Transverse Kähler structures, Sasaki-Ricci flow, and holomorphic Hamiltonian vector fields on Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q*}$

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Abstract

We investigate compact Sasaki manifolds in view of transverse Kähler geometry. We study the deformations of the Sasaki-Einstein structure under the transverse Kähler-Ricci flow. In the frame of contact geometry, we describe the construction of Hamiltonian holomorphic vector fields and Hamiltonian functions. The general results are applied to the five-dimensional Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$.

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1 Introduction

In the last time Sasakian geometry, as an odd-dimensional analogue of Kähler geometry, has become of high interest in connection with some modern developments in mathematics (see e.g. [1] and the references therein). A Sasakian structure sits between two Kähler structures, namely, the one of its metric cone and the one on the 1-dimensional foliation generated by the Reeb vector field.

In theoretical physics, the interest in Sasaki-Einstein geometries [2] has arisen in the context of AdS/CFT correspondence in maximally supersymmetric theories. We also note the relevance of contact geometries in irreversible thermodynamics, statistical physics, systems with dissipation, etc., see e.g. [3].

The concept of Ricci flow was introduced by Hamilton [4] representing a method to continuously deform a Riemannian manifold. Recently, the method was applied to Sasaki manifolds to generate new Sasaki structures [5].

The completely integrable Hamiltonian systems in the symplectic setting are best described by the famous Arnold-Liouville theorem and its generalization. The construction of an analogous theory of complete integrability in contact geometry has been done more recently [6, 7].

*Paper dedicated to the memory of Professor Oliviu Gherman.
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In this paper, we investigate the Sasaki-Ricci flow equation on five-dimensional Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$. For this purpose we introduce a set of local complex coordinates to parametrize the transverse holomorphic structure and the Sasakian analogue of the Kähler potential from the Kähler geometry. Finally, we introduce the Hamiltonian holomorphic vector fields and describe their deformations under the Sasaki-Ricci flow.

The paper is organized as follows. In the next section, we recall some definitions and well-known results concerning the Sasaki-Einstein spaces and Sasaki-Ricci flow. In Sections 3 and 4, the general results are applied to the five-dimensional Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$. In spite of the complexity of the Sasaki-Ricci flow equation, we are able to find some particular explicit analytical solutions. Section 5 is devoted to the Hamiltonian holomorphic vector fields and Hamiltonian functions on spaces $T^{1,1}$ and $Y^{p,q}$. In Section 6, we provide some closing remarks.

2 Preliminaries

Here we recall the definitions and main facts about Sasakian structures and their deformations under the Sasaki-Ricci flow.

2.1 Sasaki manifolds

Definition 1 *A $(n+1)$-dimensional Riemannian manifold (M, g) is Sasakian if its metric cone $C(M) = M \times \mathbb{R}^+$ with metric $\bar{g} = dr^2 + r^2 g$ is Kähler, with r the coordinate on $\mathbb{R}^+ = (0, +\infty)$.*

Moreover, if the Sasaki space is Einstein ($Ric_g = 2ng$), then the Kähler metric cone is Ricci flat ($Ric_{\bar{g}} = 0$), i.e. a Calabi-Yau manifold.

Let J denote the complex structure on $C(M)$. On $C(M)$ we have a vector field $\bar{\xi}$ and a 1-form $\bar{\eta}$ defined by

$$\bar{\xi} = Jr\frac{\partial}{\partial r} \text{ and } \bar{\eta}(\cdot) = \frac{1}{r^2}\bar{g}(\bar{\xi}, \cdot)\,, \tag{1}$$

respectively. The vector field $\bar{\xi}$ restricted to M is called the *characteristic vector field* or the *Reeb vector field* (let us note it by ξ).

Let now $\mathcal{D} = \ker \eta$, where η is the restriction of $\bar{\eta}$ to M. We have the g-splitting of the tangent bundle TM of M:

$$TM = \mathcal{D} \oplus L_\xi\,, \tag{2}$$

where L_ξ is the trivial line bundle generated by ξ.

Restrict J to $\mathcal{D}$ and extend it to an endomorphism $\Phi \in \mathrm{End}(TM)$ by setting $\Phi\xi = 0$. Φ satisfies

$$\Phi^2 = -\mathbb{1} + \eta \otimes \xi\,, \tag{3}$$

and

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y)\,, \tag{4}$$

for any smooth vector fields X, Y on M.

We have a global 2-form Ω^T on M coming from the contact 1-form η

$$\Omega^T = \frac{1}{2}d\eta\,. \tag{5}$$

We get that $(\mathcal{D}, \Phi_{|\mathcal{D}}, d\eta)$ gives M a transverse Kähler structure with Kähler form Ω^T and transverse metric g^T given by

$$g^T(X,Y) = d\eta(X, \Phi Y)\,, \tag{6}$$

for any smooth vector fields X, Y on M and related to the Sasakian metric g on M by

$$g = g^T + \eta \otimes \eta\,. \tag{7}$$

From the transverse metric g^T one can define a connection ∇^T on $\mathcal{D}$ which is torsion free such that $\nabla^T g^T = 0$. Moreover, tha Sasaki-Einstein manifold is transverse Kähler-Einstein [1, 2].

Every $(2n+1)$-dimensional Sasaki manifold is locally generated by a real-valued function h of $2n$ variables, called the Sasaki potential, which is the analogue of the Kähler potential. One can introduce local coordinates $(x, z^1, \dots, z^n)$ on a small neighborhood of $U = I \times V$ of M with $I \in \mathbb{R}$ and $V \in \mathbb{C}^n$. In the chart U we may write [8]

$$\begin{aligned}
\xi &= \frac{\partial}{\partial x}\,,\\
\eta &= dx + i\sum_{j=1}^{n}(h_{,j}\mathrm{d}z^j) - i\sum_{\bar{j}=1}^{n}(h_{,\bar{j}}d\bar{z}^j)\,,\\
d\eta &= -2i\sum_{j,\bar{k}=1}^{n} h_{,j\bar{k}}dz^j \wedge d\bar{z}^k\,,\\
g &= \eta\otimes\eta + g^T = \eta\otimes\eta + 2\sum_{j,\bar{k}=1}^{n} h_{,j\bar{k}}dz^j\mathrm{d}\bar{z}^k\,,\\
\Phi &= -i\sum_{j=1}^{n}[(\partial_j - ih_{,j}\partial_x)\otimes dz^j] + i\sum_{\bar{j}=1}^{n}(\partial_{\bar{j}} + ih_{,\bar{j}}\partial_x)\otimes d\bar{z}^j]\,,
\end{aligned} \tag{8}$$

where $h_{,j} = \frac{\partial}{\partial z^j}h$ and $h_{,j\bar{k}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^k}h$. The Sasaki potential $h : U \to \mathbb{R}$ does not depend on x, i.e. $\partial_x h = 0$.

2.2 Deformations of Sasaki structures and Sasaki-Ricci flow

In what follows we consider deformations of the Sasaki structures which preserve the Reeb vector field ξ. For this purpose it is necessary to introduce the basic forms.

Definition 2 *A r-form α on M is called* basic *if*

$$\iota_\xi \alpha = 0\,, \quad \mathcal{L}_\xi\,\alpha = 0\,, \tag{9}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to the vector field ξ.

A basic r-form of type (p,q), $r = p+q$, has the form

$$\alpha = \alpha_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q} dz^{i_1}\wedge\cdots\wedge dz^{i_p}\wedge d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_q}\,, \tag{10}$$

where $\alpha_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}$ does not depend on x. In particular a function φ is basic if and only if $\xi(\varphi) = 0$. That is the case of the Sasaki potential h.

Let φ be a basic function and consider the deformation of the contact form η:

$$\tilde{\eta} = \eta + d_B^c \varphi\,, \tag{11}$$

where $d_B^c = \frac{i}{2}(\bar{\partial}_B - \partial_B)$ with

$$\partial_B = \sum_{j=1}^{n} dz^j \frac{\partial}{\partial z^j}\,, \quad \bar{\partial}_B = \sum_{j=1}^{n} d\bar{z}^j \frac{\partial}{\partial \bar{z}^j}\,. \tag{12}$$

This deformation implies that other fundamental tensors are also modified:

$$\begin{aligned} \tilde{\Phi} &= \Phi - (\xi \otimes (d_B^c \varphi)) \circ \Phi\,, \\ \tilde{g} &= d\tilde{\eta} \circ (\mathbb{1} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}\,, \\ d\tilde{\eta} &= d\eta + d_B d_B^c \varphi\,. \end{aligned} \tag{13}$$

To introduce the transverse Kähler-Ricci flow, also called Sasaki-Ricci flow, we consider the flow $(\xi, \eta(t), \Phi(t), g(t))$ with initial data $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$ generated by a basic function $\varphi(t)$. The Sasaki-Ricci flow equation is [5, 9]

$$\frac{\partial g^T}{\partial t} = -Ric^T_{g(t)} + (2n+2) g^T(t)\,, \tag{14}$$

where Ric^T is the transverse Ricci curvature. In the case of the deformation (11) with a basic function φ, in local coordinates, the Sasaki-Ricci flow can be expressed as a parabolic Monge-Ampère equation

$$\frac{\partial \varphi}{\partial t} = \ln \det(g^T_{j\bar{k}} + \varphi_{j\bar{k}}) - \ln(\det g^T_{j\bar{k}}) + (2n+2)\varphi\,. \tag{15}$$

3 Sasaki-Ricci flow on $T^{1,1}$ and $Y^{p,q}$ spaces

In what follows we study the Sasaki-Ricci flow on five-dimensional Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$. We present some explicit, analytical solutions of the Sasaki-Ricci flow equation on these spaces.

3.1 Sasaki-Einstein space $T^{1,1}$

The Sasaki-Einstein space $T^{1,1}$ is one the most renowned example of homogeneous Sasaki-Einstein space in five dimensions.

The standard metric on this manifold is [10, 11]

$$ds^2 = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,, \tag{16}$$

where $\theta_i \in [0, \pi)$, $\phi_i \in [0, 2\pi)$, $i = 1, 2$ and $\psi \in [0, 4\pi)$. The contact 1-form η is

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1\, d\phi_1 + \cos\theta_2\, d\phi_2)\,, \tag{17}$$

and the Reeb vector field has the form

$$\xi = 3\frac{\partial}{\partial \psi}\,. \tag{18}$$

Writing metric (16) with contact form (17), we get for the transverse metric

$$g^T = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2)\,. \tag{19}$$

As on $T^{1,1}$ the transverse structure is locally isomorphic to a product $S^2\times S^2$, for each S^2 sphere the complex coordinates z^j are related to the spherical coordinates as

$$z^j = \tan\frac{\theta_j}{2}e^{i\phi_j}\,, \quad j=1,2\,. \tag{20}$$

The Sasaki potential of the transverse metric g^T is

$$h = \frac{1}{3}\sum_{j=1}^{2}\log(1+z^j\bar{z}^j) - \frac{1}{6}\sum_{j=1}^{2}\log(z^j\bar{z}^j)\,. \tag{21}$$

In the case of the of the Sasaki-Einstein space $T^{1,1}$, a distinguished class of solutions of the Sasaki-Ricci flow equation is represented by the following families of basic functions [12, 13]

$$\varphi(t) = (e^{6t}-1)\sum_{j=1,2}\left[c_j(\ln z^j + \ln\bar{z}^j) + d_j(\ln^2 z^j + \ln^2\bar{z}^j)\right]\,, \tag{22}$$

with c_j, d_j arbitrary constants and the complex coordinates z^j are given in (20).

In terms of angular coordinates we have

Proposition 1 *The families of contact forms*

$$\tilde{\eta} = \eta + \frac{e^{6t}-1}{2}\sum_j\left[-c_j d\phi_j + d_j\frac{\phi_j}{\sin\theta_j}d\theta_j + d_j\log\tan\frac{\theta_j}{2}d\phi_j\right]\,, \tag{23}$$

with arbitrary real constants c_j, d_j , $j=1,2$, represents deformations of the canonical contact stacture of $T^{1,1}$.

Regarding the deformed metrics and other tensors, they can be evaluated using (13).

3.2 Sasaki-Einstein space $Y^{p,q}$

The metric of the Sasaki-Einstein space $Y^{p,q}$ is given by the line element [11]

$$\begin{aligned} ds^2 = {}& \frac{1-y}{6}(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{w(y)q(y)}{36}(d\beta - \cos\theta\, d\phi)^2 \\ &+ \frac{1}{9}\left[d\psi + \cos\theta\, d\phi + y(d\beta - \cos\theta\, d\phi)\right]^2\,, \end{aligned} \tag{24}$$

where

$$\begin{aligned} w(y) &= \frac{2(a-y^2)}{1-y}\,, \\ q(y) &= \frac{a-3y^2+2y^3}{a-y^2}\,, \\ f(y) &= \frac{a-2y+y^2}{6(a-y^2)}\,. \end{aligned} \tag{25}$$

In the case of the space $Y^{p,q}$ the contact 1-form η is

$$\eta = \frac{1}{3}d\psi + \frac{1}{3}y\, d\beta + \frac{1-y}{3}\cos\theta\, d\phi\,, \tag{26}$$

and the Reeb vector field is

$$K_\eta = 3\frac{\partial}{\partial\psi}\,. \tag{27}$$

A detailed analysis of the metric $Y^{p,q}$ showed that it is globally well-defined and there are a countable infinite number of Sasaki-Einstein manifolds characterized by two relatively prime positive integers p, q with $p < q$. If $0 < a < 1$ the cubic equation

$$Q(y) = a - 3y^2 + 2y^3 = \frac{1-y}{2}w(y)\,q(y) = 0\,, \tag{28}$$

has three real roots, one negative (y_1) and two positive, the smallest being y_2. The coordinate y ranges between the two smaller roots of the cubic equation (28), i.e. $y_1 \leq y \leq y_2$.

The angular coordinates span the ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 2\pi$. In order to specify the range of the variable β, we note that it is connected with another variable α

$$\beta = -(6\alpha + \psi)\,. \tag{29}$$

The range of α is

$$0 \leq \alpha \leq 2\pi\ell\,, \tag{30}$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}\,. \tag{31}$$

The Reeb Killing vector field (27) has compact orbits when ℓ is a rational number and the corresponding $Y^{p,q}$ manifold is called quasi-regular. If ℓ is irrational the orbits of the Reeb vector field do not close densely filling the orbits of a torus and the Sasaki-Einstein manifold is said to be irregular.

For what follows it is useful to evaluate the following integrals

$$f_1(y) = \exp\left(\int \frac{1}{H^2(y)}dy\right) = \sqrt{(y-y_1)^{-\frac{1}{y_1}}(y_2-y)^{-\frac{1}{y_2}}(y_3-y)^{-\frac{1}{y_3}}}\,, \tag{32}$$

$$f_2(y) = \exp\left(\int \frac{y}{H^2(y)}dy\right) = \frac{1}{\sqrt{Q(y)}}\,, \tag{33}$$

where

$$H^2(y) = \frac{1}{6}w(y)\,q(y) = \frac{1}{3}\frac{Q(y)}{1-y}\,. \tag{34}$$

We introduce a local set of transverse complex coordinates addressing the transverse Kähler structure of $Y^{p,q}$:

$$\begin{aligned} z^1 &= \tan\frac{\theta}{2}e^{i\phi}\,, \\ z^2 &= \frac{\sin\theta}{f_1(y)}e^{i\beta}\,. \end{aligned} \tag{35}$$

In terms of the complex coordinates (35) the Sasaki-Kähler potential is [13, 14]

$$h = \frac{1}{3}\left[\left(1 + \frac{1}{z^1\bar{z}^1}\right)f_2(y)\right] + \frac{1}{6}\ln(z^1\bar{z}^1)\,. \tag{36}$$

Note that the additional term restores the correct form of the contact form η of the space $Y^{p,q}$ without altering the transverse part of the metric.

It is interesting to note that also in the case of the $Y^{q,q}$ space, the corresponding class of solutions of the Sasaki-Ricci flow equation can be put in the same form as in (22), but, of course, with the complex coordinates z^j given by (35).

In terms of angular coordinates we have the following families of deformations of the contact structure of $Y^{p,q}$:

Proposition 2 *The families of basic functions*

$$\begin{aligned}\varphi(t) = (e^{6t}-1)\Bigg[\frac{c_1}{2}\phi^2 + d_1 \ln\tan\frac{\theta}{2} - \frac{c_1}{2}\left(\ln\tan\frac{\theta}{2}\right)^2 \\ +\frac{c_2}{2}\beta^2 + d_2\ln\rho - \frac{c_2}{2}(\ln\rho)^2\Bigg]\,,\end{aligned} \tag{37}$$

with c_j, d_j arbitrary constants, stand as solutions of the transverse Kähler-Ricci flow equation on the manifold $Y^{p,q}$.

The corresponding deformed contact structures remain Sasaki-Einstein with the contact forms

$$\begin{aligned}\tilde{\eta} = \eta + \frac{e^{6t}-1}{2}\Bigg[\frac{c_1\phi}{\sin\theta}d\theta + \left(-d_1 + c_1\ln\tan\frac{\theta}{2}\right)d\phi \\ +\frac{c_2\beta}{\rho}\,d\rho + (-d_2 + c_2\ln\rho)d\beta\Bigg]\,.\end{aligned} \tag{38}$$

4 Hamiltonian holomorphic vector fields

Definition 3 *[9, 15]*

A complex vector field X on a Sasaki manifold M, commuting with the Reeb vector field ξ is called Hamiltonian holomorphic vector field if

1. *its projection onto the normal bundle $\bar{X}$ is transversally holomorphic;*

2. *the basic function, called Hamiltonian function,*

$$u_X = i\eta(X)\,, \tag{39}$$

satisfies

$$\bar{\partial}_B u_X = -\frac{i}{2}\iota(X)d\eta\,. \tag{40}$$

In the foliation chart of $(x, z^1, \dots z^n)$, the Hamiltonian holomorphic field X is written as

$$X = \left(-iu_x + i\sum_{jl} h^{j\bar{l}}\frac{\partial u_x}{\partial\bar{z}^l}h_j\right)\xi - \sum_{jl} h^{j\bar{l}}\frac{\partial u_x}{\partial\bar{z}^l}\frac{\partial}{\partial z^j}\,. \tag{41}$$

Indeed, let us assume that the Hamiltonian vector field has the the form

$$X = A\frac{\partial}{\partial x} + \sum_j X^j\frac{\partial}{\partial z^j}\,. \tag{42}$$

From (39) we get

$$u_X = i\,A - X^j h_j\,. \tag{43}$$

which means that

$$A = -iu_X - i\sum_j X^j h_j\,. \tag{44}$$

On the other hand, from (40) we have

$$\bar{\partial}_B u_X = -\sum_{jl} X^j h_{j\bar{l}} d\bar{z}^l\,. \tag{45}$$

Therefore

$$X^j = -\sum_{jl} h^{j\bar{l}} \frac{\partial u_X}{\partial \bar{z}^l}\,, \tag{46}$$

where $h^{j\bar{l}}$ is the inverse of $h^{j\bar{l}}$.

We note that on any compact manifold, the set of all global holomorphic vector fields is a finite Lie algebra and spans the tangent space at every point. Let us assume that two holomorphic Hamiltonian vector fields are written in a local base as in equation (42):

$$\begin{aligned} X &= A\frac{\partial}{\partial x} + \sum_j X^j \frac{\partial}{\partial z^j}\,,\\ Y &= B\frac{\partial}{\partial x} + \sum_j Y^j \frac{\partial}{\partial z^j}\,. \end{aligned} \tag{47}$$

Their commutator is

$$[X,Y] = \sum_{jl}\left[X^j\frac{\partial Y^l}{\partial z^j} - Y^j\frac{\partial X^l}{\partial z^j}\right]\frac{\partial}{\partial z^l}\,. \tag{48}$$

The corresponding Hamiltonian function is

$$u_{[X,Y]} = \sum_{jl}\left[X^j\frac{\partial Y^l}{\partial z^j} - Y^j\frac{\partial X^l}{\partial z^j}\right] h_l\,. \tag{49}$$

In what follows we shall present two distinguished examples of Hamiltonian vector fields.

1. Let us consider a Hamiltonian vector field proportional with the Reeb vector field:

$$X = C\xi = C\frac{\partial}{\partial x}\,, \tag{50}$$

   where C is an arbitrary constant. From (43) we get that the corresponding Hamiltonian function is:

$$u_X = i\eta(X) = iC\,. \tag{51}$$

   We remark that under a deformation of the contact form η with a basic function, as in equation (11), the hamiltonian function remains unaltered since the basic function ϕ does not depend on x.

2. Let us consider that the holomorphic Hamiltonian vector fields are proportional with $\frac{\partial}{\partial z^j}$, $j = 1, 2$:
$$X = C^j \frac{\partial}{\partial z^j} \,. \tag{52}$$
Usinq (43) we get that the Hamiltonian function is
$$u_X = -C^J h_j \,, \tag{53}$$
with the corresponding derivative h_j of the potential h evaluated for the spaces $T^{1,1}$ or $Y^{p,q}$ using equations (21), (36) respectively.

Finally we evaluate the deformation of the holomorphic Hamiltonian function under a deformation (11).

Proposition 3 *By the deformation* (11)*, the Hamiltonian function* u_X *is deformed to* $u_X + X\varphi$*.*

Indeed, taking into account the definition (39) of the Hamiltonian function, the perturbed Hamiltonian is
$$\tilde{u}_X = i\tilde{\eta}(X) \,. \tag{54}$$
Using the class of deformations (22) we get the that following modification of the Hamiltonian function:
$$u_x \longrightarrow \sum_{j=1,2} C^j \left\{ -h_j + (e^{6t} - 1)\Big[\frac{c_j}{z^j} + 2d_j \frac{\ln z^j}{z^j}\Big] \right\} . \tag{55}$$
Remark 1 *By the Cartan formula, the Lie derivative of a symplectic (close) form* ω
$$\mathcal{L}_X \omega = \iota_X d\omega + d(\iota_X \omega) = d(\iota_X \omega) \,. \tag{56}$$
The Lie derivative of the transverse Kähler form
$$\omega^T = i g^T_{j\bar{l}} dz^j \wedge d\bar{z}^l = 2i h_{j\bar{l}} dz^j \wedge d\bar{z}^l \,, \tag{57}$$
is
$$\mathcal{L}\omega^T = d(\iota_X \omega^T) = (\partial + \bar{\partial})(\iota_X \omega^T) = -2i \frac{\partial u_X}{\partial z^i \partial \bar{z}^l} dz^i \wedge d\bar{z}^l \,. \tag{58}$$
Therefore the Hamiltonian vector field X (41) *does not preserve the symplectic form on local orbit spaces unless the Hamiltonian function* u_X *is a real valued function.*

5 Concluding remarks

In this paper, we examine the transverse Kähler structure of the Sasaki manifolds. Using the Sasaki-Ricci flow, we perturb the Sasakian structure, keeping the Reeb vector field fixed , but modifying the contact form with basic functions.

Starting with the five-dimensional Sasaki-Einstein manifolds $T^{1,1}$ and $Y^{p,q}$, as a seed, we generate new families of Sasakian structures. We are able to find some particular explicit analytical solutions of the Sasaki-Ricci flow equation depending on some arbitrary constants.

It is interesting to analyze the integrable Hamiltonian systems in the setting of contact geometry and to evaluate the perturbations of the Hamiltonian functions under the Sasaki-Ricci flow.

It is worth extending the study of Sasaki-Ricci flow on higher-dimensional Sakaki-Einstein spaces [2] as well as other contact spaces with 3-Sasaki structures [16] or mixed 3-structures [17].

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