8]This is valid only in the probe limit, where the backreaction of the scalar field on the geometry can be neglected, i.e. the coupling λ is small.

[9]We will consider perturbations around the original undeformed theory, thus $\phi_0(x) = 0$.

Clearly, one can obtain these profiles by using the bulk-to-boundary propagator in the Schrödinger spacetime, namely[10] [27]

$$\phi_1(r,\vec{x},t) = \lambda \int d^d x_1 \int\limits_{-\infty}^{+\infty} d\tau_1\, K(r,\vec{x},\tau;\vec{x}_1,\tau_1)\,\hat{\phi}_0(\vec{x}_1,\tau_1) = i\lambda\, e^{-\frac{i\pi d}{2}} r^{d-\Delta+2} \tag{18}$$

for the first scalar field configuration and

$$\phi_2(r,\vec{x},t) = \lambda \int d^d x_1 \int\limits_{0}^{\tau} d\tau_1\, K(r,\vec{x},\tau;\vec{x}_1,\tau_1)\,\hat{\phi}_0(\vec{x}_1,\tau_1) = \frac{i\lambda\, e^{-\frac{i\pi d}{2}}}{\Gamma(a)}\, r^{d-\Delta+2}\,\Gamma\left(a,\frac{\mu}{t}\right) \tag{19}$$

for the second scalar field configuration. Here, $\Gamma(a,\mu/t)$ is the incomplete gamma function and we have further defined the parameters

$$a = \Delta - \frac{d}{2} - 1, \quad \mu = \frac{1}{2} M r^2. \tag{20}$$

One also notes, that the field $\hat{\phi}_0(\vec{x}_1,\tau_1) = 1$ is the source function on the boundary at $r = 0$ (see [19]), and the bulk-to -boundary propagator reads [24]

$$K(r,\vec{x},\tau;\vec{x}_1,\tau_1) = \frac{i\left(\frac{M}{2}\right)^{\Delta-1} e^{-i\pi\frac{\Delta}{2}}}{\pi^{d/2}\,\Gamma\left(\Delta-\frac{d}{2}-1\right)}\,\frac{\theta(\tau-\tau_1)}{(\tau-\tau_1)^{\Delta}}\, r^{\Delta}\, e^{\frac{iM}{2}\frac{r^2+(\vec{x}-\vec{x}_1)^2}{\tau-\tau_1}}. \tag{21}$$

In the final expression for $\phi_2(x)$ we have restored the real time $\tau \to -it$, so that subsequent calculations with the incomplete gamma functions will attain the correct properties. Furthermore, the integral converges only for $a > 0$ and $M > 0$.

Knowing the profiles of the fields $\phi_{1,2}$ we can write the HFIM integrals according to (17), adopted for the Schrödinger case (4), namely

$$\delta I_k = \frac{1}{2\kappa} \lim_{r\to\tilde{\varepsilon}} \int d^d x \left(\int\limits_{-T}^{-\epsilon} dt \sqrt{|\gamma|}\, n_\mu\, g^{\mu\nu} \phi_k\, \partial_\nu \phi_k + \int\limits_{\epsilon}^{T} dt \sqrt{|\gamma|}\, n_\mu\, g^{\mu\nu} \phi_k\, \partial_\nu \phi_k \right), \tag{22}$$

where γ is the metric on the boundary, $\tilde{\varepsilon} > 0$ is the regulator in the holographic direction r near the boundary $r \to 0$, n_μ is the normal outward vector to the boundary, $\epsilon > 0$ is a regulator around $t = 0$, and T is a cut-off at temporal infinity. On the boundary only the component n_r is non-zero, thus $\sqrt{|\gamma|}\, n_r g^{rr} = L^d\, r^{-1-d}$. However, one has to account for the fact that ϕ_1 and ϕ_2 span different ranges for t, which leads to

$$\delta I_1 = \frac{L^d V_{\mathbb{R}^d}}{2\kappa} \lim_{r\to\tilde{\varepsilon}} \left(\int\limits_{-T}^{-\epsilon} dt\, r^{-1-d}\, \phi_1(r)\, \partial_r \phi_1(r) + \int\limits_{\epsilon}^{T} dt\, r^{-1-d}\, \phi_1(r)\, \partial_r \phi_1(r) \right), \tag{23}$$

$$\delta I_2 = \frac{L^d V_{\mathbb{R}^d}}{2\kappa} \lim_{r\to\tilde{\varepsilon}} \int\limits_{\epsilon}^{T} dt\, r^{-1-d}\, \phi_2(t,r)\, \partial_r \phi_2(t,r). \tag{24}$$

[10]The integration is performed over the Euclidean time τ_1 for $\tau_1 \leq \tau$.

The non-trivial solutions to these integrals have been found in [27]. The final step is to expand the overlap (15) and write down the expression for the holographic Fisher information metric in the Schrödinger spacetime, i.e.

$$G_{\lambda\lambda} = a_0(\mu)(T-\epsilon) + \sum_{i=1}^{7}\Big[\, a_i(\mu,T)\,\Gamma^p\Big(\alpha_i, \frac{q\mu}{T}\Big) + \, b_i(\mu,\epsilon)\,\Gamma^p\Big(\alpha_i, \frac{q\mu}{\epsilon}\Big)\Big], \tag{25}$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function with

$$\alpha_1 = \alpha_4 = a-2, \quad \alpha_2 = a, \quad \alpha_3 = a-1, \tag{26}$$

$$\alpha_5 = 2a-4, \quad \alpha_6 = 2a-2, \quad \alpha_7 = 2a-3. \tag{27}$$

and

$$p = \begin{cases} 1, & i = 4 \div 7, \\ 2, & i = 1 \div 3. \end{cases} \qquad q = \begin{cases} 1, & i = 1 \div 4, \\ 2, & i = 5 \div 7. \end{cases} \tag{28}$$

The holographic Fisher information metric (25) is equipped with three different divergences near the Schrödinger boundary at $r = 0$, namely $\epsilon \to 0$, $T \to \infty$ and $\mu \to 0$. However, on the boundary the limit $T \to \infty$ is not mandatory and the cutoff T could be considered finite. Furthermore, one also has the relation $b_i(\mu,\epsilon) = -a_i(\mu, T \to \epsilon)$, where

$$\begin{aligned}
a_0 &= \frac{L^d V_{\mathbb{R}^d}}{\mu^a 2^{a+1}\kappa} M^a \left[2c_0 + c_2\left(\Gamma^2(a) - 2\Gamma(2a)\,\mathcal{B}_{1/2}(a,a)\right)\right], \\
a_1 &= \frac{L^d V_{\mathbb{R}^d}}{4\mu^{a-1}\kappa} c_1 (a-1)(a-2)^2, \quad a_2 = \frac{T L^d V_{\mathbb{R}^d}}{2^{a+1}\mu^a\kappa} M^a c_2, \\
a_3 &= -\frac{L^d V_{\mathbb{R}^d}}{2^{a+1}\mu^{a-1}\kappa} M^a c_2 (a-1) \quad a_4 = \frac{T^{2-a} e^{-\frac{\mu}{T}} L^d V_{\mathbb{R}^d}}{2\mu\kappa} c_1 (a-1)(a-2), \\
a_5 &= -\frac{L^d V_{\mathbb{R}^d}}{2^{2a-3}\mu^{a-1}\kappa} c_1 (a-1)(a-2), \quad a_6 = \frac{L^d V_{\mathbb{R}^d}}{2^{3a-1}\mu^{a-1}\kappa}\left(2^a c_1 - 2c_2 M^a\right), \\
a_7 &= \frac{L^d V_{\mathbb{R}^d}}{2^{2a-2}\mu^{a-1}\kappa} c_1 (a-1), \quad a = \Delta - \frac{d}{2} - 1, \\
c_0 &= \frac{e^{-i\pi d}}{2}(d - \Delta + 2), \quad c_1 = \frac{M^a\, e^{-id\pi}}{2^{a-1}\,\Gamma^2(a)}, \quad c_2 = -\frac{2c_0}{\Gamma^2(a)},
\end{aligned} \tag{29}$$

with $\mathcal{B}_{1/2}(a,a)$ being the incomplete beta function. We have explicitly kept the coefficients by different names a's and b's to reflect the different asymptotic behavior of the arguments of the incomplete gamma functions.

3.3 Asymptotic behaviour of HFIM near the boundary

In order to see if the holographic correspondence holds in the non-relativistic case, one has to show that HFIM (25) asymptotically approaches QFIM (9), near the boundary at $r = 0$. In this case, we effectively have two competing divergences, namely $\mu \to 0$, when approaching $r = 0$ along r, and $\epsilon \to 0$ along t, near $t = 0$. Therefore, we can look at different situations, e.g. one in which μ goes to zero asymptotically faster than ϵ, and the other case, where ϵ goes faster to zero than μ. In both cases we have two possibilities for the cut-off T, i.e. $T \to \infty$ or finite T.

After some careful analysis, one finds that the case of dominant[11] ϵ over μ at $T \to \infty$, does lead to the desired outcome near the boundary. To show this, one can use the

[11]This means that ϵ approaches zero sufficiently faster than μ.

asymptotic expansion

$$\Gamma\left(a,z\right)\sim z^{a-1}e^{-z}\sum_{k=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-k)}z^{-k},\quad |z|\to\infty, \tag{30}$$

which defines the explicit divergence structure of the b_i terms, while for the a_i terms we can use

$$\Gamma\left(a,z\right)=\Gamma(a),\quad z\to 0,\quad a>0. \tag{31}$$

Therefore, Eq. (25) acquires the following asymptotic form near the boundary $r=0$

$$\begin{aligned}\frac{G_{\lambda\lambda}}{\epsilon^{1-a}} &= A_1\frac{T\,\epsilon^{a-1}}{\mu^a}-A_2\left(\frac{\epsilon}{\mu}\right)^a+A_3\frac{T^{2-a}\,e^{-\frac{\mu}{T}}\,\epsilon^{a-1}}{\mu}+A_4\left(\frac{\epsilon}{\mu}\right)^{a-1}\\ &+e^{-\frac{2\mu}{\epsilon}}\sum_{i=1}^{7}\sum_{k=0}^{\infty}\gamma_i(a,k)\left(\frac{\epsilon}{\mu}\right)^{\delta_i(a,k)},\end{aligned} \tag{32}$$

where $A_{1,2,3,4}$, γ_i and δ_i are some finite constants, which do not dependent on the regulators. Due to the fact that ϵ is dominant, all γ_i terms vanish by the suppressing factor of $e^{-\frac{2\mu}{\epsilon}}$. Obviously the divergence structure of the A_i terms depend on the range spanned by the parameter a. For example, the first term has a divergence structure $T\,\epsilon^{a-1}\mu^{-a}$ for $T\to\infty,\mu\to 0,\epsilon\to 0$. Therefore, one can take $A_1T\,\epsilon^{a-1}\mu^{-a}=K=\mathit{const}$ for $a>1$, thus it can be considered regular. This leads to a vanishing term $A_3T^{2-a}e^{-\frac{\mu}{T}}\epsilon^{a-1}\mu^{-1}\to 0$. The terms with $\epsilon^a\mu^{-a}$ and $\epsilon^{a-1}\mu^{1-a}$ also vanish for $a>1$, because $\epsilon\to 0$ is dominant. Therefore, one can recover the divergence structure of the dual CFT quantum Fisher information metric (9) from (32) for $a>1$, namely

$$G_{\lambda\lambda}=K\epsilon^{1-a}=K\epsilon^{2+\frac{d}{2}-\Delta},\quad a>1, \tag{33}$$

where K is the HFIM normalization constant. Imposing $C=K$, where C is defined in Eq. (10), leads to

$$K=\kappa\frac{(a_0+a_2\Gamma^2(a))(2a+d+2)\Gamma(a-1)\,e^{\frac{i\pi}{2}(d-a-1)}}{2^{a-1}L^d(2-2a+d)\left(\Gamma(a)\Gamma(a+1)-\Gamma(2a+1)B_{1/2}(a,a)\right)}, \tag{34}$$

which gives a complete match between HFIM and QFIM near the boundary at $r=0$. This analysis suggests that the the holographic principle extends to models with non-relativistic symmetries.

4 Final Remarks

In this work we have considered a duality between dipole field theory and string theory in Schrödinger spacetime, which is a particular example of gauge/gravity correspondence for systems with non-relativistic symmetries. We focused our investigation on the derivation of the quantum Fisher information metric on the space of coupling constants over the conformal field theory and its dual holographic counterpart in the bulk of spacetime.

In the dual gauge theory side, we have used the 2-point correlation function between two primary operators in order to calculate the quantum Fisher metric. It showed relatively simple divergence structure along the time direction and dependence only on the conformal dimension Δ of the operators and the discrete non-relativistic particle number M.

On the gravity side of the correspondence, we have adapted the perturbative method for the computation of the holographic Fisher information metric, suggested by [6], to find HFIM in the dual bulk Schrödinger geometry. The results for the HFIM showed complicated divergence structure with some additional convergence conditions. By analyzing the asymptotic behaviour of the bulk holographic Fisher information metric, we were able to reduce it to the quantum Fisher metric near the boundary of the spacetime. The exact match of these quantities from both sides of the gauge/gravity correspondence further proved that one can extend the holographic principle to dual systems with non-relativistic symmetries.

Acknowledgments

We would like to thank the organizers of the event for their hospitality and especially Radu Constantinescu and Goran Djordjevic. This work was partially supported by the Bulgarian NSF grants H28/5 and DN18/1. H. D. gratefully acknowledges the support from the program JINR - “Bulgaria” at Bulgarian Nuclear Regulatory Agency.

References

[1] J. M. Maldacena, Int. J. Theor. Phys. **38** (1999), 1113-1133, [arXiv:hep-th/9711200 [hep-th]].

[2] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96** (2006), 181602, [arXiv:hep-th/0603001 [hep-th]].

[3] V. E. Hubeny, M. Rangamani and T. Takayanagi, JHEP **07** (2007), 062, [arXiv:0705.0016 [hep-th]].

[4] M. Alishahiha, Phys. Rev. D **92** (2015) no.12, 126009, [arXiv:1509.06614 [hep-th]].

[5] S. Banerjee, J. Erdmenger and D. Sarkar, JHEP **08** (2018), 001, [arXiv:1701.02319 [hep-th]].

[6] A. Trivella, Class. Quant. Grav. **34** (2017) no.10, 105003, [arXiv:1607.06519 [hep-th]].

[7] M. Guica, F. Levkovich-Maslyuk and K. Zarembo, J. Phys. A **50** (2017) no.39, 39, [arXiv:1706.07957 [hep-th]].

[8] U. Gursoy and C. Nunez, Nucl. Phys. B **725** (2005), 45-92, [arXiv:hep-th/0505100 [hep-th]].

[9] D. Z. Freedman and U. Gursoy, JHEP **11** (2005), 042, [arXiv:hep-th/0506128 [hep-th]].

[10] U. Gursoy, JHEP **05** (2006), 014, [arXiv:hep-th/0602215 [hep-th]].

[11] N. P. Bobev and R. C. Rashkov, Phys. Rev. D **74** (2006), 046011, [arXiv:hep-th/0607018 [hep-th]].

[12] N. P. Bobev and R. C. Rashkov, Phys. Rev. D **76** (2007), 046008, [arXiv:0706.0442 [hep-th]].

[13] N. P. Bobev, H. Dimov and R. C. Rashkov, Bulg. J. Phys. **35** (2008), 274-285 [arXiv:hep-th/0506063 [hep-th]].

[14] C. S. Chu, G. Georgiou and V. V. Khoze, JHEP **11** (2006), 093, [arXiv:hep-th/0606220 [hep-th]].

[15] Sachdev, S., Ye, J., Phys. Rev. Lett. 70 (1993), 3339, [arXiv:cond-mat/9212030].

[16] Y.-I. Shin, C. H. Schunck, A. Schirotzek, and W. Ketterle, "Phase diagram of a two-component Fermi gas with resonant interactions," Nature 451 no. 7179, (2008) 689–693, arXiv:cond-mat.soft/0709.3027.

[17] A. Adams, K. Balasubramanian and J. McGreevy, JHEP **11** (2008), 059, [arXiv:0807.1111 [hep-th]].

[18] D. T. Son, Phys. Rev. D **78** (2008), 046003, [arXiv:0804.3972 [hep-th]].

[19] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101** (2008), 061601, [arXiv:0804.4053 [hep-th]].

[20] A. O. Barut, "Conformal group – Schroedinger group – dynamical group – the maximal kinematical group of the massive Schroedinger particle," Helvetica Physica Acta 46 no. 4, (1973) 496–503.

[21] V. K. Dobrev, Int. J. Mod. Phys. A **29** (2014), 1430001, [arXiv:1312.0219 [hep-th]].

[22] A. Golubtsova, H. Dimov, I. Iliev, M. Radomirov, R. C. Rashkov and T. Vetsov, JHEP **08** (2020), 090, [arXiv:2004.13802 [hep-th]].

[23] J. Alvarez-Jimenez, A. Dector and J. D. Vergara, JHEP **03** (2017), 044, [arXiv:1702.00058 [hep-th]].

[24] A. Volovich and C. Wen, JHEP **05** (2009), 087, [arXiv:0903.2455 [hep-th]].

[25] R. G. Leigh and N. Nguyen hoang, JHEP **11** (2009), 010, [arXiv:0904.4270 [hep-th]].

[26] D. Bak and A. Trivella, JHEP **09** (2017), 086, [arXiv:1707.05366 [hep-th]].

[27] H. Dimov, M. Radomirov, R. C. Rashkov and T. Vetsov, [arXiv:2009.01123 [hep-th]].