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Remarks on fields with a holographic dual

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Abstract

Recent progress in holographic correspondence uncovered remarkable relations between key characteristics of the theories on both sides of duality and certain integrable models.

In this paper we consider simple application of the Fredholm method to obtain important characteristics of holographic duality. We consider two different lines along which one can develop techniques using Fredholm method and Fredholm operators. The first one is focused on obtaining dispersion relations (anomalous dimensions in the dual theory) for the class of pulsating string solutions. The second direction is the study of deformations by single/double-trace operators. In this case we give sketchy qualitative description of how the Fredholm operators can be use.

1 Introduction

For the last forty years there have been attempts to realize a correspondence between gauge theories with large number of colors and string theory. Having graviton in its closed string spectrum at the same time, it is challenging to somehow collect all the interactions under a common "roof" - string theory. Over the years, this idea has its highs and downs. It is amazing that the physical community always discovers new ideas and challenges and keeps that dream alive.

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The discovery of string/gauge theory duality (in particular, AdS/CFT correspondence or holographic correspondence [1, 2]) has been profound boost in our thinking not only about those theories but also about fundamental laws of Nature and in particular the quantum features of spacetime itself. For instance, the idea of "emergent spacetime" however has been putted forward and seriously developed last years bringing together seemingly unrelated issues. Another attractive point is the appearance of strong evidences that gauge theories, some of which already experimentally proven, are related to string theory. Along with these, over the last years the ideas of holographic correspondence received wide applications in various areas of contemporary theoretical physics - from condensed matter and relativistic fluids to cosmological models.

Currently String/gauge theory duality is one of the hottest topics. String(gravity)/gauge theory duality, along with its tremendous success, poses also a number of conceptual issues. Such as, for instance, the following ones: (a) if gravity(string) theory is dual to certain gauge theory, it should be possible to re- construct any of them from the others! How can this holographic correspondence be explicitly realized? (b) if the above statement is true, the (quantum) gravity should be encoded in the boundary theory! How then is this information stored on the boundary and how can it be extracted? Moreover, can we think of space-time, ergo gravity as an emergent phenomenon?

The promotion of the idea that collective dynamics of more fundamental non-gravitational degrees of freedom is entangled to produce emergent gravity poses even more fundamental questions about emergence phenomena- emergence of what, from what or what kind of emergence?

One should note that, along with all other things, one of the most attractive features of this duality is that it offers a way to resolve many puzzles of the weak/strong coupling phenomena.

Let us first briefly remind the basic points of the conjecture of AdS/CFT correspondence following mainly the Witten's arguments.

The main components of the correspondence are defined as follows.

- Consider a spacetime M supplied with a boundary $\partial M = \Sigma$.
- The fields in the bulk we denote by Φ_i (the index stands for arbitrary tensorial structure) while the fields on the boundary are ϕ_k.
- The metric of M is $g_{\mu\nu}$ while the induced metric on the boundary ∂M is $g_{|\partial M} = \gamma$.
- Consider a radial slice of the spacetime Σ_ρ at fixed ρ. The fields on this slice will be also denoted by ϕ_k.
- At given slice Σ_ρ one can consider the amplitude

$$\Psi_{\Sigma_\rho}[\phi_i] = \int_{\Phi_i|\Sigma_\rho=\phi_i} D\Phi_i e^{iS_{B,M}(\Phi_i)}, \tag{1.1}$$

  where $S_{B,M}$ is the bulk action and the integral is evaluated with Dirichlet boundary conditions for the fields on Σ_ρ.

- One can define and compute the quantum Hamilton-Jacobi functional

$$S(\phi_i) = \Gamma(\Phi_i)|_{\frac{\delta\Gamma}{\delta\Phi_i}=0,\, \Phi_{i|\partial M}=\phi_i}. \tag{1.2}$$

- The fields on the boundary ϕ_i are characterized by their tensor structure and the conformal transformations of the boundary metric, namely

$$\gamma \to \rho^{-2}\gamma \quad \Longrightarrow \quad \phi_i \to \rho^{d-\Delta_i}\phi_i. \tag{1.3}$$

  The quantity Δ_i is associated with the conformal dimension of a primary operator $\mathcal{O}_i$ on the CFT side of the correspondence (actually it is coupled to it).

- For any given CFT one can define the generating functional of connected correlation functions

$$Z_{CFT}[\phi_i] = \langle e^{\int_\Sigma \phi_i \mathcal{O}_i}\rangle. \tag{1.4}$$

Before stating the correspondence let us make a short remark. It has been shown explicitly that in the case of asymptotically flat spaces this Dirichlet amplitude is nothing but the S-matrix functional! The standard S-matrix elements can be extracted by taking derivatives of the quantum Hamilton-Jacobi functional. Thus, one can think of $\Psi_{\Sigma_\rho}[\phi_i] = e^{iS(\phi_i)}$ as *on-shell* amplitude defining connected S-matrix elements. Note also that in the flat space the quantum Hamilton-Jacobi functional is the generating functional of connected S-matrix elements.

The AdS/CFT correspondence states the equality of

$$Z_{CFT}[\phi_i] = \Psi_{\Sigma_0}[\phi_i], \tag{1.5}$$

where Σ_0 is the (asymptotic) boundary of the spacetime.

Let us comment briefly on the bulk reconstruction in this setup.

- In principle Ψ_{Σ_ρ} *represents the quantum spacetime* but *only trough the dependence on the boundary metric*!

- Changing the *radial slice* changes the induced metric on Σ_ρ! Thus, knowing Σ_ρ *for all* ρ (i.e. all possible γ) allows to reconstruct the semi-classical spacetime!

- On the other hand, assuming the correspondence, *the variation of the boundary means moving the radial slice in the bulk*!

If our boundary lies at infinity, the induced metric on the slice Σ_ρ scales as $\rho^{-2}\gamma$, where γ is a representative of the conformal class of the metric induced on the boundary. Thus, it is natural suggest that the asymptotic property of the spacetime encoded in Ψ is given by the behavior of the wave functional $\Psi[\rho^{-2}\gamma]$ in the limit when $\rho \to 0$.

Let us briefly comment on the effect of deforming the dual CFT with a multi-trace operator. In general, this deformation schematically can be described by the following modification of the action

$$S_0 \to S_W = S_0 + W[\hat{\mathcal{O}}(\phi_i)], \tag{1.6}$$

where ϕ_i are the fundamental degrees of freedom. When $\hat{\mathcal{O}}(\phi_i)$ is *single-trace* it has the from

$$\hat{\mathcal{O}}(\phi_i) = \mathrm{Tr}\left(\prod_i \phi_i\right). \tag{1.7}$$

According to the AdS/CFT correspondence one writes

$$\mathbb{Z}[J] = \langle e^{\int J\hat{\mathcal{O}}}\rangle_{CFT} = \int \mathcal{D}[\phi_i]e^{-S_0[\phi_i]}e^{J\mathcal{D}[\phi_i]} \simeq \int \mathcal{D}[\varphi]e^{-S_{grav}[\varphi,J]} \sim e^{-S_{grav}[\varphi;J]}. \tag{1.8}$$

The source J is classical one and the expectation value is taken with respect to the undeformed action. Note that $\hat{\mathcal{O}}$ is the operator acting on the Hilbert space states while $\mathcal{O}$ is just the function entering the path integral!

Following the AdS/CFT correspondence logic, one has the chain of relation

$$\begin{aligned}
\mathbb{Z}^W_{CFT}[J] &= \langle e^{J\hat{\mathcal{O}}}\rangle^W_{CFT} \equiv \langle e^{J\hat{\mathcal{O}}-W[\hat{\mathcal{O}}]}\rangle_{CFT} \\
&= \int \mathcal{D}[\phi_i] e^{-S_0[\phi_i]-W[\mathcal{O}]+J\mathcal{O}} = \int \mathcal{D}[\phi_i] e^{-S_0[\phi_i]} e^{-W[\mathcal{O}]} e^{J\mathcal{O}} \\
&= e^{-W[\frac{\delta}{\delta J}]} \langle e^{J\hat{\mathcal{O}}}\rangle_{CFT} = e^{-W[\frac{\delta}{\delta J}]} \int \mathcal{D}[\varphi] e^{-S_{grav}[\varphi;J]} \\
&= \int \mathcal{D}\varphi e^{-S_{grav}[\varphi;J;W]} \sim e^{-S_{grav}[\varphi_{cl};J;W]}. \qquad (1.9)
\end{aligned}$$

The functional dependence on W above is important and it depends on the specific fields entering the theory, see for some comments on this point [3] for instance.

Below we sketch briefly how these arguments work.

Let us briefly comment on the non-local effective action contributions. As discussed in [4], the non-local part, $\Gamma[\phi, g]$ satisfies

$$\frac{1}{\sqrt{g}} \left(g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} - \beta^I(\phi) \frac{\delta}{\delta \phi^I} \right) \Gamma[\phi, g] = 4\text{-derivatives terms}. \qquad (1.10)$$

The Callan-Symanzik equations for expectation values of local operators can be obtained by varying above relation with respect to fields ϕ^I. After the variation is done, one puts the fields to their constant average value given by the couplings of the gauge theory. Taking the metric to be $g_{\mu\nu} = a^2 \eta_{mu\nu}$ and defining

$$\int g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} = a \frac{\partial}{\partial a}, \qquad \int \frac{\delta}{\delta \phi^I} = \frac{\partial}{\partial \phi^I}, \qquad (1.11)$$

a little algebra is needed to obtain the standard Callan-Simanzyk equations

$$\left(a \frac{\partial}{\partial a} - \beta^I \frac{\partial}{\partial \phi^I} \right) \langle \mathcal{O}_{I_1}(x_1) \dots \mathcal{O}_{I_n}(x_n) \rangle - \gamma_{I_i}{}^{J_i} \langle \mathcal{O}_{I_1}(x_1) \dots \mathcal{J}_{\rangle}(x_i) \mathcal{O}_{I_n}(x_n) \rangle = 0. \qquad (1.12)$$

Here the anomalous scaling dimensions of the operators $\mathcal{O}_I$ are given by

$$\gamma_I{}^J = \nabla_I \beta^J. \qquad (1.13)$$

⊙ On any CFT$_d$, there is a mapping between operators $\mathcal{O}_a(x)$ and states on a Hilbert space of he theory on $R \times S^{d-1}$,

$$\mathcal{O}_a(x) \leftrightarrow |\mathcal{O}_{a S^{d-1}}\rangle.$$

⊙ The eigenvalue for the translation generator along the R direction (τ) is Δ_a = scaling dimension of $\mathcal{O}_a$ such that,

$$\mathcal{O}_a(\lambda x) = \lambda^{-\Delta_a} \mathcal{O}_a(x) \quad \Longrightarrow \quad e^{\tau \mathcal{H}_\tau} |\mathcal{O}_a\rangle = e^{-i\Delta_a} |\mathcal{O}_a\rangle,$$

where $\mathcal{H}_\tau$ is the Hamiltonian corresponding to the dilatation operator in radial quantization.

$$\Delta_a \, (\text{inCFT}) = L \, E_a \, (\text{globalAdS}). \qquad (1.14)$$

From string side the *anomalous dimensions* are determined by the *dispersion relations*!

To solve the problems in holography various approaches and techniques have been used. However, to my opinion the usage of Fredholm method and Fredholm determinants is somehow underestimated. The purpose of this paper is to bring attention on this method and demonstrate on simple examples how it could be used.

2 Aspects of holographic correspondence and Fredholm alternative

2.1 Basics of Fredholm equations

In this subsection we will list very sketchy, and without details and proofs, the basic properties of Fredholm equations and method for solving them. We will focus *only* on those which will be used in what follows. More details can be found in any good book on Integral equations, as well as in few references with more specific focus say [5]- [11].

Fredholm integral equations are two types

- Fredholm's integral equation of the I-st kind:

$$\int_a^b K(x,y)u(y)dy = f(x) \qquad \hat{K}u = f \qquad \text{non-homogeneous}$$

$$\int_a^b K(x,y)u(y)dy = 0 \qquad \hat{K}u = 0 \qquad \text{homogeneous eqn.}$$

- Fredholm's integral equation of the II-nd kind: (λ - parameter)

$$u(x) = \lambda \int_a^b K(x,y)u(y)dy + f(x); \quad \hat{K}u = f \quad \text{non-homogeneous}$$

Generically the Fredholm has a solution if the parameter λ is *not* a spectral eigenvalue

$$u(x) = f(x) + \int_a^b \frac{D(x,y;\lambda)}{D(\lambda)} f(y)dy, \tag{2.1}$$

where $D(x,y;\lambda)$ and $D(\lambda)$ are called Fredholm minors and Fredholm determinant and they are entire functions of the complex variable λ. *The eigenvalues satisfy $D(\lambda) = 0$. They are isolated and since $D(\lambda)$ is entire only a finite number of them can lie in a bounded region.*

The Fredholm minors and Fredholm determinant have representation in power series

$$D(x,y;\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A_n(x,y)\lambda^n, \qquad D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n \lambda^n$$

The coefficients A_n with $A_0 = K(x,y)$

$$A_n(x,y) = \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} K(x,y) & K(x,y_1) & \cdots & K(x,y_n) \\ K(y_1,y) & K(y_1,y_1) & \cdots & K(y_1,y_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(y_n,y) & K(y_n,y_1) & \cdots & K(y_n,y_n) \end{vmatrix} dy_1 dy_2 \dots dy_n.$$

The coefficients B_n ($B_0 = 1$)

$$B_n = \underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} K(y_1, y_1) & K(y_1, y_2) & \cdots & K(y_1, y_n) \\ K(y_2, y_1) & K(y_2, y_2) & \cdots & K(y_2, y_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(y_n, y_1) & K(y_n, y_2) & \cdots & K(y_n, y_n) \end{vmatrix} dy_1 dy_2 \ldots dy_n.$$

Beside using Fredholm minors and determinants, there are also two other methods of solving Fredholm equations deserving attention.

Iterated Kernels. We start with the equation

$$u(x) - \lambda \int_a^b K(x, y) u(y) dy = f(x), \qquad a \leq x \leq b. \tag{2.2}$$

As we discussed, for small enough λ and under certain square integrability conditions there exist a solution as Neumann series

$$u(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \psi_n(x), \tag{2.3}$$

Substitute (2.3) into (2.2) and compare the terms of equal powers in λ. The comparison yields

$$\psi_1(x) = \int_a^b K(x, y) f(y) dy$$

$$\psi_2(x) = \int_a^b K(x, y) \psi_1(y) dy = \int_a^b K_2(x, y) f(y) dy$$

$$\psi_3(x) = \int_a^b K(x, y) \psi_2(y) dy = \int_a^b K_3(x, y) f(y) dy \tag{2.4}$$

$$\ldots \tag{2.5}$$

The kernels $K_n(x, y)$ are called *iterated kernels* and are defined as

$$K_n(x, y) = \int_a^b K(x, z) K_{n-1}(z, y) dz, \quad n \geq 2, \quad K_1(x, y) = K(x, y). \tag{2.6}$$

Actually one can write

$$K_n(x, y) = \int_a^b K_l(x, z) K_{n-l}(z, y) dz, \quad 1 \leq l < n. \tag{2.7}$$

Written in terms of the original kernel, the n-th one is

$$K_n(x, y) = \int_a^b \int_a^b \cdots \int_a^b K(x, s_1) K(s_1, s_2) \ldots K(s_{n-1}, y) ds_1 ds_2 \ldots ds_{n-1}. \tag{2.8}$$

Successive Approximations. The method uses quite similar to the iterated kernels concept. In this case one should use the recurrent formula

$$u_n(x) = f(x) + \lambda \int\limits_a^b K(x,y) u_{n-1}(y) dy, \qquad u_0(x) = f(x), \tag{2.9}$$

with the very same output.

The Resolvent. The resolvent of the integral equation can be defined via the iterated kernels by the formula

$$R(x,y;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,y). \tag{2.10}$$

If

$$|\lambda| < \frac{1}{B}, \quad B = \sqrt{\int\limits_a^b \int\limits_a^b K^2(x,y) dx dy}$$

$$\int\limits_a^b K^2(x,y) dy \leq A, \quad a \leq A \leq b, \tag{2.11}$$

then the Neumann series converges absolutely and uniformly on $[a,b]$.

Thus, a solution of a Fredholm equation of the second kind is expressed by the formula

$$u(x) = f(x) + \lambda \int\limits_a^b R(x,y) f(y) dy. \tag{2.12}$$

Thus, the problem reduces to obtaining the Resolvent.

2.2 Pulsating strings, anomalous dimensions and Fredholm method

As discussed earlier, the duality connects perturbative and non-perturbative corners of given theories. Thus, to obtain anomalous dimensions of operators in field theory at strong coupling we can use perturbative regime in its holographic dual. In this respect the Fredholm method becomes valuable tool in holographic correspondence.

Let us discuss a simple example of application of Fredholm integral equation in the perturbation theory. We frequently have certain symmetries and, after appropriate reduction, the problem of finding wave function reduces to solving ordinary differential equation. To see how the method works let us consider the Schrödinger radial equation

$$\frac{d^2\psi(r)}{dr^2} + \left[q^2 - V_{unprt}(r) - \lambda V_{prt}(r)\right] \psi(r) = 0, \tag{2.13}$$

where q is a certain quantum number and λ is small perturbation parameter. The form of $V_{prt}(r)$ is fairly complicated in general and the exact solutions is hard to obtain. Another rewriting suggests different point of view

$$\frac{d^2\psi(r)}{dr^2} + \left[q^2 - V_{unprt}(r)\right] \psi(r) = \lambda V_{prt}(r)\, \psi(r), \tag{2.14}$$

namely, the solution of (2.14) can be considered as the solution of the following *Fredholm equation*

$$\psi(r) = \phi(r) + \lambda \int\limits_0^\infty K(r,r')\psi(r')dr'. \tag{2.15}$$

In (2.15) the symbol $\phi(r)$ stands for the analytic solution of the unperturbed problem, namely (2.14) with vanishing right hand side,

$$\frac{d^2\phi(r)}{dr^2} + \left[q^2 - V_{unprt}(r)\right]\phi(r) = 0. \tag{2.16}$$

Also, $K(r,r')$ is the integral kernel

$$K(r,r') = G(r,r')V_{prt}(r'). \tag{2.17}$$

The Green's function satisfies the equation

$$\left[\frac{d}{dr} + q^2 - V_{unprt}(r)\right] G(r,r') = \delta(r-r'). \tag{2.18}$$

Then the solution can be represented as a perturbation series

$$\psi(r) = \sum_{k=0}^\infty \lambda^k \psi^{(k)}(r), \tag{2.19}$$

where

$$\psi^{(0)}(r) = \phi(r), \quad \psi^{(1)}(r) = \int\limits_0^\infty K(r,r')\phi(r')dr', \quad \psi^{(k)}(r) = \int\limits_0^\infty K(r,r')\psi^{(k-1)}(r')dr'. \tag{2.20}$$

By making use of the interated kernel method described above, one writes

$$\psi^{(n)}(r) = \int\limits_0^\infty K_n(r,r')\phi(r')dr'. \tag{2.21}$$

- One can construct the resolvent as series expansion in λ

$$K(r,r',\lambda) = \sum_{k=0}^\infty \lambda^k K_{k+1}(r,r'). \tag{2.22}$$

According to (2.19) and (2.22) the solution (2.15) can be written as

$$\psi(r) = \phi(r) + \int\limits_0^\infty K(r,r',\lambda)\phi(r')dr'. \tag{2.23}$$

Let us summarize the discussion so far. The wave function is perturbatively expressed in (2.19). Assume that the system is governed by a Hamiltonian H with quantized eigenvalues labeled by principal quantum number n. In the same way we will enumerate the eigenfunctions Ψ_n and the energies E_n expressing them as series in λ

$$E_n = \sum_{k=0}^\infty \lambda^k E_n^{(k)}, \qquad \psi_n = \sum_{k=0}^\infty \lambda^k \psi_n^{(k)}. \tag{2.24}$$

Next, for any ψ_n of quantum number n we are free to further impose orthogonality conditions

$$\langle\psi_n^{(0)}|\psi_n^{(m)}\rangle = \delta_{m,0}, \tag{2.25}$$

which in particular means normalization of $\psi_n^{(0)}$

$$\langle\psi_n^{(0)}|\psi_n^{(0)}\rangle = 1. \tag{2.26}$$

In AdS/CFT correspondence the dispersion relation on bulk theory side determine anomalous dimensions of the gauge theory operators. Let us take a look at the energies in our example. Taking H_0 to be the unperturbed Hamiltonian, perturbation theory energies are

$$\begin{aligned}
&H_0\,\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}\\
&H_0\psi_n^{(1)} + V_{prt}\psi_n^{(0)} = E_n^{(0)}\psi_n^{(1)} + E_n^{(1)}\psi_n^{(0)}\\
&H_0\psi_n^{(2)} + V_{prt}\psi_n^{(1)} = E_n^{(0)}\psi_n^{(2)} + E_n^{(1)}\psi_n^{(1)} + E_n^{(2)}\psi_n^{(0)}\\
&\ldots
\end{aligned} \tag{2.27}$$

The orthogonality relation (2.25) leads to the tower

$$\begin{aligned}
E_n^{(0)} &= \langle\psi_n^{(0)}|H_0|\psi_n^{(0)}\rangle,\\
E_n^{(1)} &= \langle\psi_n^{(0)}|V_{prt}|\psi_n^{(0)}\rangle,\\
E_n^{(2)} &= \langle\psi_n^{(0)}|V_{prt}|\psi_n^{(1)}\rangle,\\
E_n^{(3)} &= \langle\psi_n^{(0)}|V_{prt}|\psi_n^{(2)}\rangle,\\
&\ldots\\
E_n^{(m)} &= \langle\psi_n^{(0)}|V_{prt}|\psi_n^{(m-1)}\rangle,
\end{aligned} \tag{2.28}$$

Since we already have the first order corrected wave function, we are able to analytically find the second order correction to the energy as

$$E^{(2)} = \int_0^\infty \phi^*\hat{V}_{pert}\psi^{(1)}dr. \tag{2.29}$$

Let us turn to applications in holographic theories calculating the corrections to the classical energy using the approach initiated in [12,13][1]. Consider a circular string, which pulsates on S^5 by expanding and contracting its length. In this case the metric of S^5 and relevant part of AdS_5 are given by

$$ds^2 = R^2\left(\cos^2\theta d\Omega_3^2 + d\theta^2 + \sin^2\theta d\psi^2 + d\rho^2 - \cosh^2\rho dt^2\right), \tag{2.30}$$

where $R^2 = 2\pi\alpha'\sqrt{\lambda}$ with λ the 't Hooft coupling. One can obtain the simplest pulsating string solution by identifying the target space time with the worldsheet one, $t=\tau$, and setting $\psi = m\sigma$, which corresponds to a string stretched along ψ direction. We also set the ansatz for $\theta=\theta(\tau)$ and $\rho=\rho(\tau)$. Hence, the Nambu-Goto action reduces to

$$S = m\sqrt{\lambda}\int dt\sin\theta\sqrt{\cosh^2\rho - \dot{\theta}^2}. \tag{2.31}$$

[1] In the examples below, the Fredholm operator actually reduces to Volterra's one.

In order to obtain the solution and the string spectrum it is useful to pass to Hamiltonian formulation. For this purpose, after identifying the canonical momenta,

$$\Pi_\rho = \frac{m\sqrt{\lambda}\,\sin\theta\,\dot\rho}{\sqrt{\cosh^2\rho - \dot\rho^2 - \dot\theta^2}}, \qquad \Pi_\theta = \frac{m\sqrt{\lambda}\,\sin\theta\,\dot\theta}{\sqrt{\cosh^2\rho - \dot\rho^2 - \dot\theta^2}}, \tag{2.32}$$

we can write the Hamiltonian in the form [12]

$$H = \cosh\rho\sqrt{\Pi_\rho^2 + \Pi_\theta^2 + m^2\lambda\sin^2\theta}. \tag{2.33}$$

If the string is placed at the origin of AdS_5 space ($\rho = 0$), we see that the squared Hamiltonian have a form similar to a point particle. Here, the last term in (2.33) can be considered as a perturbation. Therefore one can first find the wave function for a free particle in the above geometry

$$-\frac{\cosh\rho}{\sinh^3\rho}\frac{d}{d\rho}\left(\cosh\rho\sinh^3\rho\frac{d}{d\rho}\psi(\rho,\theta)\right) - \frac{\cosh^2\rho}{\sin\theta\cos^3\theta}\frac{d}{d\theta}\left(\sin\theta\cos^3\theta\frac{d}{d\theta}\psi(\rho,\theta)\right) = E^2\psi(\rho,\theta). \tag{2.34}$$

The solution to this equation is

$$\Psi_{2n}(\rho,\theta) = (\cosh\rho)^{-2n-4}P_{2n}(\cos\theta), \tag{2.35}$$

where $P_{2n}(\cos\theta)$ are spherical harmonics on S^5 and the energy spectrum is given by

$$E_{2n} = \Delta = 2n + 4. \tag{2.36}$$

Next step consists in finding the states Ψ_{2n}. Since we are looking for the first few corrections, we will give here only $\Psi_{2n}^{(1)}$. Since the perturbation term does not depend on ρ and due to the normalization condition, the corrections does not involve ρ-part of Ψ_{2n}. Thus, we will give below only θ-dependent part $\tilde\Psi_{2n}^{(1)}(\theta)$

$$\tilde\Psi_{2n}^{(1)}(\theta) = \int G(\theta,\theta')m\sin^2\theta'\,P_{2n}(\cos\theta')d\mu(\theta') = A_{2n}P_{2n}(x), \tag{2.37}$$

where

$$A_{2n} = \frac{(4n-1)(4n+3) - 4n^2(4n+3) - (4n+1)^2(4n-1)}{4n+1}. \tag{2.38}$$

The first order correction to the energy in perturbation theory now yields

$$\delta E^2 = \int_0^{\pi/2} d\theta\,\Psi_{2n}^{*(0)}(\theta)\,m^2\sin^2\theta\,\Psi_{2n}^{(0)}(\theta) = \frac{m^2}{2}. \tag{2.39}$$

and, taking into account the first order correction contribution, the anomalous dimension turns out to be

$$\Delta - 4 = 2n\left[1 + \frac{1}{2}\frac{m^2\lambda}{(2n)^2}\right]. \tag{2.40}$$

Another more involving example is the case of generalized pulsating strings [14]. Let us consider a circular pulsating string expanding and contracting on S_5, which has a center of mass that is moving on an S_3 subspace. We will assume that the string is with fixed spatial coordinates in AdS_5 (exept the time variable), so the relevant metric is:

$$ds^2 = R^2\,(-dt^2 + d\gamma^2 + \cos^2\gamma\, d\chi^2 + \sin^2\gamma\, d\,\Omega_3^2), \tag{2.41}$$

where $d\Omega_3^2 = g_{ij}\, d\varphi^i\, d\varphi^j$ is the metric on the S_3 subspace, i. e. $g_{ij} = diag(1, \cos^2\varphi^1, \sin^2\varphi^1)$ and $R^2 = 2\pi\alpha'\sqrt{\lambda}$. If we identify t with τ and use following classical ansätz:

$$\gamma = \gamma(\tau),\ \chi = \chi(\tau),\ \varphi^i = n^i\sigma + \alpha^i(\tau),\ i = 1, 2, 3, \tag{2.42}$$

the Nambu-Goto action

$$S\ =\ -\frac{1}{2\pi\alpha'}\int d\tau\, d\sigma\sqrt{-det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{2.43}$$

then reduces to

$$\begin{aligned} S\ =\ &-\sqrt{\lambda}\int d\tau\, d\sigma\ \sin\gamma \\ &\times\ \sqrt{g_{ij}n^i n^j\,(1-\dot\gamma^2-\dot\chi^2\cos^2\gamma) + \sin^2\gamma\left[(g_{ij}n^i\dot\alpha^j)^2 - (g_{ij}\dot\alpha^i\dot\alpha^j)\,(g_{ij}n^i n^j)\right]}. \end{aligned} \tag{2.44}$$

Now we are going to apply the procedure for calculation of the anomalous dimensions developed in [12]. For this purpose, we find first the canonical momenta of our system. Straightforward calculations give for the momenta

$$\Pi_\gamma = \frac{\sqrt{\lambda}\,\sin\gamma\,(g_{ij}n^i n^j)\,\dot\gamma}{\sqrt{g_{ij}n^i n^j\,(1-\dot\gamma^2-\dot\chi^2\cos^2\gamma) + \sin^2\gamma\left[(g_{ij}n^i\dot\alpha^j)^2 - (g_{ij}\dot\alpha^i\dot\alpha^j)\,(g_{ij}n^i n^j)\right]}}, \tag{2.45}$$

$$\Pi_\chi = \frac{\sqrt{\lambda}\,\sin\gamma\,(g_{ij}n^i n^j)\,\cos^2\gamma\,\dot\chi}{\sqrt{g_{ij}n^i n^j\,(1-\dot\gamma^2-\dot\chi^2\cos^2\gamma) + \sin^2\gamma\left[(g_{ij}n^i\dot\alpha^j)^2 - (g_{ij}\dot\alpha^i\dot\alpha^j)\,(g_{ij}n^i n^j)\right]}}, \tag{2.46}$$

$$\Pi_{\alpha^k} = \frac{\sqrt{\lambda}\,\sin^3\gamma\,[(g_{ij}n^i n^j)\,\dot\alpha^s - (g_{ij}n^i\dot\alpha^j)\,n^s]\,g_{sk}}{\sqrt{g_{ij}n^i n^j\,(1-\dot\gamma^2-\dot\chi^2\cos^2\gamma) + \sin^2\gamma\left[(g_{ij}n^i\dot\alpha^j)^2 - (g_{ij}\dot\alpha^i\dot\alpha^j)\,(g_{ij}n^i n^j)\right]}}. \tag{2.47}$$

Solving for the derivatives in terms of the canonical momenta and substituting back into the Hamiltonian, we find

$$H\ =\ \sqrt{\Pi_\gamma^2 + \frac{\Pi_\chi^2}{\cos^2\gamma} + \frac{g_{ij}\Pi^i\Pi^j}{\sin^2\gamma} + \lambda\,(g_{ij}n^i n^j)\,\sin^2\gamma}, \tag{2.48}$$

or

$$H^2\ =\ \Pi_\gamma^2 + \frac{\Pi_\chi^2}{\cos^2\gamma} + \frac{g_{ij}\Pi^i\Pi^j}{\sin^2\gamma} + \lambda\,(g_{ij}n^i n^j)\,\sin^2\gamma. \tag{2.49}$$

Since we consider high energies, one can think of this Hamiltonian as of square root of a point particle one[2]. Hence, we can consider the potential terms as a perturbation. The potential itself has the form

$$V(\varphi^1, \gamma)\ =\ \lambda\left[\left((n^1)^2 + (n^2)^2\right) + \left((n^3)^2 - (n^2)^2\right)\,\sin^2\varphi^1\right]\,\sin^2\gamma. \tag{2.50}$$

[2] When we apply quasiclassical quantization we are dealing actually with a family of solutions. Each of them looks like ¨almost¨ as for point particle (for more comments see for instance [12, 13]

The above perturbation to the free action will produce the corrections to the energy and therefore, to the anomalous dimension.

Thus, we proceed with the consideration of the free wave-functions on S_5 and S_3, and then use the perturbation theory to first order to find the correction of order λ. The total S_5 angular momentum quantum number will be denoted by L and the total angular momentum quantum number on S_3 is J. In these notations we have for S_3 and S_5 :

$$\begin{aligned}\triangle(S_3) &= \frac{1}{\sin\varphi^1\cos\varphi^1}\frac{\partial}{\partial\varphi^1}\left[\sin\varphi^1\cos\varphi^1\frac{\partial}{\partial\varphi^1}\right]+ \\ &\qquad \frac{1}{\cos^2\varphi^1}\frac{\partial^2}{\partial(\varphi^2)^2}+\frac{1}{\sin^2\varphi^1}\frac{\partial^2}{\partial(\varphi^3)^2}, \qquad (2.51)\end{aligned}$$

$$\triangle(S_5) = \frac{1}{\sin^3\gamma\cos\gamma}\frac{\partial}{\partial\gamma}\left[\sin^3\gamma\cos\gamma\frac{\partial}{\partial\gamma}\right]+\frac{1}{\cos^2\gamma}\frac{\partial^2}{\partial\chi^2}+\frac{1}{\sin^2\gamma}\triangle(S_3). \qquad (2.52)$$

Next step is to find the wave function for our system. For this purpose we consider the Schrodinger equations which take for three-sphere the form:

$$\begin{aligned}\Delta_{S^3} &= \left[\frac{1}{\cos\varphi^1\,\sin\varphi^1}\frac{d}{d\varphi^1}\left(\cos\varphi^1\,\sin\varphi^1\frac{d}{d\varphi^1}\right)-\frac{m^2}{\cos^2\varphi^1}-\frac{l^2}{\sin^2\varphi^1}+J(J+2)\right] \\ \Delta_{S^5}(\varphi^1)U(\varphi^1) &= 0, \qquad (2.53)\end{aligned}$$

and for five-sphere correspondingly

$$\begin{aligned}\Delta_{S^5} &= \left[\frac{1}{\cos\gamma\,\sin^3\gamma}\frac{d}{d\gamma}\left(\cos\gamma\,\sin^3\gamma\frac{d}{d\gamma}\right)-\frac{M^2}{\cos^2\gamma}-\frac{J(J+2)}{\sin^2\gamma}+L(L+4)\right] \\ \Delta_{S^3}(\gamma)U(\gamma) &= 0. \qquad (2.54)\end{aligned}$$

After separation of variables, the three-sphere (ortho-normalized) wave function is given by

$$\Psi^J_{m,l}(\omega) = \sqrt{\frac{2(2k+l+m+1)\,k!\,\Gamma(k+l+m+1)}{\Gamma(k+l+1)\,\Gamma(k+m+1)}}\;\omega^{\frac{l}{2}}\,(1-\omega)^{\frac{m}{2}}\,P_k^{(l,m)}(1-2\omega), \qquad (2.55)$$

Here J, k, l are quantum numbers associated with the three-sphere.

On the other hand, the normalized wave-function of the five-sphere takes the form

$$\Psi^L_{M,J}(z) = \sqrt{\frac{2(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}}\;z^{\frac{\alpha-1}{2}}\,(1-z)^{\frac{\beta}{2}}\,P_n^{(\alpha,\beta)}(1-2z). \qquad (2.56)$$

The potential depende only on γ and can also be rewritten in terms of the new variables. We find for it

$$V(\varphi^1,\gamma) = \lambda\left[\left((n^1)^2+(n^2)^2\right)+\left((n^3)^2-(n^2)^2\right)\omega\right]z = V(\omega,z). \qquad (2.57)$$

Since the variables are separated, one can apply Fredholm method for the two variables at once. After straightforward calculations the final result for the first order correction to E^2 is found to be

$$E^2_{(1)} = \lambda\left[\left((n^1)^2+(n^2)^2\right)+\frac{\left((n^3)^2-(n^2)^2\right)}{(2k+l+m+1)}\left(\frac{(k+l)\,(k+l+m)}{(2k+l+m)}\right.\right.$$

$$+\frac{(k+1)(k+m+1)}{(2k+l+m+2)}\Bigg)\Bigg]\frac{1}{(2n+\alpha+\beta+1)}\times$$
$$\times\left[\frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta)}+\frac{(n+1)(n+\beta+1)}{(2n+\alpha+\beta+2)}\right]. \quad (2.58)$$

Here we used the same notation as in [14] just to demonstrate the perfect agreement of presented technique with the known results.

In conclusion, we find very useful the usage of Fredholm method not only for finding corrections to the energy, but also for building relevant states which can be useful for computing correlation functions.

2.3 Deformations and Fredholm determinant

In this subsection we will only give an idea of how Fredholm determinants appear when we deform in a certain way the original theory.

Lets us focus on the deformations by single- and double-trace operators and how it reflects on the other side of holographic duality. While the single-trace deformations are old and fairly well (in many cases) understood problem, double-trace deformations still surprises the physics theory community.

Let's come to the problem. Deformations in one side of duality translates to the other one, as discussed in the beginning of this manuscript. Below we will stick to the notation in the Introduction. To account for such deformations we consider the partition function of a theory with a gravity dual

$$\mathbb{Z}_\rho[J]=\int\mathcal{D}e^{-S[\phi]-\int\frac{\rho}{2}\mathcal{O}^2+\int J\mathcal{O}}=\langle e^{-\int\frac{\rho}{2}\mathcal{O}^2+\int J\mathcal{O}}\rangle_0,$$

where $\mathcal{O}$ is a single trace operator, ρ is a constant (matrix), but J is generically not.

Next step is to use Hubbard-Stratonovich transformation

$$\mathbb{Z}_\rho[J]=\sqrt{\det(-(\rho)^{-1}\mathbf{1})}\int\mathcal{D}\sigma\langle e^{\int(\frac{1}{2}(\rho)^{-1}\sigma^2+\sigma\mathcal{O}+J\mathcal{O}^2)}\rangle,$$

with the assumption that higher point functions of $\mathcal{O}$ are suppressed in $1/N$ and the expectation value can be considered as

$$\langle e^{(\sigma+J)\mathcal{O}}\rangle_0\approx e^{\frac{1}{2}\langle(\int(\sigma+J)\mathcal{O})^2\rangle_0}.$$

To bring $\mathbb{Z}_\rho[J]$ into pure square involving J, we introduce operators of Fredholm type

$$\hat{B}[\sigma](x)=\int\langle\mathcal{O}(x)\mathcal{O}(z)\rangle_0\sigma(z),\quad \hat{K}=1+\rho\,\hat{B},\quad \hat{U}=\frac{\hat{B}}{1+\rho\,\hat{B}}$$

Simple and straightforward calculations shows that the deformed partition function becomes

$$\mathbb{Z}_\rho[J]=\exp\frac{1}{2}\big[\langle J\,|\hat{U}J\rangle-\operatorname{tr}\log\hat{K}\big].$$

The two-point function in deformed theory can be also easily extracted

$$\int\mathcal{D}e^{-S[\phi]-\int\frac{\rho}{2}\mathcal{O}^2+\int J\mathcal{O}}=\mathbb{Z}_\rho[J]=\exp\frac{1}{2}\big[\langle J\,|\hat{U}J\rangle-\operatorname{tr}\log\hat{K}\big]$$
$$\Longrightarrow\qquad\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_\rho=\frac{\partial^2\log\mathbb{Z}_\rho[J]}{\partial J(x)\partial J(0)}=U(x,0).$$

Let us assume now that the double-trace operators have a support over a finite interval. Such theories are for example, dipole theories dual to Schrödinger spaces. The deformed action takes the form

$$\tilde{S} = S_0 + \int \Big(J(x)\mathcal{O}(x) + \int_{x'} \rho(x-x')\mathcal{O}(x)\mathcal{O}(x') \Big).$$

where $\rho(x-x')$ is the source for the bi-local double trace operator. Such pictures exactly matches above consideration and we can define immediately the corresponding Fredholm-like operator

$$\tilde{B}[\mathcal{O}](x) = \int_{x'} \rho(x-x')\mathcal{O}(x')$$

It is straightforward to compute the effective action which has the form

$$\Gamma[\mathcal{O}] = (J,\mathcal{O}) - W_\rho[J], \qquad \mathcal{O}(x) = \frac{\delta W_\rho[J]}{\delta J(x)}.$$

Then

$$\rho(x-x') = \frac{\delta^2 \Gamma[\mathcal{O}]}{\delta\mathcal{O}(x)\delta\mathcal{O}(x')}.$$

the analysis proceeds according to the above prescription.

Recently deformations by irrelevant double-trace operators in two dimensions attracted a lot of attention [15]. It has been proven that deformation with operators of the form $T\bar{T}$, where T is the stress tensor, are integrable [15, 16]. Some of the approaches [17, 18] are pretty similar in spirit of what has been discussed above. Indeed, if we treat single- and double-trace deformations at once, it is useful to introduce ρ, 2×2 matrix that sources $T\bar{T}$, $T\bar{J}$, $J\bar{T}$, $J\bar{J}$ deformations. Now we are interesting in the generating functional in the deformed theory

$$e^{-W_\rho[\tilde{J}]} = \int D\varphi e^{-S[\varphi] + \int \tilde{J}^A \mathcal{O}_A - \frac{1}{2}\rho^{AB}\mathcal{O}_A\mathcal{O}_B}, \tag{2.59}$$

where $\tilde{J}^A$ denote the sources in the deformed theory that couple to $\mathcal{O}_A$ and φ denotes the fundamental degrees of freedom in the CFT, over which the path integral is performed, weighted by the action $S[\varphi]$. To this end, let us rewrite (2.59) as

$$\begin{aligned} e^{-W_\rho[\tilde{J}]} = \int D\sigma^A \int D\varphi e^{-S[\varphi] + \int \sigma^A\mathcal{O}_A + \int \tilde{J}^A\mathcal{O}_A - \int \sigma^A\mathcal{O}_A - \frac{1}{2}\rho^{AB}\mathcal{O}_A\mathcal{O}_B} \\ = \int D\sigma^A e^{-W[\sigma^A]} e^{-\int(\sigma^A - \tilde{J}^A)\mathcal{O}_A - \frac{1}{2}\rho^{AB}\mathcal{O}_A\mathcal{O}_B} \\ = \int D\sigma^A e^{-W[\sigma^A] + \frac{1}{2}\int(\sigma^A - \tilde{J}^A)(\rho^{-1})_{AB}(\sigma^B - \tilde{J}^B)}. \end{aligned} \tag{2.60}$$

In the last term of (2.60) we used the trick of inserting identity

$$\sqrt{\det \rho^{-1}} \int D\tilde{\sigma}^A e^{\frac{1}{2}\tilde{\sigma}^A(\rho^{-1})_{AB}\tilde{\sigma}^B} = 1, \tag{2.61}$$

and performed the shift

$$\tilde{\sigma}^A = \sigma^A - \tilde{J}^A + \rho^{AB}\mathcal{O}_B. \tag{2.62}$$

The undeformed theory has as sources $\sigma^A = J^A$ while for the deformed theory the sources are $\tilde{\sigma}^A$. Evaluated at the saddle point one finds

$$-\frac{\delta W[\sigma^A]}{\delta \sigma^A}\Big|_{\sigma^a_*} = \langle \mathcal{O}_A \rangle = -(\rho^{-1})_{AB}(\sigma^B_* - \tilde{J}^B), \tag{2.63}$$

where

$$\sigma^A_* = J^A = \tilde{J}^A + \rho^{AB}\langle \mathcal{O}_A \rangle \qquad \text{or} \quad \tilde{J}^A = J^A - \rho^{AB}\langle \mathcal{O}_B \rangle. \tag{2.64}$$

The two generating functions are related as follows

$$W_\rho[\tilde{J}^A] = W[J^A] - \frac{1}{2}\int \rho^{AB}\langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle. \tag{2.65}$$

As it is clear from the above sketchy description, the Fredholm determinant techniques can be successfully applied to find many of features of given theory such like correlation functions, entanglement entropies, complexities etc. We will present qualitative and quantitative discussion of these issue in the near future [20].

3 Concluding remarks

In this paper we consider simple application of the Fredholm method to obtain important characteristics of holographic duality. After some basic information about Fredholm equations, we focused on dispersion relations of pulsating strings in $AdS_5 \times S^5$ background. The later define the anomalous dimensions of the gauge theory operators, which are at strong coupling regime and there is no other relevant tools fo obtaining them. This problem is actually studied and we have chosen it to compare results and demonstrate the effectiveness of the Fredholm method. It is interesting to note that along with obtaining physically relevant results, the method allows to construct a system of states $\psi_n^{(k)}$ associated with ψ_n, which could be considered as refinement gining kind of "fine structure". The later may be used to compute entanglement and may provide additional important information.

Another issue has been the deformations of holographic theories by higher-trace operators. We have shown that the partition function for double-trace operators can be obtained by applying certain Fredholm-like operator. This can be used to obtain correlation functions in the deformed theory as well. One can think of deforming theories in two dimensions with higher projective invariants, starting with Schwarzian [19]. This is expected to bring a bridge between higher projective invariants, tau-functions and Fredholm determinants, see [10]. We will return to some issues raised in this paper in the near future [20].

To conclude, it seems that there are many unexplored directions of applying Fredholm method which could be used to solve highly non-trivial problems.

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