Fermionic extensions of KdV equation

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Abstract

The integrable cases of fermionic extensions of the KdV equation are reviewed, using Hirota bilinear formalism extended to super space. The supersymmetric and the non-supersymmetric continuous and discrete KdV equations are presented with their super-bilinearisations and supersoliton solutions.

1 Introduction

It is well known that the study of KdV equation by Kruskal and his collaborates was the starting point of the modern theory of integrable systems or soliton theory [1]. Yu. Manin and A. Radul gave the first systematic construction of an integrable supersymmetric hierarchy more then thirty years ago [2]. The simplest and the most important reduction of the supersymmetric KP hierarchy, given by Manin and Radul, is the supersymmetric KdV equation. Still, the complete integrability of supersymmetric equations has no definitive answer so far both classically and quantum mechanically. Supersymmetry is an extra symmetry imposed by construction, which means, roughly speaking, an invariance with respect to a kind of interchanging between the bosonic and fermionic fields [2, 3]. Some of the most interesting results obtained so far about integrability of supersymmetric nonlinear equations are about Darboux transformations [4, 5], bi-Hamiltonian structure [6], prolongation structures [7] and Painlevé property [8], the appearance of nonlocal integrals of motion [9], lack of unique bilinear formulation of integrable hierarchies [10, 11, 12, 13, 14], inelastic interactions of supersolitons[15], dressing of fermionic phases [16, 17], peculiar properties of super-Painlevé equations [18] etc.

The study of supersoliton interaction was also extended to lattice equations, where the supersymmetry is broken due to discretization. The equations can still be analyzed using Bäcklund-Darboux transformations, Lax pair or super-bilinear formalism [19, 20, 21, 22, 23].

In this paper we are going to focus on the results obtained on the fermionic extensions of the KdV equation. The first completely integrable super-extension of KdV was given independently by Kuperschmidt and Kulish [24, 25] before the Manin-Radul [2] paper. The Kuper-KdV (known as super fermionic KdV equation) is not supersymmetric and served as the first example of integrability in the Grassmann algebra. Kuperschmidt himself established the Lax pair and bi-Hamiltonian structure, Kulish and Zeitlin [26] adapted he IST scheme to Kuper-KdV equation in the case of a Grassmann algebra with only one generator, and only recently the supersolitons were constructed using super-Hirota bilinear formalism [27].

The paper is organized as follows: in Section 2 the notion of supersymmetry is briefly discussed, together with the supersymmetric KdV hierarchy and its bilinearisation with the aide of the super-Hirota operators, in Section 3 and 4 the continuous susy-KdV and the Kuper-KdV are presented, with the bilinear form and supersoliton solutions. In Section 5, the asymptotology of interaction for susy-KdV and Kuper-KdV are presented. In Section 6 other supersymmetric and non-supersymmetric KdV-type equations are briefly presented and Section 7 focuses on discrete super-KdV and its solutions.

2 Supersymmetry and the supersymmetric KdV hierarchy

The supersymmetric extension of a nonlinear evolution equation is a system of coupled equations of bosonic $u(x,t)$ and fermionic fields $\xi(x,t)$, related through a transformation:

$$u \to u + \delta u, \quad \delta u = \lambda \xi_x$$

$$\xi \to \xi + \delta\xi, \quad \delta\xi = \lambda u,$$

which leaves the system invariant (λ is a fermionic parameter).

In the limit where the fermionic field $\xi(x,t)$ is zero, the supersymmetric extension reduces to the initial equation. The fermionic and the bosonic fields are described, in the classical context, by anticommuting (odd) and respectively commuting (even) functions in an infinitely generated Grassmann algebra.

The mathematical formulation of these concepts imply the extension of the classical space (x,t) to a superspace (x,t,θ), and also the extension of the pair of fields (u,ξ) to a superfield $\Phi(x,t,\theta)$, which can be bosonic, $\Phi(t,x,\theta) = u(x,t) + \theta\xi(x,t)$ or fermionic $\Phi(t,x,\theta) = \xi(x,t) + \theta u(x,t)$.

We work in $\mathcal{N} = 1$ SUSY, which means that we have only one Grassmann variable θ and that we consider only space supersymmetry invariance [28]: $x \to x - \lambda\theta, \quad \theta \to \theta + \lambda$. The transformation above is generated by an operator $Q = \partial_\theta - \theta\partial_x$, which anticommutes with the covariant derivative $\mathcal{D} = \partial_\theta + \theta\partial_x$ (notice that $\mathcal{D}^2 = \partial_x$, as $\theta^2 = 0$).

Equations written in terms of the superfield $\Phi(x,t,\theta)$ and the covariante derivative $\mathcal{D} = \partial_\theta + \theta\partial_x$ are supersymmetric invariant. Different supersymmetric extensions of nonlinear equations can be constructed using the superspace formalism.

One of the most tractable methods in solving such equations is the super Hirota bilinear formalism. The extension of the Hirota operator has been introduced by A. Carstea in [16] and is defined by its action on a pair of Grassmann-valued functions (f,g) in the following form:

$$\mathbf{S}_x f \cdot g = (\mathcal{D}f)g - (-1)^{|f|} f(\mathcal{D}g),$$

where $\mathcal{D} = \partial_\theta + \theta\partial_x$ is the covariant derivative and $|f|$ is the Grassmann parity of the function f, which is zero if the function is bosonic and 1 if the function is fermionic with f, g general odd and even functions.

Some of the most important properties of the super-Hirota operator are:

$$\begin{aligned} \mathbf{S}_x^{2N} f \cdot g &= \mathbf{D}_x^N f \bullet g, \\ \mathbf{S}_x^{2N+1} f \cdot g &= \mathbf{S}_x \mathbf{D}_x^N f \bullet g. \end{aligned}$$

3 Continuous supersymmetric KdV (susy-KdV)

The supersymmetric fermionic extension of continuous KdV equation was constructed more then 30 years ago [2]. The equation, known in the literature as supersymmetric

KdV equation (susy-KdV), has the following form:

$$\Phi_t + \mathcal{D}^6\Phi + 3\mathcal{D}^2(\Phi\mathcal{D}\Phi) = 0, \tag{1}$$

where $\Phi(x,t,\theta)$ is an odd superfield and $\mathcal{D}$ is the super-covariant derivative.

Considering the odd superfield $\Phi(x,t,\theta) = \psi(x,t) + \theta u(x,t)$, with $\psi(x,t)$ odd (fermionic) function and $u(x,t)$ even (bosonic) function, the nonlinear form of susy-KdV (1), on the components, is [28]:

$$\begin{aligned} u_t + 6uu_x + u_{xxx} - 3\psi\psi_{xx} &= 0 \\ \psi_t + 3(\psi u)_x + \psi_{xxx} &= 0. \end{aligned} \tag{2}$$

In order to construct the super-bilinear form of (1) we consider the nonlinear substitution for the superfield:

$$\Phi(t,x,\theta) = 2\mathcal{D}^3 \log\tau(t,x,\theta)$$

with $\tau = F + \theta G$ a bosonic tau function (F bosonic and G fermionic) and the super-bilinear operator $\mathbf{S}$. We obtain [20]:

$$(\mathbf{S}_x\mathbf{D}_t + \mathbf{S}_x^7)\tau \bullet \tau = 0.$$

In terms of F and G, the super-bilinear form or susy-KdV is:

$$\begin{aligned} (\mathbf{D}_t + \mathbf{D}_x^3)G \bullet F &= 0 \\ (\mathbf{D}_t\mathbf{D}_x + \mathbf{D}_x^4)F\cdot F - 2(\mathbf{D}_t + \mathbf{D}_x^3)G\bullet G &= 0 \end{aligned} \tag{3}$$

Remark: The same substitution can be used to write the full supersymmetric hierarchy proposed by McArthur and Yung:

$$\begin{aligned} \Phi_{t1} &= -\Phi_x \\ \Phi_{t3} &= -(\Phi_{xx} - 3\Phi\mathcal{D}\Phi)_x \\ \Phi_{t5} &= -(\Phi_{xxxx} - 5\Phi_x x\mathcal{D}\Phi + 10\Phi(\mathcal{D})^2 - 5(\mathcal{D}\Phi_x)\Phi_x - 5\Phi\mathcal{D}\Phi_{xx})_x, \end{aligned}$$

where the second equation is the supersymmetric KdV equation.

Using the same ansatz, $\phi = 2\mathcal{D}^3\log\tau$, the full supersymmetric KdV hierarchy has been bilinearized [14]:

$$\begin{aligned} (\mathbf{S}_x^7 - 4\mathbf{S}_x\mathbf{D}_{t_3})\tau\bullet\tau &= 0 \\ (\mathbf{S}_x^{11} + 20\mathbf{S}_x^5\mathbf{D}_{t_3} - 96\mathbf{S}_x\mathbf{D}_{t_5})\tau\bullet\tau &= 0 \\ (\mathbf{S}_x^{15} + 560\mathbf{S}_x^3\mathbf{D}_{t_3}^2 - 336\mathbf{S}_x^5\mathbf{D}_{t_5} - 960\mathbf{S}_x\mathbf{D}_{t_7})\tau\bullet\tau &= 0 \\ (\mathbf{S}_x^{15} + 140\mathbf{S}_x^9\mathbf{D}_{t_3} - 1344S_x^5\mathbf{D}_{t_5} - 7680\mathbf{S}_x\mathbf{D}_{t_7})\tau\bullet\tau &= 0. \end{aligned}$$

The first equation of above is the super bilinear form of supersymmetric KdV.

Now we can compute directly the supersoliton solutions of the bilinear system (3) as combinations of exponentials $\exp(k_i x - \omega_i t)$, where k_i are commuting (even) invertible Grassmann numbers and $\omega_i = \omega_i(k_i)$ is the dispersion relation. For the fermionic (odd) tau function $G(x,t)$, odd parameters ζ_i have to be considered. Accordingly, every supersoliton is characterized by the following triplet $(k_i, \zeta_i, \omega_i(k_i,\zeta_i))$. Of course, nobody imposes the number of odd parameters for a supersoliton, but we consider the simplest case here, where any supersoliton is characterized by only one k_i and ζ_i.

On the components, the 2-supersoliton solution of (3) has the following form:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}(1 + \frac{2}{k_2 - k_1}\zeta_1\zeta_2)e^{\eta_1+\eta_2}, \tag{4}$$
$$G = \zeta_1 e^{\eta_1} + \zeta_2 e^{\eta_2} + A_{12}(\alpha_{12}\zeta_1 + \alpha_{21}\zeta_2)e^{\eta_1+\eta_2},$$

where ζ_i are odd Grassmann parameters and:

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \quad i = 1, 2$$

$$A_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad \alpha_{ij} = \frac{k_i + k_j}{k_i - k_j}, \quad i, j = 1, 2.$$

As $\tau = F + \theta G$, the 2-supersoliton is, as shown in [15], [20]:

$$\tau_2 = 1 + e^{\eta_1+\theta\zeta_1} + e^{\eta_2+\theta\zeta_2} + A_{12}\left(1 + \frac{2}{k_2 - k_1}\zeta_1\zeta_2\right) e^{\eta_1+\eta_2+\theta(\alpha_{12}\zeta_1+\alpha_{21}\zeta_2)}.$$

The interaction of supersolitons differ of the interaction of ordinary solitons mainly by the appearance of the fermionic correction $\frac{2}{k_2-k_1}\zeta_1\zeta_2$ multiplying A_{12} and also by the dressing factor α_{ij} that appears in the exponential. In addition, α_{ij} appears for any pair of supersolitons in the interaction, which can be seen more clearly in the 3-supersoliton solution form:

$$\tau_3 = 1 + \sum_{i=1}^{3} e^{\eta_i+\theta\zeta_i} + \sum_{i<j}^{3} A_{ij}\left(1 + \frac{2}{k_j - k_i}\zeta_i\zeta_j\right) e^{\eta_i+\eta_j+\theta(\alpha_{ij}\zeta_i+\alpha_{ji}\zeta_j)} +$$
$$+ A_{12}A_{13}A_{23}\left(1 + \frac{2}{k_2 - k_1}\alpha_{13}\alpha_{23}\zeta_1\zeta_2\right)\left(1 + \frac{2}{k_3 - k_2}\alpha_{21}\alpha_{31}\zeta_2\zeta_3\right) \times$$
$$\times \left(1 + \frac{2}{k_3 - k_1}\alpha_{12}\alpha_{32}\zeta_1\zeta_3\right) e^{\;\eta_1+\eta_2+\eta_3+\theta(\alpha_{12}\alpha_{13}\zeta_1+\alpha_{21}\alpha_{23}\zeta_2+\alpha_{31}\alpha_{32}\zeta_3)}.$$

It is known that in the bosonic context the construction of the multisoliton solution is quite difficult, but it has been observed that once a general 3-soliton solution is constructed, then it can be proved by induction that the general N-soliton can be constructed as well. The existence of the 3-soliton solution has been used in the classification of completely integrable bilinear equations [29] as an integrability criterion. This criterion of integrability we consider to remain valid also for the fermionic extensions of the solitonic equations. From the 3-supersoliton solution it is likely that the N-supersoliton solution has the form[20]:

$$\tau_N = \sum_{\mu\in\{0,1\}} \exp\left(\sum_{i=1}^{N} \mu_i(\eta_i + \theta\zeta_i \prod_{m\neq i} \alpha_{im}) + \sum_{i<j} \mu_i\mu_j(\ln A_{ij} + \frac{2\zeta_i\zeta_j}{k_j - k_i} \prod_{k\neq i,j} \alpha_{ik}\alpha_{jk})\right), \tag{5}$$

where the products on α_{im} and $\alpha_{ik}\alpha_{jk}$ are considered to be 1 for $N = 1$ and $N = 2$.

4 Kuperschmidt KdV (Kuper-KdV)

The super or fermionic KdV equation of Kuperschmidt (Kuper-KdV), which, as mentioned before, is not supersymmetric, has the expression:

$$\begin{aligned} u_t - 6uu_x + u_{xxx} + 12\xi\xi_{xx} &= 0 \\ \xi_t + 4\xi_{xxx} - 6u\xi_x - 3u_x\xi &= 0 \end{aligned} \tag{6}$$

with $u(x,t)$ a bosonic function and $\xi(x,t)$ a fermionic one.

Even though the complete integrability was established in 1984 by the existence of the Lax pair by Kuperschmidt itself in [24], the superbilinear form and the multi supersolitons were constructed only recently by Carstea and the author in [27].

Considering the following nonlinear substitutions $u = -2\partial_x^2 \log F(x,t)$ and $\xi(x,t) = G(x,t)/F(x,t)$, where $G(x,t)$ is a Grassmann odd (anticommuting) function and $F(x,t)$ is a Grassmann even (commuting) function, we obtain the bilinear form, as shown in [27]:

$$\begin{aligned} (\mathbf{D}_t\mathbf{D}_x + \mathbf{D}_x^4)F \bullet F + 6\mathbf{D}_x G \bullet G &= 0 \\ (\mathbf{D}_t + \mathbf{D}_x^3)G \bullet F + 3\mathbf{D}_x K \bullet F &= 0 \\ \mathbf{D}_x^2 G \bullet F - KF &= 0 \end{aligned}$$

where $K(x,t)$ is an auxiliary odd function. It has no role in the solution but is crucial for bilinearisation.

The 3-supersoliton solution of Kuper-KdV has the following form:

$$G = \sum_{i=1}^{3} \zeta_i e^{\frac{\eta_i}{2}} + \sum_{i\neq j\neq l}^{3} \zeta_i \alpha_{ij} A_{ij} e^{\frac{\eta_i}{2}+\eta_j}(1 + \alpha_{il}A_{il}A_{jl}e^{\eta_l}) + \zeta_1\zeta_2\zeta_3 M_{123} e^{\frac{\eta_1+\eta_2+\eta_3}{2}}$$

$$\begin{aligned} F = 1 + \sum_{i=1}^{3} e^{\eta_i} + \sum_{i<j}^{3} A_{ij} e^{\eta_i+\eta_j} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3} + \\ + \sum_{i<j\neq l} \frac{16\zeta_i\zeta_j A_{ij}}{(k_i-k_j)^3} e^{\frac{\eta_i+\eta_j}{2}} (1 + A_{il}A_{jl}e^{\eta_k}) \end{aligned}$$

$$\begin{aligned} K = \sum_{i=1}^{3} \zeta_i \frac{k_i^2}{4} e^{\frac{\eta_i}{2}} + \sum_{i\neq j\neq l}^{3} \zeta_i \frac{k_i^2}{4} \alpha_{ij} A_{ij} e^{\frac{\eta_i}{2}+\eta_j}(1 + \alpha_{il}A_{il}A_{jl}e^{\eta_l}) + \\ + \zeta_1\zeta_2\zeta_3 M_{123} Q_{123} e^{\frac{\eta_1+\eta_2+\eta_3}{2}} \end{aligned}$$

where:

$$A_{ij} = \left(\frac{k_i-k_j}{k_i+k_j}\right)^2, \quad \alpha_{ij} = \frac{k_i+k_j}{k_i-k_j}$$

$$M_{123} = -8\prod_{i\neq j\neq l}^{3} (k_i-k_l)\left(\frac{\alpha_{ij}A_{ij}}{(k_i-k_j)^2} + \frac{\alpha_{lj}A_{lj}}{(k_l-k_j)^2}\right)$$

$$Q_{123} = \frac{1}{3}\sum_{i\neq j\neq l}^{3} \left(\frac{k_l^2}{4} + \frac{A_{il}A_{jl}\alpha_{il}\alpha_{jl}(k_i-k_j)^4}{4(k_j-k_l)^2 A_{ij}A_{il}\alpha_{il} + 4(k_i-k_l)^2 A_{ij}A_{jl}\alpha_{jl}}\right).$$

5 Asymptotology of interaction for susy-KdV and Kuper-KdV

Even though the similarities between the forms of the two solutions (of susy-KdV and Kuper-KdV) are obvious, the asymptotology of interaction is completely different [27].

For susy-KdV, in the reference system of the first supersoliton, (η_1 fixed and $k_1 < k_2$) from (4) it has been shown that:

$$\lim_{t\to+\infty} F = 1 + e^{\eta_1}, \lim_{t\to+\infty} G = \zeta_1 e^{\eta_1},$$

$$\lim_{t\to-\infty} F = 1 + A_{12}(1 + \frac{2}{k_2 - k_1}\zeta_1\zeta_2)e^{\eta_1}, \lim_{t\to-\infty} G = \zeta_2 + A_{12}(\zeta_1\alpha_{12} + \zeta_2\alpha_{21})e^{\eta_1}.$$

The interaction is elastic for the bosonic component with a fermionic correction of the phase shift, which is given by $\frac{2}{k_2-k_1}\zeta_1\zeta_2$. For the fermionic component the interaction is not elastic, because not only that the amplitude is dressed by the α_{ij}, but the creation of a fermionic background in the fermionic tau function appears given by ζ_2.

In the case of Kuper-KdV, the 2-supersoliton solution is:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} + \zeta_1\zeta_2 A_{12}\frac{16}{(k_1 - k_2)^3}e^{\frac{\eta_1+\eta_2}{2}},$$

$$G = \zeta_1 e^{\frac{\eta_1}{2}} + \zeta_2 e^{\frac{\eta_2}{2}} + \zeta_1\alpha_{12}A_{12}e^{\frac{\eta_1}{2}+\eta_2} + \zeta_2\alpha_{21}A_{12}e^{\frac{\eta_2}{2}+\eta_1}.$$

Considering the reference frame of the first supersoliton (η_1 fixed and $k_1 < k_2$) the asymptotology of interaction shows that:

$$\lim_{t\to+\infty} F = 1 + e^{\eta_1}, \lim_{t\to+\infty} G = \zeta_1 e^{\eta_1},$$

$$\lim_{t\to-\infty} F = 1 + A_{12}e^{\eta_1}, \lim_{t\to-\infty} G = \zeta_2 A_{12}\alpha_{12}e^{\eta_1/2}.$$

The interaction is simpler in the Kuper-KdV case. Asymptotically, the bosonic supersoliton does not feel at all the presence of the fermionic one. The fermionic supersoliton has not only a phase shift, but also a changing of amplitude from ζ_1 to ζ_2.

Another observation of the authors in [27] was the presence of the fermionic dressing $\alpha_{ij} = (k_i + k_j)/(k_i - k_j)$ which appears both in supersymmetric KdV and Kuper-KdV. This fermionic dressing seems to be universal, as it appears in all bilinear super-equations analyzed so far in the literature. In the discrete setting the fermionic dressing appears as well in the form $\alpha_{ij} = (e^{k_i+k_j} - 1)/(e^{k_i} - e^{k_j})$ [20].

6 Other supersymmetric and non-supersymmetric KdV-type equations

In this section, four other supersymmetric (SUSY) and non-supersymmetric KdV-type equations are briefly presented [16], [30], [31].

• $\mathcal{N} = 1$ SUSY Sawada-Kotera-Ramani:

$$\Phi_t + \mathcal{D}^{10}\Phi + \mathcal{D}^2(10\mathcal{D}\Phi\mathcal{D}^4\Phi + 5\mathcal{D}^5\Phi\Phi + 15(\mathcal{D}\Phi)^2\Phi) = 0$$

Using the nonlinear substitution $\Phi = 2\mathcal{D}^3 \log \tau(t, x, \theta)$ the super-bilinear form is obtained [16]:

$$(\mathbf{S}_x \mathbf{D}_t + \mathbf{S}_x^{11})\tau \bullet \tau = 0,$$

which has the same N-supersoliton solution structure as for super-KdV, (5), except:

$$\eta_i = k_i x - k_i^5 t + \theta\zeta_i + \eta_i^{(0)}$$
$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2 \frac{k_i^2 - k_i k_j + k_j^2}{k_i^2 + k_i k_j + k_j^2}.$$

• $\mathcal{N} = 1$ SUSY Hirota-Satsuma (shallow water wave):

$$\mathcal{D}^4\Phi_t + \Phi_t \mathcal{D}^3\Phi + 2\mathcal{D}^2\Phi\mathcal{D}\Phi_t - \mathcal{D}^2\Phi - \Phi_t = 0$$

Using the nonlinear substitution $\Phi = 2\mathcal{D}\log\tau(t, x, \theta)$, one obtains the super-bilinear form [16]:

$$(\mathbf{S}_x^5 \mathbf{D}_t - \mathbf{S}_x^3 - \mathbf{S}_x \mathbf{D}_t)\tau \bullet \tau = 0$$

which has the N-supersoliton solution (5), except:

$$\eta_i = k_i x - \frac{k_i}{k_i^2 - 1} t + \theta\zeta_i + \eta_i^{(0)}$$

$$A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2 \frac{(k_i - k_j)^2 + k_i k_j[(k_i - k_j)^2 - (k_i^2 - 1)(k_j^2 - 1)]}{(k_i - k_j)^2 - k_i k_j[(k_i - k_j)^2 - (k_i^2 - 1)(k_j^2 - 1)]}.$$

• $\mathcal{N} = 1$ Holod-Pakuliak system (non-supersymmetric) [30]:

$$\begin{aligned}
u_t &= -u_{xxx} + 6uu_x + 6(\alpha\beta_{xx} - \alpha_{xx}\beta) \\
\alpha_t &= -4\alpha_{xxx} + 6u\alpha_x + 3u_x\alpha \\
\beta &= -4\beta_{xxx} + 6u\beta_x + 3u_x\beta.
\end{aligned}$$

In [30] the existence of an infinite number of motion integrals in involution is proved and the hierarchy of higher equations is constructed. The bilinear form can also be constructed if $\alpha = G_1/F$, $\beta = G_2/F$, $u = -2(\log F)_{xx}$. Considering these transformations we get [27]:

$$\begin{gathered}
(D_t D_x + D_x^4)F \cdot F + 6D_x G_1 \cdot G_2 = 0 \\
(D_t + D_x^3)G_1 \cdot F + 3D_x K_1 \cdot F = 0 \\
(D_t + D_x^3)G_2 \cdot F + 3D_x K_2 \cdot F = 0 \\
D_x^2 G_1 \cdot F - K_1 F = 0 \\
D_x^2 G_2 \cdot F - K_2 F = 0,
\end{gathered}$$

where K_1, K_2 are two auxiliary fermionic functions. The computation of soliton solutions goes on the same way as in the Kuper-KdV case, except of some phases in the exponentials. However the solutions turn out to be rather trivial, i.e. $G_1 = \pm G_2$ reducing the system to the one component case. It is an open problem how to find more general solutions.

• $\mathcal{N} = 1$ super-KdV equation of Geng-Wu (non-supersymmetric) [31]:

$$\begin{aligned}
u_t &= -u_{xxx} + 6uu_x + 12u\xi_{xx}\xi + 6u_x\xi_x\xi - 3\xi_{xxxx}\xi - 6\xi_{xxx}\xi_x \\
\xi_t &= -4\alpha_{xxx} + 6u\alpha_x + 6u_x\alpha.
\end{aligned}$$

This super-extension of the KdV hierarchy, associated with a 3×3 matrix spectral problem was analized using super-trace identity. Generalized bi-Hamiltonian structures were established and also infinite conservation laws were derived in [31].

7 Semidiscrete and discrete fermionic extensions of KdV

The semidiscrete and discrete fermionic extensions of the KdV equation have also been analyzed in the literature [19], [20].

Xue, Levi and Liu found in [19] integrable super extensions of potential discrete KdV, known as the potential semidiscrete super-KdV:

$$\begin{aligned}
\partial_t \psi_n &= 2\frac{\psi_{n+1}-\psi_{n-1}}{u_{n+1}-u_{n-1}-4p} \\
\partial_t u_t &= 2\frac{u_{n+1}-u_{n-1}}{u_{n+1}-u_{n-1}-4p} + \frac{u_{n+1}-u_{n-1}-8p}{(u_{n+1}-u_{n-1}-4p)^2}(\psi_{n+1}-\psi_{n-1})(\psi_n-\psi_{n-1})
\end{aligned} \tag{7}$$

and the potential discrete super-KdV:

$$\begin{aligned}
\psi_{n+1,m+1}-\psi_{n,m} &= \frac{2(p_1+p_2)(\psi_{n+1,m}-\psi_{n,m+1})}{2(p_2-p_1)+u_{n+1,m}-u_{n,m+1}} \\
u_{n+1,m+1}-u_{n,m} &= \frac{2(p_1+p_2)(u_{n+1,m}-u_{n,m+1})}{2(p_2-p_1)+u_{n+1,m}-u_{n,m+1}} - \\
-\frac{(p_1+p_2)(4(p_2-p_1)+u_{n+1,m}-u_{n,m+1})}{(2(p_2-p_1)+u_{n+1,m}-u_{n,m+1})^2}&(\psi_{n+1,m}-\psi_{n,m+1})(\psi_{n,m+1}-\psi_{n,m}),
\end{aligned} \tag{8}$$

where ψ_n, u_n (depending on n and t) and $\psi_{n,m}$, $u_{n,m}$ (depending on n and m) are fermionic and respectively bosonic fields with values in the odd and even sector of the Grassmann algebra. The authors established integrability of (7), (8) by displaying the Bäcklund transformations and Lax pair.

7.1 Two semidiscrete versions of super-KdV in bilinear formalism

The nonlocality of the lattice makes the supersymmetry implementation to be quite difficult, so a direct approach is recommended. In [20] the Hirota super bilinear form for (7) was constructed (where the constant $4p$ was rescaled to 1 by taking $\omega_n \to 4p\omega_n$ and $\partial_t \to 4p\partial_t$):

$$\begin{aligned}
&\mathbf{D}_t\gamma_n \bullet f_n - (\gamma_{n+1}f_{n-1}-\gamma_{n-1}f_{n+1}) = 0 \\
&\mathbf{D}_t g_n \bullet f_n - (g_{n+1}f_{n-1}-g_{n-1}f_{n+1}) + \mathbf{D}_t\gamma_n \bullet \gamma_n - 2\gamma_{n+1}\gamma_{n-1} = 0 \\
&(g_{n+1}f_{n-1}-g_{n-1}f_{n+1}) - (f_{n+1}f_{n-1}-f_n^2) - \frac{1}{2}\mathbf{D}_t\gamma_n \bullet \gamma_n + \gamma_{n+1}\gamma_{n-1} = 0.
\end{aligned} \tag{9}$$

Here $\gamma_n(t)$ is a Grassmann fermionic (odd) tau function and $g_n(t)$, $f_n(t)$ are Grassmann bosonic (even) tau functions. The corespondence between (7) and the bilinear form can be proven considering $\omega_n(t) = g_n(t)/f_n(t)$ and $\psi_n(t) = \gamma_n(t)/f_n(t)$.

The 3-supersoliton solution was found in [20] of the following form:

$$\gamma_n = \sum_{i=1}^{3} \zeta_i e_i^{\eta} + \sum_{i<j}^{3} a_{ij} e^{\eta_i+\eta_j}(\alpha_{ij}\zeta_i + \alpha_{ji}\zeta_j) + \tag{10}$$
$$+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(\alpha_{12}\alpha_{13}\zeta_1 + \alpha_{21}\alpha_{23}\zeta_2 + \alpha_{31}\alpha_{32}\zeta_3) +$$
$$+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(\delta_{12}\alpha_{13}\alpha_{23}\alpha_{31}\alpha_{32} - \delta_{13}\alpha_{12}\alpha_{32}\alpha_{21}\alpha_{23} + \delta_{23}\alpha_{21}\alpha_{31}\alpha_{12}\alpha_{13})\zeta_1\zeta_2\zeta_3,$$
$$g_n = \sum_{i=1}^{3} b_i e_i^{\eta} + \sum_{i<j}^{3}(b_i + b_j)a_{ij}e^{\eta_i+\eta_j}(1 + \delta_{ij}\zeta_i\zeta_j) + (b_1 + b_2 + b_3)a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3} +$$
$$+(b_1 + b_2 + b_3)a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(\delta_{12}\alpha_{13}\alpha_{23}\zeta_1\zeta_2 + \delta_{23}\alpha_{21}\alpha_{31}\zeta_2\zeta_3 + \delta_{13}\alpha_{12}\alpha_{32}\zeta_1\zeta_3) \tag{11}$$
$$f_n = \sum_{i=1}^{3} b_i e_i^{\eta} + \sum_{i<j}^{3} a_{ij}e^{\eta_i+\eta_j}(1 + 2\delta_{ij}\zeta_i\zeta_j) + a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3} +$$
$$+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(2\delta_{12}\alpha_{13}\alpha_{23}\zeta_1\zeta_2 + 2\delta_{23}\alpha_{21}\alpha_{31}\zeta_2\zeta_3 + 2\delta_{13}\alpha_{12}\alpha_{32}\zeta_1\zeta_3), \tag{12}$$

where:

$$a_{ij} = \left(\frac{e^{k_i} - e^{k_j}}{e^{k_i+k_j} - 1}\right)^2, \quad \omega_i = 2\sinh k_i, \quad \eta_i = k_i n + \omega_i t,$$
$$\alpha_{ij} = \left(\frac{e^{k_i+k_j} - 1}{e_i^k - e_j^k}\right)^2, \quad b_i = \frac{e_i^k - 1}{e_i^k + 1}, \quad \delta_{ij} = \frac{\alpha_{ij}}{b_i + b_j}.$$

The structure of the supersoliton solution is similar to the soliton solution of the continuous case, the important difference is the displaying of the dressing factor in the fermionic component of the supersoliton.

Considering a slightly different bilinear form:

$$\mathbf{D}_t\gamma_n \bullet f_n - (\gamma_{n+1}f_{n-1} - \gamma_{n-1}f_{n+1}) = 0 \tag{13}$$
$$\mathbf{D}_t g_n \bullet f_n - (g_{n+1}f_{n-1} - g_{n-1}f_{n+1}) - \frac{1}{2}\mathbf{D}_t\gamma_n \bullet \gamma_n + \gamma_{n+1}\gamma_{n-1} = 0$$
$$(g_{n+1}f_{n-1} - g_{n-1}f_{n+1}) - (f_{n+1}f_{n-1} - f_n^2) + \gamma_{n+1}\gamma_{n-1} = 0,$$

in [20] was found a new integrable form of semidiscrete super-KdV equation:

$$\begin{aligned}\partial_t\psi_n &= \frac{\psi_{n+1} - \psi_{n-1}}{1 + u_{n-1} - u_{n+1}}\\ \partial_t u_t &= \frac{u_{n+1} - u_{n-1}}{1 + u_{n+1} - u_{n-1}} + \frac{(1 + u_{n-1} - u_{n+1})(\psi_n\psi_{n+1} - \psi_n\psi_{n-1}) + \psi_{n+1}\psi_{n-1}}{(1 + u_{n-1} - u_{n+1})^2}\end{aligned}$$

The 3-supersoliton solution is similar but with a more complicated dressing in the supersoliton interaction:

$$\gamma_n = \sum_{i=1}^{3} \zeta_i e_i^{\eta} + \sum_{i<j}^{3} a_{ij} e^{\eta_i+\eta_j}(\alpha_{ij}\zeta_i + \alpha_{ji}\zeta_j) +$$
$$+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(\alpha_{12}\alpha_{13}\zeta_1 + \alpha_{21}\alpha_{23}\zeta_2 + \alpha_{31}\alpha_{32}\zeta_3) +$$
$$+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(\delta_{12}\alpha_{13}\alpha_{23}\alpha_{31}\alpha_{32} - \delta_{13}\alpha_{12}\alpha_{32}\alpha_{21}\alpha_{23} + \delta_{23}\alpha_{21}\alpha_{31}\alpha_{12}\alpha_{13})\zeta_1\zeta_2\zeta_3,$$
$$g_n = \sum_{i=1}^{3} b_i e_i^{\eta} + \sum_{i<j}^{3}(b_i + b_j)a_{ij}e^{\eta_i+\eta_j}(1 + \delta_{ij}\zeta_i\zeta_j) + (b_1 + b_2 + b_3)a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3} +$$

$$+(b_1+b_2+b_3)a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(\Delta_{12}\alpha_{13}\alpha_{23}\zeta_1\zeta_2+\Delta_{23}\alpha_{21}\alpha_{31}\zeta_2\zeta_3+\Delta_{13}\alpha_{12}\alpha_{32}\zeta_1\zeta_3),$$

$$f_n=\sum_{i=1}^{3}b_i e_i^{\eta}+\sum_{i<j}^{3}a_{ij}e^{\eta_i+\eta_j}(1+2\delta_{ij}\zeta_i\zeta_j)+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}+$$

$$+a_{12}a_{13}a_{23}e^{\eta_1+\eta_2+\eta_3}(2\Delta_{12}\alpha_{13}\alpha_{23}\zeta_1\zeta_2+2\Delta_{23}\alpha_{21}\alpha_{31}\zeta_2\zeta_3+2\Delta_{13}\alpha_{12}\alpha_{32}\zeta_1\zeta_3),$$

where:

$$\Delta_{ij}=\delta_{ij}\left(\frac{(b_i+b_k)+(b_j+b_k)}{b_1+b_2+b_3}\right)|_{k\neq i,j}.$$

7.2 Potential discrete super-KdV and discrete super-KdV

After solving the potential semidiscrete super-KdV and semidiscrete super-KdV, starting from the semidiscrete superbilinear form:

$$\begin{aligned}
&\mathbf{D}_t\gamma_n\bullet f_n-(\gamma_{n+1}f_{n-1}-\gamma_{n-1}f_{n+1})=0\\
&\mathbf{D}_t g_n\bullet f_n-(g_{n+1}f_{n-1}-g_{n-1}f_{n+1})-\frac{m_1}{2}\mathbf{D}_t\gamma_n\bullet\gamma_n+m_2\gamma_{n+1}\gamma_{n-1}=0\\
&(g_{n+1}f_{n-1}-g_{n-1}f_{n+1})-(f_{n+1}f_{n-1}-f_n^2)+\frac{m_3}{2}\mathbf{D}_t\gamma_n\bullet\gamma_n+m_4\gamma_{n+1}\gamma_{n-1}=0,
\end{aligned}\tag{14}$$

which can be particularized to both (9) and (13), the integrable time discretizations are found in [20], using the Hirota bilinear method, one of the most powerful methods of finding integrable discretizations [32, 33]. The method implies that starting with a correctly bilinearized integrable differential or differential-difference integrable system (in the sense of allowing the construction of a general multisoliton solution), in the first step, one has to replace differential Hirota operators with discrete ones preserving the gauge invariance. The resulting bilinear fully discrete system is not necessarily integrable, so in the second step, the multisoliton solution must be found [33]. If this exists, then the discrete bilinear system is integrable and, in the final step, the nonlinear form has to be recovered.

The integrable discretization of (14) is:

$$\begin{aligned}
\gamma_n^{m+1}f_n^m-\gamma_n^m f_n^{m+1}&=h(\gamma_{n+1}^{m+1}f_{n-1}^m-\gamma_{n-1}^m f_{n+1}^{m+1}),\\
g_n^{m+1}f_n^m-g_n^m f_n^{m+1}-h(g_{n+1}^{m+1}f_{n-1}^m-g_{n-1}^m f_{n+1}^{m+1})&=m_1\gamma_n^{m+1}\gamma_n^m-hm_2\gamma_{n+1}^{m+1}\gamma_{n-1}^m,\\
h(g_{n+1}^{m+1}f_{n-1}^m-g_{n-1}^m f_{n+1}^{m+1})-h(f_{n+1}^{m+1}f_n^m-f_n^m f_n^{m+1})&=m_3\gamma_n^{m+1}\gamma_n^m-hm_4\gamma_{n+1}^{m+1}\gamma_{n-1}^m.
\end{aligned}$$

For $m_1=m_2=-2$ and $m_3=-1, m_4=1$ the above system admits 3-supersoliton solution with the same form as (10)-(12), but with different quantities for:

$$\eta_i=k_i n+\omega_i hm,\quad b_i=\frac{(1+h)(e_i^k-1)}{(e_i^k+1)},\quad e^{h\omega_i}=\frac{h-e_i^k}{e_i^k(he_i^k-1)}.$$

The nonlinear form of the above superbilinear system is found in [20] as:

$$\begin{aligned}
\psi_n^{m+1}-\psi_n^m&=h\frac{\psi_{n+1}^{m+1}-\psi_{n-1}^m}{1-u_{n+1}^{m+1}+u_{n-1}^m}\\
u_n^{m+1}-u_n^m&=h\frac{u_{n+1}^{m+1}+u_{n-1}^m}{1-u_{n+1}^{m+1}+u_{n-1}^m}+h\frac{2-u_{n+1}^{m+1}-u_{n-1}^m}{(1-u_{n+1}^{m+1}+u_{n-1}^m)^2}(\psi_{n+1}^{m+1}-\psi_{n-1}^m)\times\\
&\quad\times(\psi_n^m-\psi_{n-1}^m),
\end{aligned}$$

which is exactly the form (8) proposed by Xue, Levi and Liu in [19] for potential discrete super-KdV, up to the transformation of the independent variables mentioned in Section 7.

The case for $m_1 = m_2 = -1, m_3 = 0, m_4 = 1$ ia also integrable and the 3-supersoliton solution has the same form as above, except δ_{ij}, that turns into Δ_{ij}. The nonlinear form for discrete super-KdV in:

$$\begin{aligned}
\psi_n^{m+1} - \psi_n^m &= h\frac{\psi_{n+1}^{m+1} - \psi_{n-1}^m}{1 - u_{n+1}^{m+1} + u_{n-1}^m} \\
u_n^{m+1} - u_n^m &= h\frac{u_{n+1}^{m+1} - u_{n-1}^m}{1 - u_{n+1}^{m+1} + u_{n-1}^m} + \\
&+h\frac{(1 - u_{n+1}^{m+1} + u_{n-1}^m)\psi_n^m(\psi_{n+1}^{m+1} - \psi_{n-1}^m) + \psi_{n+1}^{m+1}\psi_{n-1}^m}{(1 - u_{n+1}^{m+1} + u_{n-1}^m)^2}.
\end{aligned}$$

8 Conclusions

This review paper presents the integrable cases of fermionic extensions of the KdV equation both in continuous and discrete context (semidiscrete and fully discrete), discused in the literature. In Section 2, after a brief presentation of the notion of supersymmetry, the extension of the Hirota operator and its most important properties are presented. The next two sections focus on two fermionic extensions of the KdV equation in the continuous context: susy-KdV and Kuper-KdV. Their bilinear form and supersoliton solutions are presented. Also the supersymmetric KdV hierarchy is listed together with its bilinearisation. Section 5 compares the two equations from the point of view of the supersoliton interaction. Other supersymmetric and non-supersymmetric KdV-type equations are briefly presented in section 6. The semidiscrete fermionic extension of KdV and potential KdV and their full discretisations are presented in Section 7.

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