New travelling wave solutions for the Fisher equation with finite memory effects

Vily Marius Cimpoiasu, Rodica Cimpoiasu*
University of Craiova, 13 A.I.Cuza, 200585 Craiova, Romania

Abstract

The first integral method is employed for describing memory effects in diffusion-reaction processes. Using this method we construct new exact travelling wave solutions for the Fisher equation in the presence of a nonlinear convection term with a finite memory transport. A rich variety of solutions may be obtained, including rational wave solutions, trigonometric, cnoidal and snoidal-type solutions or kink-shaped soliton ones. The existence of these solutions is proved under certain parametric domains. The present method might also successfully solve other high-dimensional nonlinear partial differential equations with various types of nonlinearity.

Keywords: First integral method, symbolic computation, Fisher equation, finite memory effect, traveling wave solutions

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1 Introduction

Many phenomena raised in science and engineering may be modeled through nonlinear partial differential equations (NPDEs). They offer an important tool to better understand the evolution paths and allow the use of computer symbolic systems like Maple and Mathematica to perform some rather complicated and tedious algebraic calculations.

Dispersion, dissipation, diffusion, reaction and convection are examples of such phenomena whose influence on the propagation of the nonlinear waves will be considered here. A variety of powerful methods, such as the inverse scattering method [1], Bäcklund transformation [2], Darboux transformation [3], Hirota's bilinear method [4], the dressing method [5], the homogeneous balance method [6], the (G'/G)-expansion method [7], the Lie symmetry reduction [8, 9, 10, 11], the generalized conditional symmetry method [12, 13, 14], the sine-cosine method [15], various extended tanh-methods [16, 17] and the first integral method [18] have been made use of in order to obtain the travelling wave solutions of NPDEs.

The first integral method (FIM) has been successfully implemented to various NPDEs and to some fractional differential equations in many studies such as [19, 20, 21, 22].

A lot of attention has also attracted the analysis of memory effects in diffusive processes as in [23, 24, 25, 26]. Fick's law [27] constitutes the key element in the description of transport. However, this description is significantly modified when memory effects are

*Corresponding author: rodicimp@central.ucv.ro

taken into account, i.e. when the dispersal of particles is not mutually independent. This fact implies that, for a given concentration gradient, the correlation between the successive movements of diffusing particles may be understood as a delay in the flux .

The diffusion equation states that the flux of diffusing particles in whatever part of the system is proportional to the density gradient: $J(x,t) = -D\frac{\partial u(x,t)}{\partial x}$, where $u(x,t)$ is the particles'concentration and $J(x,t)$ is the flux of the diffusing particles. If we also use the continuity equation, $\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}$, we get the one-dimensional Fick's law in the form $\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$. Here D is the diffusion coefficient. The memory effect appears when the dispersal of the particle is mutually not independent [26].

When the memory effect is taken into account, then we have the following modification of Fick's law in the presence of the nonlinear convection term

$$J(x,t+\tau) = -D\frac{\partial u(x,t)}{\partial x} + vu^2, \tag{1}$$

which takes care of the adjustment of a concentration gradient at time t with a flux $J(x,t+\tau)$ at a later time $(t+\tau)$. Here τ is the delay time of the particles taken to adopt a precised direction of propagation and v is the coefficient of the nonlinear convective flux term.

By expanding J in Eq. (1) up to first order in τ, one obtain:

$$J(x,t) + \tau\frac{\partial J(x,t)}{\partial \tau} = -D\frac{\partial u(x,t)}{\partial x} + vu^2. \tag{2}$$

On the other hand, the population balance equation for the particles takes into account the conservation of the equation supplemented by a source function $f(u)$ for the particles in the form:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} + f(u), \tag{3}$$

Now, differentiate Eq. (2) with respect to x, Eq. (3) with respect to t and eliminate $J(x,t)$ from the resulting expression, we obtain:

$$u_{2t} - \beta D u_{2x} - f'(u)u_t + \beta(u_t - f(u)) + k\beta u u_x = 0, \tag{4}$$

where the denotations $\beta \equiv 1/\tau$, $k \equiv 2v$ and $f'(u) = \frac{df}{du}$. are made use of. Let us remark that Eq. (4) is an hyperbolic reaction-diffusion equation and that it is a generalization of Fisher equation for finite memory transport and nonlinear damping. It describes a transport phenomenon in which both diffusion and convection processes are of an equal importance. It reduces to Burgers equation [28] in the absence of the source term ($f(u) = 0$). For $\tau = 0$ it becomes the standard Fisher equation, while for $\tau = 0$ and $f(u) = 0$ it turns into Burgers equation [24].

Our main interest in the present work is to implement the FIM in order to prove once again its power of handling nonlinear equations, so that we could apply it to models with various types of nonlinearity. By making use of the Division Theorem for polynomials, we described this method in order to find exact travelling wave solutions for nonlinear PDEs. In Section 3, the algorithmic method with a symbolic computation is applied in detail upon Fisher equation with a finite memory effect. Some of the obtained solutions are new, to the best of our knowledge. Finally, some essential facts are pointed out in concluding remarks.

2 First integral method for Fisher equation with memory effect

The procedure for making use of the first integral method may be summarized as follows:

Step1: Consider a general NPDE for the physical field $u(t, x, u)$ given by:

$$E(u,\ u_x,\ u_t,\ u_{2x},\ u_{xt},\ u_{tt}, ...) = 0. \tag{5}$$

Using a wave variable $\xi = x \pm ct$, we could rewrite Eq. (5) in the following nonlinear ODE:

$$G(U,\ U^{'},\ U'',\ U^{'''}, ...) = 0, \tag{6}$$

where the "prime" index denotes the derivation with respect to ξ.

Step2: We assume that Eq. (6) has a solution of the form:

$$U(\xi) \equiv X(\xi) = u(x, t), \tag{7}$$

and introduce a new independent variable $Y = X^{'}(\xi)$, which leads us towards a new system of ODEs:

$$X^{'}(\xi) = Y(\xi),\ Y^{'}(\xi) = P(X(\xi), Y(\xi)). \tag{8}$$

Step3: In accordance with the qualitative theory of ODEs [29], if it is possible to obtain the first integrals for the system (8), the solutions of the system could be immediately obtained. The division theorem (DT) [30] provides us an idea about how to obtain such first integrals.

Division Theorem: Suppose that $F(x, y)$, $G(x, y)$ are polynomials in the complex domain $\mathbf{C}[x, y]$ and that $F(x, y)$ is irreducible in $\mathbf{C}[x, y]$. If $G(x, y)$ vanishes at all the zero points of $F(x, y)$, then there exists a polynomial $H(x, y)$ in $\mathbf{C}[x, y]$ such that

$$G(x, y) = H(x, y)F(x, y). \tag{9}$$

In this section, we illustrate the efficiency of the previously mentioned algorithmic method upon Eq. (1). Using the wave variable $\xi = x - ct$, the master Eq. (1) turns into the following ODE:

$$U^{''} + (A\dot{f}(U) - B + QU)U^{'} - Mf(U) = 0, \tag{10}$$

where "dot" denotes d/dU and the parameters A, B, M, Q are expressed by

$$A = \frac{c}{c^2 - \beta D},\ B = \frac{\beta c}{c^2 - \beta D},\ M = \frac{\beta}{c^2 - \beta D},\ Q = \frac{k\beta}{c^2 - \beta D} \tag{11}$$

Using (7) and (8) we get:

$$\begin{aligned} X^{'}(\xi) &= Y(\xi), \\ Y^{'}(\xi) &= (-A\dot{f}(X) + B - QX)Y + Mf(X). \end{aligned} \tag{12}$$

In accordance with the FIM, it is supposed that $X(\xi)$ and $Y(\xi)$ are non-trivial solutions of the system (12) and that $q(X, Y) = \sum_{i=0}^{i=m} a_i(X)Y^i$ is an irreducible function in the domain $\mathbf{C}[X, Y]$ such that

$$q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X)Y^i = 0, \tag{13}$$

where $a_i(X)$, $i = \overline{1,m}$ are polynomials of X and $a_m(X) \neq 0$. Eq. (13) is known as the first integral to the system (12). Due to the Division Theorem, there exists a polynomial $\rho(X) + \sigma(X)Y$ in the complex domain $\mathbf{C}[X,Y]$ such that

$$\frac{dq}{d\xi} = \frac{dq}{dX}\frac{dX}{d\xi} + \frac{dq}{dY}\frac{dY}{d\xi} = [\rho(X) + \sigma(X)Y]\sum_{i=0}^{m} a_i(X)Y^i. \tag{14}$$

3 Travelling wave solutions for Eq. (4)

In this study, let us take into consideration two different cases, by assuming that $m = 1$ and respectively $m = 2$ in Eq. (14).

3.1 Travelling wave solutions in the case $m = 1$

In this case, by equating the coefficients of Y^i, $i = 0, 1, 2$ on both sides of Eq. (14), we have:

$$\begin{aligned} \dot{a}_1(X) &= \sigma(X)a_1(X), \\ \dot{a}_0(X) &= Aa_1(X)\dot{f}(X) - Ba_1(X) + QXa_1(X) \\ &+\rho(X)a_1(X) + \sigma(X)a_0(x), \\ Ma_1(X)f(X) &= \rho(X)a_0(X). \end{aligned} \tag{15}$$

Since $a_i(X)$, $i = 0, 1$ are polynomials, we may deduce from the first equation of (15) that $a_1(X)$ is a constant and $\sigma(X) = 0$. For simplicity reasons, let us consider $a_1(X) = 1$.The system (15) becomes:

$$\begin{aligned} \dot{a}_0(X) &= A\dot{f}(X) - B + QX + \rho(X), \\ Mf(X) &= \rho(X)a_0(X). \end{aligned} \tag{16}$$

Let us choose $f(X) = e + sX + wX^2$, $a_0(X) = q + rX + vX^2$, $\rho(X) = p = const.$ and substitute them, as well as the parameters A, B, M, Q from (11) into the system (16). Equating the coefficients of X^j, $j = \overline{0,2}$ to zero, we get an algebraic system which admits the solution:

$$\begin{aligned} q &= -\frac{2ewk(s+\beta)}{4\beta Dw^2 - k^2s^2}, \; r = -\frac{2swk(s+\beta)}{4\beta Dw^2 - k^2s^2}, \\ v &= -\frac{2w^2k(s+\beta)}{4\beta Dw^2 - k^2s^2}, \; c = -\frac{\beta(4w^2D + sk^2)}{2wk(s+\beta)}, \end{aligned} \tag{17}$$

where we have to impose the constraints $\frac{w^2}{s^2} \neq \frac{k^2}{4\beta D}$, $s \neq -\beta < 0$, s, $w \neq 0$, $k, D > 0$, $\frac{D}{k^2} \neq -\frac{s}{4w^2}$, $\frac{4w^2D+sk^2}{w(s+\beta)} < 0$. The solution (17) is expressed in terms of the coefficients β, D, k, e, s, w involved into the concerned model (4).

Combining (17) with (13) and then with (12), we have to solve the integral:

$$X'(\xi) = -q - rX(\xi) - vX^2(\xi). \tag{18}$$

Let us consider $\Delta = r^2 - 4qv = \frac{-4w^2k^2(s+\beta)^2(4ew-s^2)}{(4\beta Dw^2-k^2s^2)^2}$. Under the conditions (17), there cases are to be discussed.

Case (i) : $\Delta = 0 \Leftrightarrow s^2 = 4ew$. Then a rational-type solution of (18) is obtained:

$$X_1(\xi) = -\frac{1}{v(\xi - \xi_0)} - \frac{r}{2v}. \tag{19}$$

Case (ii) : $\Delta > 0 \Leftrightarrow s^2 > 4ew$. Then the hyperbolic-type solutions of (18) are pointed out:

$$
\begin{aligned}
X_2(\xi) &= -\frac{r}{2v} - \frac{\sqrt{\Delta}}{2v}\tanh\left[\frac{\sqrt{\Delta}}{2}(\xi-\xi_0)\right],\\
X_3(\xi) &= -\frac{r}{2v} + \frac{\sqrt{\Delta}}{2v}\coth\left[\frac{\sqrt{\Delta}}{2}(\xi-\xi_0)\right].
\end{aligned} \tag{20}
$$

Case (iii) : $\Delta < 0 \Leftrightarrow s^2 < 4ew$. Eq. (18) admits a trigonometric solution with the form:

$$
X_4(\xi) = -\frac{r}{2v} - \frac{\sqrt{-\Delta}}{2v}\tan\left[\frac{\sqrt{-\Delta}}{2}(\xi-\xi_0)\right]. \tag{21}
$$

3.2 Travelling wave solutions in the case $m = 2$

Suppose that $m = 2$, by equating the coefficients of Y^i, $i = 0, 1, 2, 3$ on both sides of Eq. (14), we have:

$$
\begin{gathered}
\dot{a}_2(X) = \sigma(X)a_2(X),\\
\dot{a}_1(X) = 2Aa_2(X)\dot{f}(X) - 2Ba_2(X) + 2QXa_2(X) + \sigma(X)a_1(X),\\
\dot{a}_0(X) = Aa_1(X)\dot{f}(X) - Ba_1(X) + QXa_1(X)\\
-2Mf(X)a_2(X) + a_1(X)\rho(X) + \sigma(X)a_0(X),\\
Ma_1(X)f(X) = a_0(X)\rho(X),
\end{gathered} \tag{22}
$$

where the parameters A, B, M, Q are provided by (11).

Since $a_i(X)$, $i = 0,\ 1,\ 2$ are polynomials, the first equation from (22) leads to the conditions $a_2(X) = const.$ and $\sigma(X) = 0$. For simplicity reasons, let us consider $a_2(X) = 1$; therefore, the remaining equations could be written as:

$$
\begin{gathered}
\dot{a}_1(X) = 2A\dot{f}(X) - 2B + 2QX,\\
\dot{a}_0(X) = Aa_1(X)\dot{f}(X) - Ba_1(X)\\
+QXa_1(X) - 2Mf(X) + a_1(X)\rho(X),\\
Ma_1(X)f(X) = a_0(X)\rho(X).
\end{gathered} \tag{23}
$$

Considering again $\deg(f(X)) = 2$ and balancing the degrees of $\rho(X)$, $a_0(X)$ and $a_1(X)$, we concluded that $\deg(\rho(X)) = 0$ only. The degrees of $a_0(X)$ and $a_1(X)$ could be evaluated as follows: (1) $\rho(X) = 0$ implies $a_1(X) \equiv 0$ and $\deg(a_0(X)) = 3$ while (2) $\rho(X) = v = const. \neq 0$ leads to $\deg(a_0(X)) = 4$ and $\deg(a_1(X)) = 2$. We ought to solve the previous system in these two cases separately.

Case 1: Suppose $\rho(X) = 0$, $a_1(X) \equiv 0$, $a_0(X) = d + gX + hX^2 + nX^3$ and $f(X) = e + sX + wX^2$. Substituting them together with (11) into the former two equations of the system (23), equating the coefficients of various powers of X, we get:

$$
e = \frac{\beta g}{2h},\ w = \frac{3\beta n}{2h},\ D = \frac{hc^2+\beta^2}{h\beta},\ k = \frac{9cn}{2h}, \tag{24}
$$

available for the arbitrary non-zero parameters β, c, s, d, g, h, n.

Combining the conditions (24) with Eq. (14) and with the first equation from (12), we conclude that the travelling wave solutions of Eq. (4) could be attributed to the resolution of the following first order integrable ODE:

$$[X'(\xi)]^2 = -d - gX - hX^2 - nX^3. \tag{25}$$

With the change of variable and the one of parameters:

$$z = (-n)^{1/3}X,\ d_2 = (-h)(-n)^{-2/3},\ d_1 = (-g)(-n)^{-1/3},\ d_0 = -d, \tag{26}$$

the ODE (25) becomes:

$$\pm(-n)^{1/3}(\xi - \xi_0) = \int \frac{dz}{\sqrt{z^3 + d_2 z^2 + d_1 z + d_0}}. \tag{27}$$

The complete discrimination system [31] for the third degree polynomial $P(z) = z^3 + d_2 z^2 + d_1 z + d_0$ is provided by:

$$\Delta = -27\left[\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_2}{3}\right]^2 - 4\left(d_1 - \frac{d_2^2}{3}\right)^3,\ D_1 = d_1 - \frac{d_2^2}{3}. \tag{28}$$

Here are the following four cases to be discussed. Let us only present the corresponding solutions and leave the details to be seen in [32].

Case (i) : $\Delta = 0$ and $D_1 < 0.$, We have $P(z) = (z-\alpha)^2(z-\gamma)$, $\alpha \neq \gamma$. If $z > \gamma$, the solutions are given as follows:

$$\begin{gathered} X = (-n)^{-1/3}\left[(\alpha-\gamma)\tanh^2\left(\frac{\sqrt{\alpha-\gamma}}{2}(-n)^{1/3}(\xi-\xi_0)\right) + \gamma\right], \\ X = (-n)^{-1/3}\left[(\alpha-\gamma)\coth^2\left(\frac{\sqrt{\alpha-\gamma}}{2}(-n)^{1/3}(\xi-\xi_0)\right) + \gamma\right], \\ \alpha > \gamma, \\ X = (-n)^{-1/3}\left[(-\alpha+\gamma)\tan^2\left(\frac{\sqrt{-\alpha+\gamma}}{2}(-n)^{1/3}(\xi-\xi_0)\right) + \gamma\right], \\ \alpha < \gamma. \end{gathered} \tag{29}$$

Case (ii) : $\Delta = 0$ and $D_1 = 0$. We have $P(z) = (z-\alpha)^3$. The solution admits the form:

$$X = 4(-n)^{-2/3}(\xi - \xi_0)^{-2} + \alpha. \tag{30}$$

Case (iii) : $\Delta > 0$ and $D_1 < 0$. We see that $P(z) = (z-\alpha)(z-\gamma)(z-\zeta)$. Let us suppose that $\alpha < \gamma < \zeta$.

When $z \in (\alpha, \gamma)$ we have:

$$X = (-n)^{-1/3}\left[\alpha + (\gamma-\alpha)sn^2\left(\frac{\sqrt{\zeta-\alpha}}{2}(-n)^{1/3}(\xi-\xi_0), l\right)\right]. \tag{31}$$

When $z > \zeta$, we have:

$$X = (-n)^{-1/3}\left[\frac{\zeta - \gamma sn^2\left(\frac{\sqrt{\zeta-\alpha}}{2}(-n)^{1/3}(\xi-\xi_0), l\right)}{cn^2\left(\frac{\sqrt{\zeta-\alpha}}{2}(-n)^{1/3}(\xi-\xi_0), l\right)}\right], \tag{32}$$

where $l^2 = \frac{\gamma-\alpha}{\zeta-\alpha}$.

Case (iv) : $\Delta < 0$. We could introduce $P(z) = (z-\alpha)(z^2+\mu z+\varphi)$, $\mu^2 - 4\varphi < 0$. Futhermore, we obtain:

$$X = (-n)^{-1/3}\left[\alpha - \sqrt{\alpha^2+\mu\alpha+\varphi} + \frac{2\sqrt{\alpha^2+\mu\alpha+\varphi}}{1+cn\left((\alpha^2+\mu\alpha+\varphi)^{1/4}(-n)^{1/3}(\xi-\xi_0),l\right)}\right],$$
$$l^2 = \frac{1}{2}\left(1 - \frac{\alpha+\frac{\mu}{2}}{\sqrt{\alpha^2+\mu\alpha+\varphi}}\right). \tag{33}$$

Case 2: Suppose $\rho(X) = v \neq 0$, $a_1(X) = q+rX+aX^2$, $a_0(X) = d+gX+hX^2+nX^3+pX^4$ and $f(X) = e+sX+wX^2$. Substituting these expressions into the system (23) and then equating the coefficients of various powers of X, wherefrom an algebraic system with 11 equations results. From theformer 6 equations it could be useful to express a, q, r, p, h, g with respect to n, v, e, s, w, A, B, M, Q. This partial result, together with (11) would then be inserted into the remaining equations. Should we solve the final algebraic system through the Maple program, an interesting solution would be pointed out:

$$\begin{aligned} s = n = 0,\ e = \frac{-2v\beta c}{vc+4\beta},\ d = \frac{4\beta v^2(vc+2\beta)^2}{(vc+4\beta)^3}, \\ k = \frac{-2w(vc-2\beta)}{v\beta},\ D = \frac{c(c^2v^2+\beta vc-4\beta^2)}{v\beta(vc+2\beta)}, \end{aligned} \tag{34}$$

available for the non-zero arbitrary constants v, w, β and for the wave velocity $c \neq \left\{\frac{-4\beta}{v}, \frac{\pm 2\beta}{v}, 0, \frac{-\beta(v\mp\sqrt{17}|v|)}{2v^2}\right\}$..

Futhermore, the conditions (34) generate for a, q, r, p, h, g the concrete expressions:

$$\begin{aligned} a = \frac{w(vc+2\beta)}{\beta c},\ g = r = 0,\ q = \frac{-2v(vc+2\beta)}{vc+4\beta}, \\ h = \frac{-4vw(vc+2\beta)^2}{c(vc+4\beta)^2},\ p = \frac{w^2(vc+2\beta)^2}{\beta c^2(vc+4\beta)}. \end{aligned}$$

In order to obtain the travelling wave solutions of Eq. (4), we need to solve (13), particularized to our results, as it follows:

$$\begin{aligned} \frac{4\beta v^2(vc+2\beta)^2}{(vc+4\beta)^3} - \frac{4vw(vc+2\beta)^2}{c(vc+4\beta)^2}X^2 + \frac{w^2(vc+2\beta)^2}{\beta c^2(vc+4\beta)}X^4 \\ + \left[\frac{-2v(vc+2\beta)}{vc+4\beta} + \frac{w(vc+2\beta)}{\beta c}X^2\right]Y + Y^2 = 0, \end{aligned} \tag{35}$$

which leads, when taking into account that $X'(\xi) = Y(\xi)$, to the ODEs:

$$X'(\xi) - \frac{\left[vc+4\beta \pm \sqrt{(vc+4\beta)}\right](2v\beta c - w(vc+4\beta)X^2)(vc+2\beta)}{2\beta c(vc+4\beta)^2} = 0. \tag{36}$$

Solving it, a kink-shaped soliton solution is obtained:

$$X(\xi) = \sqrt{\frac{2vc\beta}{w(vc+4\beta)}}\tanh\left[\chi(\xi+\xi_0)\right], \tag{37}$$

with the parameters ξ_0 and $\chi = (vc+2\beta)[vc+4\beta \pm \sqrt{vc(vc+4\beta)}]\sqrt{\frac{w}{2c\beta(vc+4\beta)^3}}$. A similar result, but making use of the extended tanh-method, may be seen in [24].

4 Concluding remarks

Many phenomena in the physical, chemical and biological sciences are described either by the interaction of diffusion and reaction or by the interaction between convection and diffusion. In this work, the first integral method hss been successfully made use of in order to establish the travelling wave solutions of Fisher equation with a finite memory effect, into which both convection as well as diffusion play important role. Kink-shaped soliton solutions as well as other ones as well as other ones expressed through trigonometric, rational or Jacobi elliptic functions were found. Our results do generalize the ones reported in [24] which have been obtained through the generalized tanh-function method. The existence of the solutions (19)-(21) is demonstrated under certain parametric domain. It has become obvious that the wave speed c depends upon time delay $\tau = 1/\beta$ and, as well as upon s, w, D and k. In other words, the speed of the travelling wavefront depends upon the reaction, diffusion and convection coefficients. The solutions obtained here may be used in order to explain biological and physical phenomena due to the fact that, in many biological and physical systems, dispersal is influenced by the diffusion coefficient D as well as by the convection coefficient k.

The performance of the FIM was found to be simultaneously reliable and effective; it also provides a larger number of solutions. The availability of computer systems like Maple allows us to solve complicated and tedious algebraic calculation. Therefore, it is readily applicable to high-dimensional NPDEs with various types of nonlinearity.

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