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# Radial diffusion of inhomogeneous sheared stochastic magnetic field lines in tokamak plasma

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**Abstract**

The model developed in our paper considers a stochastic magnetic field that contains two linear deterministic terms representing the gradient of the magnetic field and the shear term and also one fluctuating term that are described by the dimensionless function $b_x(X, Y, Z)$ taken to be Gaussian processes and that are perpendicular to the main magnetic field $B_0$. We have calculated the radial diffusion coefficient $D_{xx}(z)$ of magnetic field lines for various parameters: the magnetic Kubo number $K_M$, the inhomogeneous parameter $K_B$ and the magnetic shear Kubo number $K_S$.

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## 1 Introduction

An important feature in plasma physics is the study of the diffusion of magnetic field lines in tokamak. A simplified model used in our paper is a Taylor expansion of the stochastic magnetic field that contains two linear deterministic terms representing the gradient of the magnetic field and the magnetic shear term and also a fluctuating term that is described by the dimensionless function $b_x(X, Y, Z)$, taken to be a Gaussian process and that is perpendicular to the main magnetic field $B_0$. The framework has been set in a previous paper [1]. The magnetic fluctuation, even when small, can destroy the nested magnetic and thus enhancing the radial transport. Also, the presence of an inhomogeneity of the magnetic field and of the magnetic shear, can be important keys in order to explain the observed increase of the poloidal flow which is equivalent to the appearance of a transport barrier. In our paper we employed the main tools of the Deccorelation Trajectory Method whose idea concerns in the study of the Langevin system not in the whole space of the realizations of the potential fluctuations; the whole space is subdivided into *subensembles* $S$, characterized by given values of the fluctuating field components at the starting point of the trajectories. The exact expression of the radial Lagrangian correlation can be written in the form of a superposition of Lagrangian correlations in various subensembles.

The main contribution of the paper is the evaluation of radial diffusion of inhomogeneous sheared stochastic magnetic field lines, in the framework of tokamak plasma, using appropriate values of Kubo numebers (see below).

## 2 The model and the Langevin equations

The local expansion of the stochastic magnetic field that considers the perpendicular variations is:

$$\mathbf{B}(X,Y,Z) = B_0\left\{\left[1+XL_B^{-1}\right]\mathbf{e}_z + \beta b_x\left(X,Y,Z\right)\mathbf{e}_x + XL_s^{-1}\mathbf{e}_y\right\} \tag{1}$$

where $\beta$ is a dimensionless parameter measuring the amplitude of the magnetic field fluctuation relative to the main magnetic field $B_0$. There are two linear terms depending on $X$ in the right hand side of eq.(1): the shear term $XL_S^{-1}$ where $L_S$ is the shear length and the nonhomogeneous term $XL_B^{-1}$ where $L_B$ is the gradient scale length, which are the radial distance on $Ox$ axis over the magnitude of the magnetic field would double in this linear model. We will define the term $B_0\left[1+XL_B^{-1}\right]\mathbf{e}_z$ as the gradient $\mathbf{B}$ term. Because the model expression given in (1) represents a Taylor series expansion of the magnetic field it is only valid for small distances from the origin, i.e. are valid the following approximations:

$$XL_B^{-1} << 1\ ,\quad XL_S^{-1} << 1 \tag{2}$$

The magnetic field lines corresponding to the definition given in (1) are:

$$\frac{dX}{\beta B_0 b_x} = \frac{dY}{B_0 X L_S^{-1}} = \frac{dZ}{B_0\left(1+XL_B^{-1}\right)} \tag{3}$$

Using (2), i.e. considering that

$$\frac{1}{1+XL_B^{-1}} \simeq 1 - XL_B^{-1}$$

and neglecting the terms quadratic in $X$ we obtain from (3) the dimensional Langevin system of equations for the magnetic field lines

$$\frac{dX}{dZ} = \beta b_x - \beta b_x X L_B^{-1} \tag{4}$$

$$\frac{dY}{dZ} = XL_S^{-1} \tag{5}$$

where the coordinate $Z$ plays the role of time.

We introduce the dimensionless coordinates $\mathbf{x} = (x,\ y,\ z)$ which are related to the dimensional ones by the relations:

$$x = \frac{X}{\lambda_\perp},\quad y = \frac{Y}{\lambda_\perp},\quad z = \frac{Z}{\lambda_\parallel} \tag{6}$$

The magnetic field given in Eq.(1) satisfies the zero-divergence constraint $\nabla \cdot \mathbf{B} = 0$ that is imposed by Maxwell's equations. This condition is automatically fulfilled if we consider that the fluctuating magnetic field derives from the following vector potential which has only the $z$ - component:

$$\mathbf{A}(\mathbf{X};Z) = B_0\lambda_\perp\beta\ \psi(\mathbf{x};z)\ \mathbf{e}_z \tag{7}$$

We have the following relation between the fluctuating part of the magnetic field and the magnetic potential $\psi(\mathbf{x};z)$:

$$b_x[\mathbf{x}(z);z] = \left.\frac{\partial\psi(\mathbf{x}(z);z)}{\partial y}\right|_{\mathbf{x}=\mathbf{x}(z)}$$

and the dimensionless system equivalent to the one given in (4, 5) is:

$$\frac{dx}{dz} = (1 - xK_M K_B)\, b_x \equiv K_{MB}(x)\, b_x \equiv w_x \tag{8}$$

$$\frac{dy}{dz} = xK_S \tag{9}$$

where we have defined the linear function in the radial coordinate $x$

$$K_{MB}(x) = 1 - xK_M K_B \tag{10}$$

and the directly fluctuating velocity

$$w_x = K_{MB}(x) \frac{\partial \psi}{\partial y}$$

The function $K_{MB}(x)$ depends on the radial coordinate $x$ and we will suppose to be not directly fluctuating quantity as the component of the magnetic field are. In the system (8-9) the following Kubo numbers are introduced:

1. The magnetic Kubo number $$K_M = \beta \frac{\lambda_\parallel}{\lambda_\perp} \tag{11}$$

2. The shear Kubo number $$K_S = \frac{\lambda_\perp}{L_S} \tag{12}$$

3. The inhomogeneous Kubo number $$K_B = \frac{\lambda_\parallel}{L_B} \tag{13}$$

The Langevin Eqs.(8 - 9) will be used in order to calculate the running and asymptotic diffusion coefficient of the magnetic field lines for different values of the Kubo numbers. The Lagrangian correlation (which is the main tool for determining the running and asymptotic diffusion coefficient) of the directly fluctuating "velocity" $w_x[\mathbf{x}(z);\ z]$ is defined as:

$$L_{xx}(z) = \langle w_x(\mathbf{x}(0); 0) w_x\left[\mathbf{x}(z); z\right]\rangle \tag{14}$$

or

$$L_{xx}(z) = K_{MB}^2\left[x(z)\right] \langle b_x(\mathbf{x}(0); 0) b_x\left[\mathbf{x}(z); z\right]\rangle$$

where $\langle ... \rangle$ denotes the ensemble average over the realizations of the fluctuating magnetic field component and the function $K_{MB}^2\left[x(z)\right]$ was factorized from the average because is not a directly fluctuating quantity. The running diffusion coefficient matrix is calculated using (14) as:

$$D_{xx}(z) = \int_0^z d\zeta L_{xx}(\zeta) \tag{15}$$

provided that the stochastic field is "stationary"; the corresponding asymptotic diffusion coefficient is then:

$$D_{xx}^{as} = \lim_{z \to \infty} D_{xx}(z) \tag{16}$$

An important simplification of the calculus can be done if a relation between the Lagrangian correlation and the corresponding Eulerian one can be established. Unfortunately, until now, does not exist a general exact relation between these two types of correlations, valid for both weak and strong turbulence regime. For a weak magnetic turbulence regime ($K_M < 1$) an approximate formula which relates the two types of correlations already exists: this is the celebrated Corrsin approximation [8] which includes the quasilinear and the Bohm approximations. The Corrsin approximation which amounts to decouple the stochastic magnetic field line position to the stochastic magnetic field is reproduced here for convenience:

$$L_{xx}(z) = \int d\mathbf{x} \langle w_x(\mathbf{x}(0); 0) w_x [\mathbf{x}(z); z] \, \delta [\mathbf{x} - \mathbf{x}(z)] \rangle \simeq$$

$$\overset{Corrs}{\simeq} \int d\mathbf{x} \langle w_x(\mathbf{x}(0); 0) w_x [\mathbf{x}(z); z] \rangle \langle \delta [\mathbf{x} - \mathbf{x}(z)] \rangle \tag{17}$$

where at least in some asymptotic sense, the exact propagator $\delta [\mathbf{x} - \mathbf{x}(z)]$ is approximated by its ensemble average.

# 3 The DCT tools

In our paper we closely follow the results obtained in [5]. The DCT method main idea concerns in the study of the Langevin system (8-9) not in the whole space of the realizations of the potential fluctuation; the whole space is subdivided into *subensembles* $S$, characterized by given values of the potential and of the fluctuating field component at the starting point of the trajectories. The exact expression of the radial Lagrangian correlation can be written in the form of a superposition of Lagrangian correlations in various subensembles. The validity of the approximation involved in DCT method can be assessed by *a posteriori* comparison with experiment and simulations, as is done in all theories of strong turbulence.

The DCT method is now systematically developed for the present problem. We first define a set of subensembles $S$ of the realizations of the stochastic sheared magnetic field that are defined by given values of the potential $\psi$ and magnetic field fluctuation $\mathbf{b}$ in the point $\mathbf{x} = 0$ at the "moment" $z = 0$:

$$\psi(\mathbf{0};\ 0) = \psi^0, \ \ b_x(\mathbf{0};\ 0) = b_x^0 \tag{18}$$

The correlation of the Lagrangian fluctuating components of the magnetic field can be represented as a sum over the subensembles $S$ of the correlations $L_{xx}^S(z)$ calculated in each subensemble:

$$L_{xx}(z) = \int d\psi^0 db_x^0 P(b_x^0,\ \psi^0) K_{MB}^2 [x(z)] \langle b_x(\mathbf{0};\ 0) b_x [\mathbf{x}(z);\ z] \rangle^S \tag{19}$$

where

$$P(b_x^0,\ \psi^0) = P(b_x^0) P(\psi^0) = (2\pi)^{-1} \exp \left[ -\frac{\left(\psi^0\right)^2 + (b_x^0)^2}{2} \right] \tag{20}$$

is the probability density of $(b_x^0,\ \psi)$ having the values $\left(b_x^0, \psi^0\right)$ at $\mathbf{x} = 0$ and at the "moment" $z = 0$. Since the initial fluctuating fields in the subensemble $S$ are $b_x(\mathbf{0};\ 0) = b_x^0$ for all trajectories, the subensemble average defined in (19) is:

$$\langle b_x(\mathbf{0};\ 0) b_x [\mathbf{x}(z);\ z] \rangle^S = b_x^0 \langle b_x [\mathbf{x}(z);\ z] \rangle^S \tag{21}$$

and thus the Lagrangian correlation $L_{xx}(z)$ is simply the weighted average Lagrangian of the fluctuating field in all subensembles. We need first to calculate the average Eulerian fields $b_x$ in the subensemble $S$:

$$b_x^S(\mathbf{x};\ z) \equiv \langle b_x\left(\mathbf{x};\ z\right)\rangle^S \tag{22}$$

The next step in the DCT method is to define a deterministic trajectory in each subensemble as a solution of the system (8-9) that becomes

$$\frac{dx^S(z)}{dz} = \left(1 - x^S K_M K_B\right) b_x^S\left[\mathbf{x}^S(z);\ z\right]$$

$$\frac{dy^S(z)}{dz} = x^S K_S$$

in which the right hand sides are replaced by the average fields $b_x^S$ in the subensemble. It can be seen also that the "Kubo number" $K\left(x\right) = \left(1 - xK_M K_B\right)$ differs from a subensemble to another. We have made the following assumption: in a subensemble $S$ we factorize the average of the terms of the form

$$\langle K\left(x\right) b_x\left(\mathbf{x};\ z\right)\rangle^S \equiv K\left(x^S\right) b_x^S\left(\mathbf{x}^S;\ z\right)$$

The system used in the DCT calculations can be formally written in the following compact form

$$\frac{dx^S}{dz} = K\left(x^S\right) b_x^S \tag{23}$$

$$\frac{dy^S}{dz} = x^S K_S \tag{24}$$

We need first to calculate the average Eulerian fields $b_x$ in the subensemble $S$:

$$b_x^S(\mathbf{x};\ z) \equiv \langle b_x\left(\mathbf{x};\ z\right)\rangle^S \tag{25}$$

The DCT method is now systematically developed for the present problem. We first define a set of subensembles $S$ of the realizations of the stochastic sheared magnetic field that are defined by given values of the potential $\psi$ and magnetic field fluctuation $\mathbf{b}$ in the point $\mathbf{x} = 0$ at the "moment" $z = 0$:

$$\psi(\mathbf{0};\ 0) = \psi^0,\ \ b_x(\mathbf{0};\ 0) = b_x^0 \tag{26}$$

The fluctuating magnetic potential $\psi(\mathbf{x}; z)$ is assumed to be a Gaussian stochastic process with zero average. The second order moment of $\psi(\mathbf{x}; z)$, *i.e.,* its Eulerian autocorrelation function $M(\mathbf{x}; z)$ is assumed to have the following factorized form:

$$M(\mathbf{x}; z) = \langle \psi(\mathbf{0}; 0)\psi(\mathbf{x}; z)\rangle = M_1(\mathbf{x})M_2(z) \tag{27}$$

where:

$$M_1(\mathbf{x}) = \exp(-\frac{\mathbf{x}^2}{2}), \quad M_2(z) = \exp(-\frac{z^2}{2}) \tag{28}$$

The mixed Eulerian correlations between the potential and the fluctuating magnetic field components are defined in [6] as:

$$M_{\psi x}(\mathbf{x}; z) = \langle \psi(\mathbf{0}; 0) b_x(\mathbf{x}; z)\rangle$$

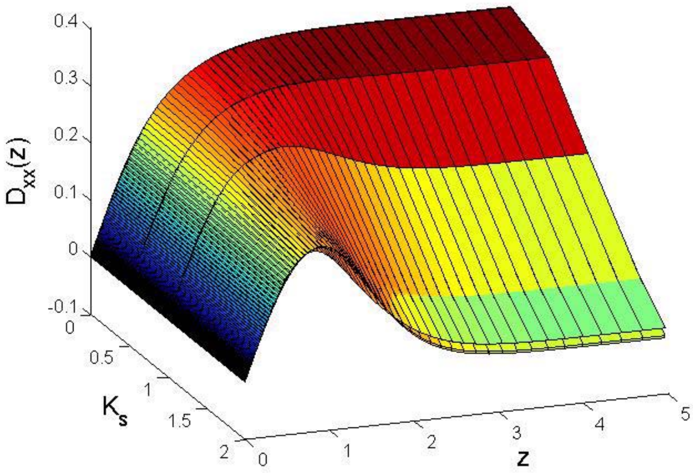


Figure 1: Radial running diffusion coefficient for $K_M = 0.5$ and various values of $K_S$

$$M_{x\psi}(\mathbf{x}; z) = \langle b_x(\mathbf{0}; 0)\psi(\mathbf{x}; z)\rangle \tag{29}$$

and the following relations between these correlations hold [6]:

$$M_{\psi x}(\mathbf{x}; z) = -M_{x\psi}(\mathbf{x}; z) = \frac{\partial M(\mathbf{x}; z)}{\partial y} = -yM(\mathbf{x}; z)$$

The dimensionless fluctuating magnetic field autocorrelation tensor components are derived from $M(\mathbf{x}; z)$ as [6], [7]:

$$M_{xx}(\mathbf{x}; z) = -\frac{\partial^2 M(\mathbf{x}; z)}{\partial y^2} = \left(1 - y^2\right) M(\mathbf{x}; z)$$

The Eulerian average of the radial fluctuation are:

$$\begin{aligned} b_x^S\left(\mathbf{x}^S;\ z\right) &= \psi^0 M_{\psi x}\left(\mathbf{x}^S;\ z\right) + b_x^0 M_{xx}\left(\mathbf{x}^S;\ z\right) \equiv \\ &\equiv \left[-\psi^0 y^S + b_x^0\left(1 - y^{S\ 2}\right)\right] M(\mathbf{x}^S; z) \end{aligned} \tag{30}$$

The Lagrangian correlation tensor has the following components:

$$L_{xx}(z) = \int\limits_{-\infty}^{\infty} d\psi^0 \int\limits_{-\infty}^{\infty} db_x^0 P(b_x^0,\ \psi^0) K_{MB}^2\left[x^S(z)\right] b_x^0 b_x^S\left[\mathbf{x}^S(z);\ z\right] \tag{31}$$

# 4 Conclusion

In this paper we have calculated the diffusion coefficients for the radial stochastic, inhomogeneous and sheared magnetic field lines. Different Kubo numbers involved in the model have influenced the diffusion. Some results were obtained in [5] for the $2D$-anisotropic case but not in the inhomogeneous case.

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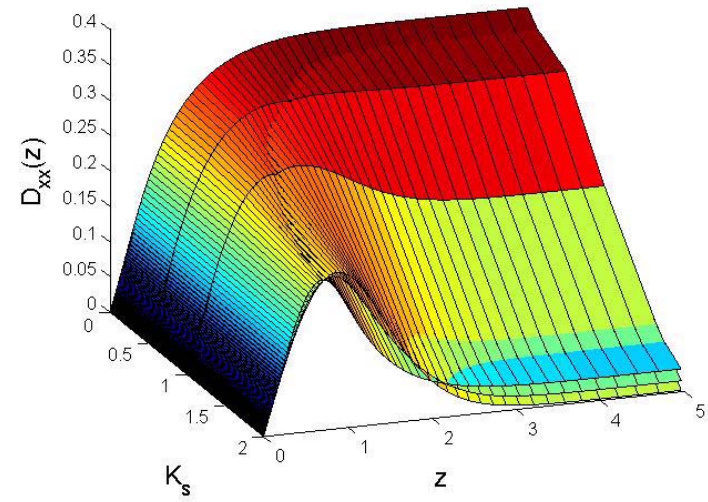


Figure 2: Radial running diffusion coefficient for $K_M = 1.0$ and various values of $K_S$

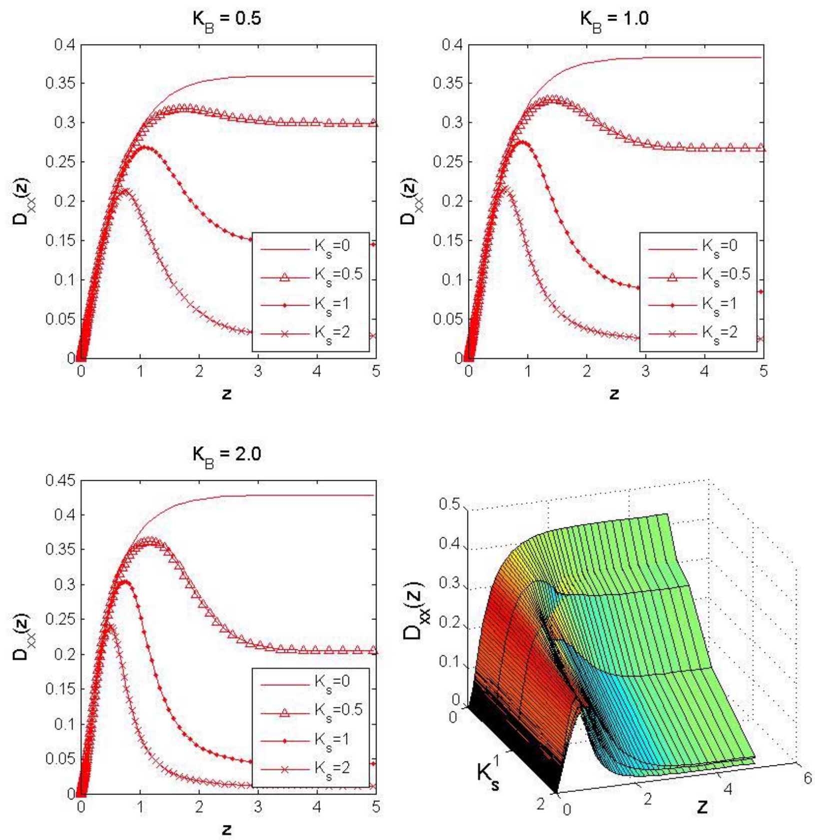


Figure 3: Radial running diffusion coefficient for $K_M = 3.0$ and various values of $K_S$ and/or $K_B$

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