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An unified frame for finding soliton and periodic solutions for the $2D$ Ricci flow model

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Abstract

Some special travelling wave solutions for the $2D$ Ricci flow model under gravity, in the conformal gauge, are constructed in this paper, by using of an uniform algoritmic method. The proposed method is able to generate soliton solutions and periodic wave solutions at the same time. These special solutions are compared to the ones obtained by making use of other known and powerful methods for solving nonlinear partial differential equations. In fact, this method is readily applicable to a large variety of nonlinear dynamical models.

Keywords: soliton solutions, periodic solution, Ricci flow model
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1 Introduction

Many problems of physics in science and engineering may be modelled through nonlinear partial differential equations. (NPDEs). In order to essentially describe the concerned phenomena, to search for efficient algorithms able to discover a rich variety of solutions to NPDEit is very important for us.

During the recent years various methods have been developed by many scientists in order to find exact explicit solutions for NPDEsAmong them: the inverse scattering method [1], the Bäcklund transformation [2], the Darboux transformation [3], Hirota's bilinear method [4], the homogeneous balance method [5], the $(G\prime/G)$-expansion method [6], the Lie symmetry reduction [7, 8], the sine-cosine method [11], various extended tanh method [12, 13] and many others.

The purpose of this paper is to present in an unified frame some travelling wave solutions of the $2D$ Ricci flow model. Ricci flow equation on two dimensional manifolds have attracted considerable attention upon physics' literature that referes to the two-dimensional black hole geometry, to the exact solutions of the renormalization group equations which describe the decay of singularities in non-compact spaces, etc. [14]. The evolution of the $2D$ Ricci flow model, within a local system of conformally flat coordinates is described by the following second-order non-linear parabolic differential equation [15]:

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2}. \tag{1}$$

This equation has been deeply studied from the algebraic point of view in [14]. The Lie point symmetries of Eq. (1) are calculated in [16] in the terms of two arbitrary functions.

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Since this does mean to deal with an infinite number of symmetry generators, we did take into consideration the linear sector of invariance, which does lead to a six-dimensional Lie algebra generated by:

$$U_1 = \partial_x, U_2 = \partial_y,\ U_3 = \partial_t,$$
$$U_4 = t\partial_x + u\partial_u,\ U_5 = x\partial_x - u\partial_u,\ U_6 = y\partial_y - u\partial_u. \tag{2}$$

In [17], we have generalized the previous results to the nonlinear heat equation. In [18] the algebra (2) imposed upon a $2D$ generalized nonlinear second-order evolution equation is taken into consideration and important particular equations compatible with the linear Lie invariance sector have been obtained. Conservation laws for the $2D$ Ricci flow model via the direct construction method (see [19] and the references therein) have been established in [20]. Some new invariant solutions have been derived.

The study of group invariant solutions for the Ricci flow model was also extended in [21], where new sets of invariant solutions have been obtained by making use of the optimal system of one-dimensional subalgebras from the Eq. (1). Using the linearization approach, special classes of solutions involving arbitrary functions are reported in [15]. Conservation laws associated to the admitted Lie algebra (2) have been constructed in [22] by using the Ibragimov's method [23] suitable for nonlinear self-adjoint differential equations.

In this paper an efficient algorithm able to find simultaneously various soliton solutions and periodic solutions of NPDEs is described in Sectiom 2. By making use of the hyperbolic function method, the solutions we are looking for come to be written into specific forms. In Section 3, we successfully solved the $2D$ Ricci flow model. Some special solutions which contain soliton solutions and periodic solutions are obtained in an unified way. Finally, some concluding remarks are pointed out.

2 Method' summary

In this section we do describe an important approach able to construct soliton solutions and periodic solutions to NPDEs. It does extend the general ansatz [9] and it does simply proceed as follows:

Step1: Consider a nonlinear evolution equation, for example one with three independent variables x, y and t given by

$$E(u,\ u_t,\ u_x,\ u_y,\ u_{tt},\ u_{tx},\ u_{ty},\ u_{xy}, ...) = 0, \tag{3}$$

where $u = u(t,\, x,\, y)$ is a physical field, E is a polynomial in $u = u(t,\, x,\, y)$ and its various partial derivatives, where highest order derivatives and nonlinear terms are involved. Combining the independent variables t, x and y into one variable ξ, we first suppose the existence of travelling wave solutions to (3) under the form:

$$u(t, x, y) = u(\xi),\ \xi = kx + my - vt. \tag{4}$$

Then Eq. (3) is reduced to a nonlinear ordinary differential equation (NODE) for $u = u(\xi)$:

$$E(u, -vu',\ ku',\ mu', ...) = 0, \tag{5}$$

where the prime denotes $d/d\xi$.

Step2: In order to seek for more exact travelling wave solutions, we suppose an uniform frame for soliton solutions and periodic solutions [10] such as:

$$u(\xi) = \sum_{i=-N}^{N} a_i f^{\ i}(\omega) + \sum_{i=1}^{N} b_i f^{\ i-1}(\omega) g(\omega) + \sum_{i=-N}^{-1} c_i f^{\ i}(w) g(\omega),\ \omega = \omega(\xi), \tag{6}$$

with the following properties

$$\frac{d\omega}{d\xi} = g(\omega),\ \frac{df(\omega)}{d\omega} = g(\omega),\ \frac{dg(\omega)}{d\omega} = f(\omega),\ f^2(\omega) - g^2(\omega) = \delta,\ \delta^2 = 1. \tag{7}$$

The parameter δ which connects the functions $f(\omega)$ and $g(\omega)$ is known. The constants a_0, $a_{\pm 1}, ...a_{\pm N}$, $b_1, ..., b_N$, $c_{-1}, ...c_{-N}$, k, m, v are to be determined later. The parameter N may be found by taking into consideration the homogeneous balance between the highest order linear term and the nonlinear term in Eq. (5).

If we choose $g(\omega) = \sinh(\omega)$ then, through the use of the separation of variables method to solve $\frac{d\omega}{d\xi} = \sinh(\omega)$, we can get $\sinh(\omega) = -\operatorname{csch}(\xi)$ and $\cosh(\omega) = -\coth(\xi)$. For similar reason, if $\frac{d\omega}{d\xi} = \cosh(\omega)$, we can obtain $\sinh(\omega) = -\cot(\xi)$ and $\cosh(\omega) = \csc(\xi)$. Doing that, the setting (6) can be splitted into the following two different forms:

$$\begin{aligned} u(\xi) = \sum_{i=-N}^{N} a_i(-\coth(\xi))^i - \sum_{i=1}^{N} b_i(-\coth(\xi))^{i-1} \operatorname{csch}(\xi) \\ - \sum_{i=-N}^{-1} c_i(-\coth(\xi))^i \operatorname{csch}(\xi), \end{aligned} \tag{8}$$

when $\delta = 1$, $f(\omega) = \cosh(\omega)$, $g(\omega) = \sinh(\omega)$, and

$$u(\xi) = \sum_{i=-N}^{N} a_i(-\cot(\xi))^i + \sum_{i=1}^{N} b_i(-\cot(\xi))^{i-1} \csc(\xi) + \sum_{i=-N}^{-1} c_i(-\cot(\xi))^i \csc(\xi), \tag{9}$$

when $\delta = -1$, $f(\omega) = \sinh(\omega)$, $g(\omega) = \cosh(\omega)$.

Step3: Substituting (6) into Eq. (5) and tacking into account the properties (7), we obtain a finite power series of $f^p(\omega)g^r(\omega)$.

Step4: Set the coefficients of $f^p(\omega)g^r(\omega))$ obtained in Step 3 to zero, and generate a set of algebraic equations for unknown parameters a_i, b_i, c_i, k, m, v.

Step5: Solving the algebraic system. When substituting its solutions into (8) and (9), we get soliton solutions and periodic solutions respectively, for the dynamical model described by (3).

2.1 Applying on the 2D Ricci flow model

In this section we are dealing with the master Eq. (1). Consider its evolution equation in terms of the travelling wave variable:

$$u(x, t) = u(\xi),\ \xi = kx + my - vt,\ k,\ m,\ v \in R^*, \tag{10}$$

where v is the wave'speed. This change of variables does lead us to an ODE for $u = u(\xi)$:

$$u^2 u' + \gamma u'' u - \gamma (u')^2 = 0,\ \gamma = \frac{km}{v}. \tag{11}$$

The balancing process between the highest degree linear term $u''u$ and the nonlinear term u^2u', gives $N = 1$. This requirement allows us to write (6) under the form:

$$u(\xi) = a_0 + a_1 f(\omega) + \frac{a_{-1}}{f(\omega)} + b_1 g(\omega) + c_{-1}\frac{g(\omega)}{f(\omega)}, \tag{12}$$

Substituting (12) into (11) and making use of (7), the left-hand side of (11) is converted into a finite power series of $f^p(\omega)g^r(\omega)$. Setting each coefficient of this series to zero, gives us a system of eleven algebraic equations for a_0, a_1, a_{-1}, b_1, c_{-1}, $\gamma = \frac{km}{v}$ as follows:

$$\begin{aligned}
3a_1^2 b_1 + b_1^3 + 2\gamma a_1 b_1 &= 0,\\
3a_1 b_1^2 + a_1^3 + \gamma(a_1^2 + b_1^2) &= 0,\\
a_1^2 c_{-1} + 2a_0 a_1 b_1 + b_1^2 c_{-1} + \gamma(a_0 b_1 + a_1 c_{-1}) &= 0,\\
2a_1 b_1 c_{-1} + a_0 b_1^2 + a_0 a_1^2 + \gamma(a_0 a_1 + b_1 c_{-1}) &= 0,\\
2a_0 a_{-1} c_{-1} + a_{-1}^2 b_1 - \delta b_1 c_{-1}^2 + \gamma\delta(a_0 c_{-1} + a_{-1} b_1) &= 0,\\
\delta(2a_0 b_1 c_{-1} + a_1 c_{-1}^2 + a_{-1} b_1^2) - \gamma\delta(c_{-1}^2 + 4a_1 a_{-1}) - \gamma a_{-1}^2 &\\
-a_{-1} c_{-1}^2 - a_1 a_{-1}^2 - a_{-1} a_0^2 &= 0,\\
\delta(2a_1 a_{-1} c_{-1} + 2a_0 a_{-1} b_1 + c_{-1}^3 + a_0^2 c_{-1}) + 4\gamma a_1 c_{-1} &\\
-\gamma\delta a_{-1} c_{-1} - 2a_{-1}^2 c_{-1} - b_1^2 c_{-1} &= 0,\\
\delta(-a_0 c_{-1}^2 + 2b_1 a_{-1} c_{-1}) - \gamma\delta a_0 a_{-1} + \gamma b_1 c_{-1} - a_0 a_{-1}^2 &= 0,\\
3\delta a_{-1} c_{-1}^2 - a_{-1}^3 - \gamma\delta a_{-1}^2 + \gamma c_{-1}^2 &= 0,\\
2\delta a_0 a_{-1} c_{-1} + \delta b_1 a_{-1}^2 - b_1 c_{-1}^2 + \gamma a_0 c_{-1} + \gamma b_1 a_{-1} &= 0,\\
3\delta a_{-1}^2 c_{-1} - c_{-1}^3 + 2\gamma a_{-1} c_{-1} &= 0.
\end{aligned} \tag{13}$$

Solving the above over-determined nonlinear algebraic equations when $\delta = 1$, we get the following types of soliton wave solutions for Eq. (8):

$$\begin{aligned}
u_{1,2}(\xi) &= a_0 - a_1[\coth(\xi) \pm \operatorname{csch}(\xi)],\ \gamma = -2a_1,\\
u_3(\xi) &= a_0 - a_1 \coth\xi,\ \gamma = -a_1,\\
u_{4,5}(\xi) &= \pm a_{-1}(1 \mp \tanh\xi),\ \gamma = -a_{-1},\\
u_{6,7}(\xi) &= \pm a_{-1}(2 \mp \coth\xi \mp \tanh\xi),\ \gamma = -a_{-1},
\end{aligned} \tag{14}$$

While substituting $\delta = -1$ into determining system (13), we may as well derive some types of real trigonometric function solutions of Eq. (8):

$$\begin{aligned}
u_8(\xi) &= a_0 - a_1 \cot\xi,\ \gamma = -a_1,\\
u_{9,10}(\xi) &= a_0 - a_1(\cot\xi \pm \csc\xi),\ \gamma = -2a_1.
\end{aligned} \tag{15}$$

In all the previous travelling wave solutions the wave variable $\xi = kx + my - \frac{km}{\gamma}t$ and k, m, a_0, $a_{-1} \in R^*$.

3 Concluding remarks

The algorithmic method applied in the present paper has proven itself to be simple and efficient. It is more general than the method proposed in [9], when the following setting has been used :

$$u(\xi) = \sum_{i=0}^{N} a_i \sinh^i(\omega) + \sum_{i=1}^{N} b_i \sinh^{i-1}(\omega)\cosh(\omega),\ \omega = \omega(\xi)$$

with $\frac{d\omega}{d\xi} = \cosh(\omega)$ or $\frac{d\omega}{d\xi} = \sinh(\omega)$.

When we whould choose to investigate the fruitful $2D$ Ricci flow model, many solutions, including solitary wave solutions (14) and periodic wave solutions (15) should be obtained simultaneously. The unified algorithm proposed here can be more general than other powerful methods developed in order to construct travelling wave solutions of NPDEs. In what concerns our chosen matter, we could mention the $\left(\frac{G'}{G}\right)$-expansion method which could generate soliton solution as a ratio of two solitary wave solutions of type $u_3(\xi)$, the tanh-coth method [12] which does lead to soliton solutions of type $u_3(\xi)$ and $u_{4,5}(\xi)$. When we have employed the famous generalized tanh function method [24], we rediscovered the results reported here. This approach does deal with the ansatz $u(x,t) = \sum_{i=0}^{M} a_i \varphi^i(\xi)$ with $a_i = const.$ and does take full advantage of Riccati equation $\varphi' = A + B\varphi^2$ with special values for parameters (A, B),more exactly $\{(1,1), (-1,-1), (1,-1), (\pm 1/2, \pm 1/2), (1/2, -1/2)\}$. Consequently, there are nonlinear PDEs which could be solved through the extended tanh function method and which could as well be solved easily through the present method.

Not only the analyzed approach does contain the hyperbolic series expansion [25, 26, 27], but also it is as well a computer-shaped method, which does allow us to perform complicated and tedious algebraic calculation on the computer. It is readily applicable to a large variety of nonlinear dynamical models and it could generate a rich variety of soliton solutions and periodic solution at the same time. We are also aware of the fact that not all among the NPDEs may be studied through our discussed method. We are investigating how the present method could be further improved in order to study other more complicated kinds of nonlinear PDEs.

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