Physics AUC, vol. 29, 17-24 (2019) PHYSICS AUC

Travelling wave solutions for the $2D$ Ricci flow model with variable coefficients through an improved tanh-method

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Abstract

In this paper an *improved tanh-function method* is applied to a new candidate: the $2D$ *Ricci flow model with variable coefficients*. Travelling wave solutions for this model are computed in an unified way. The method has been previously used for other models and its main improvements consist in the choice of a wave variable not linear in respect to t and in expressing the solution of the studied equation through finite series with time variable coefficients. Hyperbolic-type solutions, rational- type solutions or steady-state ones, related to some parameters and to time variable coefficients of the model, are pointed out. These solutions previously not reported in literature, provide control upon the dynamics of the concerned system. Graphical representations of some specific solutions support this statement.

Keywords: coefficient Ricci flow equation, travelling wave solutions, symbolic computations

PACS: 05.45.-a, 04.20.Jb

1 Introduction

Many matters of physics occurring in science and engineering may be modelled through non-linear partial differential equations (NPDEs). In order to deeply understand and evaluate the structure of the concerned phenomena, a considerable attention has been devoted to the search for efficient algorithms able to discover a rich variety of NPDEs solutions. This is partly due to the availability of computer symbolic systems like Maple and Mathematica which allow us to perform some rather complicated and tedious algebraic calculations. During recent years, various analytical methods have been developed by many scientists in order to find exact and explicit solutions for NPDEs. Among them: the inverse scattering method [1], the Bäcklund transformation [2], the Darboux transformation [3], Hirota's bilinear method [4], the dressing method [5], the homogeneous balance method [6], the Lie symmetry reduction [7, 8], the generalized conditional symmetry method [9], the sine-cosine method [10], various extended tanh-methods [11] and others.

Many among physical and biological systems are not homogeneous due to the fluctuations and non-uniformities which occur under environmental conditions. Therefore most of the nonlinear equations with interesting applications possess coefficients that vary spatially and/or temporally. In this paper we will implement an efficient (as well as generalized) version of the tanh-method in order to construct some exact wave solutions for the Variable Coefficient Ricci Flow (VCRF) equation:

$$A(t)u_t u^2 - B(t)u_{xy}u + M(t)u_x u_y = 0, \tag{1}$$

where $A(t)$, $B(t)$ and $M(t)$ are suitable functions which only depend upon the time variable.

If we choose $A(t) = B(t) = M(t) \equiv 1$, we obtain the standard $2D$ Ricci flow model [12]. A huge number of papers have studied this non-linear parabolic differential equations. Let us mention here only few among them which are directly related to the present paper. By making use of the linearization approach, special classes of solutions which involve arbitrary functions are reported in [12]. In [8], Lie point symmetries are calculated in terms of two arbitrary functions. Since this means to deal with an infinite number of symmetry generators, the linear sector of invariance has been taken into consideration, and it leads to the Lie algebra generated by:

$$\begin{gathered} U_1 = \partial_x, U_2 = \partial_y, U_3 = \partial_t, \\ U_4 = t\partial_x + u\partial_u, U_5 = x\partial_x - u\partial_u, U_6 = y\partial_y - u\partial_u. \end{gathered} \tag{2}$$

In [13] the algebra (2) has been imposed upon a $2D$ generalized nonlinear second order evolution equation and important particular equations compatible with the linear Lie invariance sector have been obtained. Conservation laws for the $2D$ Ricci flow model via the direct construction method [14], [15], have been established in [16] and some new invariant solutions have been derived. The same problem of the conservation laws associated to the Lie algebra (2) has been taken into consideration in [17] through Ibragimov's method [18] which is suitable for nonlinear self-adjoint differential equations. The study of the Lie group invariant solutions for the Ricci flow with constant coefficients has been extended in [19], by using the optimal system of one-dimensional subalgebras for Eq. (1) with $A(t) = B(t) = M(t) \equiv 1$,. New sets of invariant solutions have been obtained.

In this paper a powerful method able to construct travelling wave solutions for NPDEs is described in Section 2. In Section 3 the generalized $2D$ Ricci flow model (1) is chosen as a new candidate which could be compatible with the algorithmic approach. Wave solutions including the hyperbolic function solutions and the rational function ones which have not yet been reported in the literature, are pointed out. Section 4 is reserved for some final remarks.

2 Description of the method

In this section we will describe an efficient method namely a generalized tanh-method which allows to construct various types of solutions for NPDEs. It extends the general ansatz [20] and as we will see it works as well for our model. The method simply proceed as follows:

Step1: Let us consider a nonlinear PDE for the physical field $u(t, x, u)$ expressed by:

$$E(u, u_t, u_x, u_y, u_{tt}, u_{tx}, u_{ty}, u_{xy}, ...) = 0 \tag{3}$$

We shall assume that the solutions of Eq. (3) could be expressed under the form:

$$u(t,\ x,\ y) = \sum_{i=0}^{N} a_i(t)\phi^i(\xi),\ \xi = \xi(t,\ x,\ y), \tag{4}$$

where the functions $\xi(t,\, x,\, y)$, $a_i(t)$, $i = \overline{0, N}$ and the positive integer N are to be determined later. The crucial idea is the one of inserting the variable $\phi(\xi)$ as a solution of Riccati equation:

$$\phi^{'} = r + m\phi + s\phi^2, \tag{5}$$

where "prime" denotes $d/d\xi$. The advantage is the one that the parameters r, m, s may be used in order to exactly evaluate the types of the travellling wave solutions.

Step2: We determine N in Eq. (4) by balancing the linear term of the highest order with the highest nonlinear term in Eq. (3).

Step3: By substituting (4) and (5) into (3) yields a set of PDEs for $a_i(t)$, $i = \overline{0, N}$ and $\xi(t,\, x,\, y)$, because all of the coefficients of ϕ^i have to vanish.

Step 4: We solve the previous system of PDEs, by substituting its solutions into Eq. (4) and by making use of the special solutions for the desired Riccati equation (5). We arrive to a series of soliton solutions, periodic solutions and rational solutions for Eq. (3).

Let us remind that the Riccati equation (5) admits four types of solutions [21]:

3 Application to the VCRF model

In this section, the above mentioned algorithmic method is applied for the first time to the Ricci flow class of equations (1). Its advantage when compared to the standard tanh-function method consists in expressing the solutions of the governing equation in terms of the Riccati solutions through finite series with variable coefficients instead of constant ones. Let us now expand $u(t,\, x,\, y)$ in accordance with (4) and (5). Balacing the terms $A(t)u_t u^2$ and $B(t)u_{xy}u$, we have to require that $N = 1$. Therefore, we are able to choose the following ansatz:

$$u(t,x,y) = f(t) + g(t)\phi(\xi),\ \xi = kx + ly + \int v(t)dt, \tag{6}$$

where functions $f(t)$, $g(t)$, $v(t)$ are to be determined later and k, l are arbitrary constants.

By substituting (6) into (1) along with Eq. (5) and by vanishing the coefficients of $\phi^k(\xi)$, $k = \overline{0,4}$, we obtain the following set of PDEs with the unknown functions $f(t)$, $g(t)$, $v(t)$, $A(t)$, $M(t)$, $B(t)$, namely:

$$\begin{gathered} Asvg + (M - 2B)kls^2 = 0, \\ Ag[\dot{g} + mvg + 2svf] + [2M - 3B]sklmg^2 = 0, \\ A[2f\dot{g} + 2mvfg + svf^{\ 2} + \dot{f}g + rvg^2] \\ -B[3smf + m^2g + 2srg]kl + M[2rs + m^2]lkg^2 = 0, \\ Af[2g\dot{f} + 2rvg^2 + f\dot{g} + mgvf] \\ -B[fm^2 + 2srf + mrg]lkg + 2Mrlkmg^2 = 0, \\ Af^2[\dot{f} + rvg] - Bkmlrgf + Mklr^2g^2 = 0, \end{gathered} \tag{7}$$

where "dot" denotes d/dt.

The previously mentioned system will be solved through the symbolic Maple computation software for some specific values or relationships existing between parameters, as follows:

(1) – (2) For $r = 0$, $ms \neq 0$ and $kl \neq 0$, we come to the solutions:

$$f(t) = 0,\ g(t) = g_0,\ v(t) = \frac{sklB(t)}{g_0 A(t)},\ \forall A(t) \neq 0,\ \forall B(t) = M(t) \neq 0, \tag{8}$$

with the parameter $g_0 \neq 0$.

By substituting (8) into (6) and by taking into account the appropriate solution of Riccati Eq. (5), we can generate for the VCRF model, two different travelling wave solutions expressed as follows:

$$u_1(\xi) = \frac{-mCg_0}{s[\cosh(m\xi) - \sinh(m\xi) + C]} \tag{9}$$

or

$$u_2(\xi) = \frac{-mg_0[\sinh(m\xi) + \cosh(m\xi)]}{s[\sinh(m\xi) + \cosh(m\xi) + C]}, \tag{10}$$

with the arbitrary constant $C \neq 0$ and the wave variable

$$\xi = kx + ly + \frac{skl}{g_0}\int \frac{B(t)}{A(t)}dt. \tag{11}$$

When coming back to the original variables $(x,\ y,\ t)$ and fitting $y = 1$, the surface configurations of the wave solution (10) for two pairs of functions $A(t)$, $B(t)$ and for two sets of parametric choices, are displayed in Figure 1.

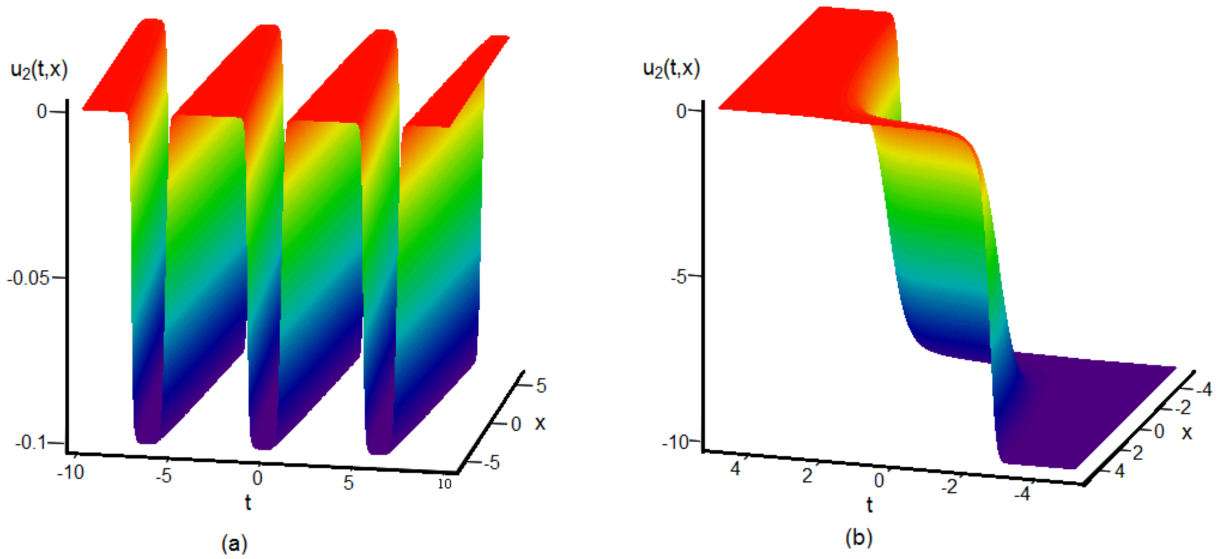

Figure 1: The surface configurations of wave solution $u_2(t,\ x,\ 1)$ are shown for particular functions and parameters: (a) $A(t) = 1$, $B(t) = \sin(t)$ and $s = 5k = 10m = 10L = 10$, $C = 3$; (b) $A(t) = 1$, $B(t) = \cosh^2(t)$ and $m = 10s = 2l = 2$, $C = -10k = 5$.

(3) For $s = 0$, $rm \neq 0$ and $kl \neq 0$, we obtain a more general solution:

$$f(t) = \frac{rg(t)}{m},\ v(t) = -\frac{\dot{g}(t)}{mg(t)},\ \forall g(t) > 0,\ \forall A(t) \neq 0,\ \forall B(t) = M(t) \neq 0. \tag{12}$$

Under these conditions the wave variable becomes $\xi = kx + ly - \frac{\ln g(t)}{m}$ and the auxiliary Eq. (5) admits the exponential form $\phi(\xi) = \frac{e^{m\xi} - r}{m}$. By substituting (12) into (6), we get the stationary solution:

$$u_3(x, y) = e^{m(kx+ly)}/m. \tag{13}$$

(4) For $r = m = 0$, $s \neq 0$ and $kl \neq 0$, we reach for another solution:

$$\begin{gathered} f(t) = 0,\ g(t) = g_1,\ v(t) = -\frac{skl}{g_1 A(t)}[2B(t) - M(t)], \\ \forall A(t) \neq 0, \forall B(t) \neq \frac{M(t)}{2}, \end{gathered} \tag{14}$$

with the parameter g_1 and the functions $B(t)$, $M(t)$ different from zero.

Let us return to (6) and review the solution available for Riccati equation.We may derive for VCRF model the rational-type wave solution:

$$u_4(\xi) = -\frac{g_1}{s\xi + C_1},\ \xi = kx + ly + \frac{skl}{g_1}\int \frac{2B(t) - M(t)}{A(t)}dt, \tag{15}$$

with C_1 arbitrary constant.

The respective shape and motion of the wave solution (15) when $y = 1$ are depicted in Figure 2 via the particular coefficient functions and parametric relations: (a) $A(t) = 1$, $B(t) = \sin^2(t)$, $M(t) = \cos^2(t)$ and $s = 50k = 15$, $C_1 = -40 = -50l$, $g_1 = 50$; (b) $A(t) = 1$, $B(t) = t^3$, $M(t) = t$ and $C_1 = 3l = 3s = 2g_1 = 30$, $k = 1$.

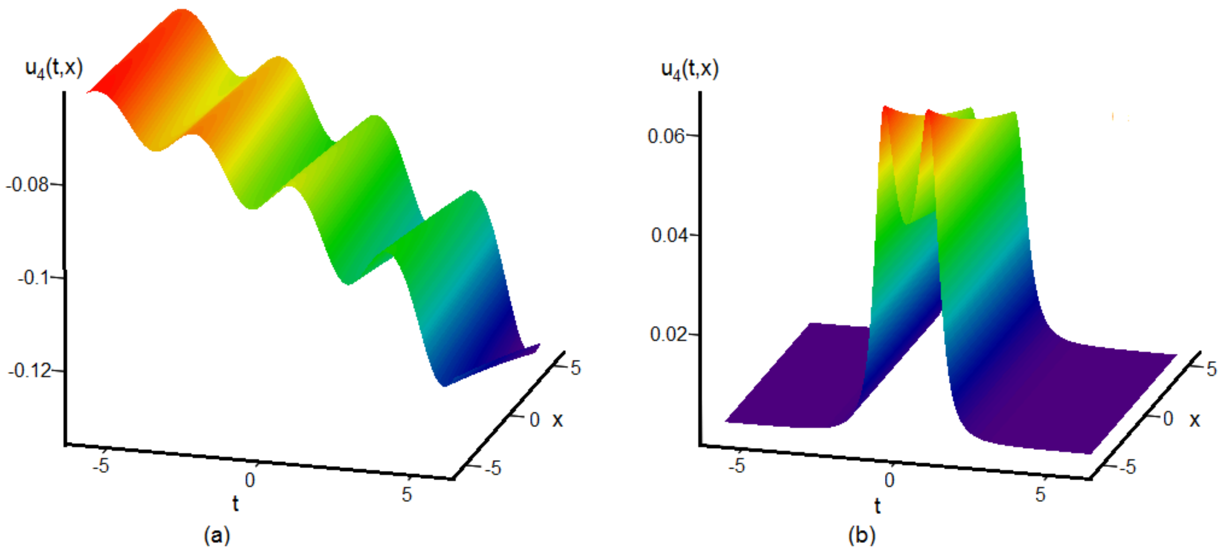

Figure 2: The surface configurations of the solution $u_4(t,\ x,\ y)$ when $y = 1$ are shown via: (a) $A(t) = 1$, $B(t) = \sin^2(t)$, $M(t) = \cos^2(t)$ and $s = 50k = 15$, $C_1 = -40 = -50l$, $g_1 = 50$; (b) $A(t) = 1$, $B(t) = t^3$, $M(t) = t$ and $C_1 = 3l = 3s = 2g_1 = 30$, $k = 1$..

4 Concluding remarks

We have applied an improved tanh-function method by taking into consideration some extensions brought to the traditional path: (i) the choice of the generalized Riccati equation with three parameters instead of one; (ii) making use of a wave variable $\xi = kx + ly + \rho(t)$ which is not compulsorily linear in t; (iii) looking for solutions of the wave equation in terms of Riccati solutions through finite series with time variable coefficients.

We have made use of this improved method in order to construct solutions of the VCRF model (1), a generalization of the $2D$ Ricci flow model in conformal gauge coming from quantum field theory. According to the values of some parameters, the concerned algorithm leads to wave solutions of hyperbolic type (9), (10), steady-state solution (13) and rational type solution (15). In all of these situations, the wave velocity admits a time variation according to the expressions of the variable coefficients $A(t)$, $B(t)$ and $M(t)$. To our best knowledge, the exact solutions reported here are new in the current literature. By employing the symbolic computation technique the method hereby described has proven itself to be simultaneously simple and efficient. Therefore, it is readily applicable to a large variety of NPDEs.

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