Reduction of Couplings and the top-bottom-Higgs-s spectrum predictions in a Finite GUT and MSSM

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Abstract

We apply the method of reduction of couplings in a Finite Unified Theory and in the MSSM. The method consists on searching for renormalization group invariant relations among couplings of a renormalizable theory holding to all orders in perturbation theory. In both cases we predict the masses of the top and bottom quarks and the light Higgs in remarkable agreement with the experiment. Moreover, we predict the masses of the other Higgses too, as well as the supersymmetric spectrum, the latter being in very confortable agreement with the LHC bounds on supersymmetric particles.

1 Introduction

The discovery of a Higgs boson [1–4] at the LHC completes the search for the particles of the Standard Model (SM), and confirms the existence of a Higgs field and the spontaneous electroweak symmetry breaking mechanism as the way to explain the masses of the fundamental particles. The over twenty free parameters of the SM, the hierarchy problem, the existence of Dark Matter, the very small masses of the neutrinos, among others, point towards a more fundamental theory, whose goal among others should be to explain at least some of these facts.

The main achievement expected from a unified description of interactions is to understand the large number of free parameters of the Standard Model (SM) in terms of a few fundamental ones. In other words, to achieve *reduction of couplings* at a more fundamental level. To reduce the number of free parameters of a theory, and thus render it

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more predictive, one is usually led to introduce more symmetry. Supersymmetric Grand Unified Theories (GUTs) are very good examples of such a procedure [5–11].

For instance, in the case of minimal $SU(5)$, because of (approximate) gauge coupling unification, it was possible to reduce the gauge couplings to one. LEP data [12] seem to suggest that a further symmetry, namely $N = 1$ global supersymmetry [10,11] should also be required to make the prediction viable. GUTs can also relate the Yukawa couplings among themselves, again $SU(5)$ provided an example of this by predicting the ratio M_τ/M_b [13] in the SM. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom and in the ways and channels of breaking the symmetry.

A natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve Gauge-Yukawa Unification (GYU) [14–23]. A symmetry which naturally relates the two sectors is supersymmetry, in particular $N = 2$ supersymmetry [24]. It turns out, however, that $N = 2$ supersymmetric theories have serious phenomenological problems due to light mirror fermions. Also in superstring theories and in composite models there exist relations among the gauge and Yukawa couplings, but both kind of theories have other phenomenological problems, which we are not going to address here.

A complementary strategy in searching for a more fundamental theory, consists in looking for all-loop renormalization group invariant (RGI) relations [25, 26] holding below the Planck scale, which in turn are preserved down to the unification scale [15–23]. Through this method of reduction of couplings [25, 26] it is possible to achieve Gauge-Yukawa Unification [14–23]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [27–29].

Although supersymmetry seems to be an essential feature for a successful realization of the above programme, its breaking has to be understood too, since it has the ambition to supply the SM with predictions for several of its free parameters. Indeed, the search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories [21, 30, 31], which involves parameters of dimension one and two.

2 Theoretical basis and the reduction of dimensionless couplings

In this section we outline the idea of reduction of couplings. Any RGI relation among couplings (which does not depend on the renormalization scale μ explicitly) can be expressed, in the implicit form $\Phi(g_1, \cdots, g_A) = \text{const.}$, which has to satisfy the partial differential equation (PDE)

$$\mu \frac{d\Phi}{d\mu} = \vec{\nabla} \cdot \vec{\beta} = \sum_{a=1}^{A} \beta_a \frac{\partial \Phi}{\partial g_a} = 0 \ , \tag{1}$$

where β_a is the β-function of g_a. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) [25, 26, 32],

$$\beta_g \frac{dg_a}{dg} = \beta_a \ , \ a = 1, \cdots, A \ , \tag{2}$$

where g and β_g are the primary coupling and its β-function, and the counting on a does not include g. Since maximally $(A-1)$ independent RGI "constraints" in the A-dimensional space of couplings can be imposed by the Φ_a's, one could in principle express all the couplings in terms of a single coupling g. The strongest requirement is to demand power series solutions to the REs,

$$g_a \;=\; \sum_n \rho_a^{(n)}\, g^{2n+1} \;, \tag{3}$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [25, 26, 32]. To illustrate this, let us assume that the β-functions have the form

$$\begin{aligned}
\beta_a &= \frac{1}{16\pi^2}\Big[\sum_{b,c,d\neq g} \beta_a^{(1)\,bcd} g_b g_c g_d + \sum_{b\neq g} \beta_a^{(1)\,b} g_b g^2\Big] + \cdots \;,\\
\beta_g &= \frac{1}{16\pi^2}\beta_g^{(1)} g^3 + \cdots \;,
\end{aligned} \tag{4}$$

where $\cdots$ stands for higher order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b, c, d. We then assume that the $\rho_a^{(n)}$'s with $n \leq r$ have been uniquely determined. To obtain $\rho_a^{(r+1)}$'s, we insert the power series (3) into the REs (2) and collect terms of $\mathcal{O}(g^{2r+3})$ and find

$$\sum_{d\neq g} M(r)_a^d\, \rho_d^{(r+1)} \;=\; \text{lower order quantities} \;,$$

where the r.h.s. is known by assumption, and

$$M(r)_a^d \;=\; 3\sum_{b,c\neq g} \beta_a^{(1)\,bcd}\, \rho_b^{(1)}\, \rho_c^{(1)} + \beta_a^{(1)\,d} - (2r+1)\, \beta_g^{(1)}\, \delta_a^d \;, \tag{5}$$

$$0 \;=\; \sum_{b,c,d\neq g} \beta_a^{(1)\,bcd}\, \rho_b^{(1)}\, \rho_c^{(1)}\, \rho_d^{(1)} + \sum_{d\neq g} \beta_a^{(1)\,d}\, \rho_d^{(1)} - \beta_g^{(1)}\, \rho_a^{(1)} \;, \tag{6}$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s can be uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

As it will be clear later by examining specific examples, the various couplings in supersymmetric theories have the same asymptotic behaviour. Therefore searching for a power series solution of the form (3) to the REs (2) is justified. This is not the case in non-supersymmetric theories, although the deeper reason for this fact is not fully understood.

The possibility of coupling unification described in this section is without any doubt attractive because the "completely reduced" theory contains only one independent coupling, but it can be unrealistic. Therefore, one often would like to impose fewer RGI constraints, and this is the idea of partial reduction [33, 34].

2.1 Reduction of dimensionful parameters

The reduction of couplings was originally formulated for massless theories on the basis of the Callan-Symanzik equation [25, 26, 32]. The extension to theories with massive parameters is not straightforward if one wants to keep the generality and the rigor on the same level as for the massless case; one has to fulfill a set of requirements coming from the renormalization group equations, the Callan-Symanzik equations, etc. along with the normalization conditions imposed on irreducible Green's functions [35]. See [36] for interesting results in this direction. Here to simplify the situation and following

Ref. [21] we would like to assume that a mass-independent renormalization scheme has been employed so that all the RG functions have only trivial dependencies of dimensional parameters.

To be general, we consider a renormalizable theory which contains a set of $(N+1)$ dimension-zero couplings, $\{\hat{g}_0, \hat{g}_1, \ldots, \hat{g}_N\}$, a set of L parameters with dimension one, $\{\hat{h}_1, \ldots, \hat{h}_L\}$, and a set of M parameters with dimension two, $\{\hat{m}_1^2, \ldots, \hat{m}_M^2\}$. The renormalized irreducible vertex function satisfies the RG equation

$$\begin{aligned} 0 &= \mathcal{D}\Gamma[\, \mathbf{\Phi}'s; \hat{g}_0, \hat{g}_1, \ldots, \hat{g}_N; \hat{h}_1, \ldots, \hat{h}_L; \hat{m}_1^2, \ldots, \hat{m}_M^2; \mu \,] , \\ \mathcal{D} &= \mu\frac{\partial}{\partial\mu} + \sum_{i=0}^{N} \beta_i \frac{\partial}{\partial \hat{g}_i} + \sum_{a=1}^{L} \gamma_a^h \frac{\partial}{\partial \hat{h}_a} + \sum_{\alpha=1}^{M} \gamma_\alpha^{m^2} \frac{\partial}{\partial \hat{m}_\alpha^2} + \sum_J \Phi_I \gamma^{\phi I}{}_J \frac{\delta}{\delta \Phi_J} . \end{aligned} \tag{7}$$

Since we assume a mass-independent renormalization scheme, the γ's have the form

$$\begin{aligned} \gamma_a^h &= \sum_{b=1}^{L} \gamma_a^{h,b}(g_0, \ldots, g_N) \hat{h}_b , \\ \gamma_\alpha^{m^2} &= \sum_{\beta=1}^{M} \gamma_\alpha^{m^2,\beta}(g_0, \ldots, g_N) \hat{m}_\beta^2 + \sum_{a,b=1}^{L} \gamma_\alpha^{m^2,ab}(g_0, \ldots, g_N) \hat{h}_a \hat{h}_b , \end{aligned} \tag{8}$$

where $\gamma_a^{h,b}, \gamma_\alpha^{m^2,\beta}$ and $\gamma_\alpha^{m^2,ab}$ are power series of the dimension-zero couplings g's in perturbation theory.

As in the massless case, we then look for conditions under which the reduction of parameters,

$$\hat{g}_i = \hat{g}_i(g) , \ (i = 1, \ldots, N) , \tag{9}$$

$$\hat{h}_a = \sum_{b=1}^{P} f_a^b(g) h_b , \ (a = P+1, \ldots, L) , \tag{10}$$

$$\hat{m}_\alpha^2 = \sum_{\beta=1}^{Q} e_\alpha^\beta(g) m_\beta^2 + \sum_{a,b=1}^{P} k_\alpha^{ab}(g) h_a h_b , \ (\alpha = Q+1, \ldots, M) , \tag{11}$$

is consistent with the RG equation (1), where we assume that $g \equiv g_0$, $h_a \equiv \hat{h}_a$ $(1 \le a \le P)$ and $m_\alpha^2 \equiv \hat{m}_\alpha^2$ $(1 \le \alpha \le Q)$ are independent parameters of the reduced theory. We find that the following set of equations has to be satisfied:

$$\beta_g \frac{\partial \hat{g}_i}{\partial g} = \beta_i , \ (i = 1, \ldots, N) , \tag{12}$$

$$\beta_g \frac{\partial \hat{h}_a}{\partial g} + \sum_{b=1}^{P} \gamma_b^h \frac{\partial \hat{h}_a}{\partial h_b} = \gamma_a^h , \ (a = P+1, \ldots, L) , \tag{13}$$

$$\beta_g \frac{\partial \hat{m}_\alpha^2}{\partial g} + \sum_{a=1}^{P} \gamma_a^h \frac{\partial \hat{m}_\alpha^2}{\partial h_a} + \sum_{\beta=1}^{Q} \gamma_\beta^{m^2} \frac{\partial \hat{m}_\alpha^2}{\partial m_\beta^2} = \gamma_\alpha^{m^2} , \ (\alpha = Q+1, \ldots, M) . \tag{14}$$

Using eq.(7) for γ's, one finds that eqs.(12-14) reduce to

$$\beta_g \, \frac{df_a^b}{dg} + \sum_{c=1}^{P} f_a^c [\, \gamma_c^{h,b} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^b \,] - \gamma_a^{h,b} - \sum_{d=P+1}^{L} \gamma_a^{h,d} f_d^b \;= 0 \; , \tag{15}$$
$$(a = P+1, \ldots, L; b = 1, \ldots, P) \; ,$$
$$\beta_g \, \frac{de_\alpha^\beta}{dg} + \sum_{\gamma=1}^{Q} e_\alpha^\gamma [\, \gamma_\gamma^{m^2,\beta} + \sum_{\delta=Q+1}^{M} \gamma_\gamma^{m^2,\delta} e_\delta^\beta \,] - \gamma_\alpha^{m^2,\beta} - \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,\delta} e_\delta^\beta \;= 0 \; , \tag{16}$$
$$(\alpha = Q+1, \ldots, M; \beta = 1, \ldots, Q) \; ,$$
$$\begin{aligned}
&\beta_g \, \frac{dk_\alpha^{ab}}{dg} + 2 \sum_{c=1}^{P} (\, \gamma_c^{h,a} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^a \,) k_\alpha^{cb} + \sum_{\beta=1}^{Q} e_\alpha^\beta [\, \gamma_\beta^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\beta^{m^2,cd} f_c^a f_d^b \\
&+ 2 \sum_{c=P+1}^{L} \gamma_\beta^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\beta^{m^2,\delta} k_\delta^{ab} \,] - [\, \gamma_\alpha^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\alpha^{m^2,cd} f_c^a f_d^b \\
&+ 2 \sum_{c=P+1}^{L} \gamma_\alpha^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,\delta} k_\delta^{ab} \,] \;= 0 \; ,
\end{aligned} \tag{17}$$
$$(\alpha = Q+1, \ldots, M; a, b = 1, \ldots, P) \; .$$

If these equations are satisfied, the irreducible vertex function of the reduced theory

$$\begin{aligned}
& \Gamma_R[\; \mathbf{\Phi}'s; g; h_1, \ldots, h_P; m_1^2, \ldots, \hat{m}_Q^2; \mu \;] \\
\equiv \; & \Gamma[\; \mathbf{\Phi}'s; g, \hat{g}_1(g), \ldots, \hat{g}_N(g); h_1, \ldots, h_P, \hat{h}_{P+1}(g,h), \ldots, \hat{h}_L(g,h); \\
& m_1^2, \ldots, \hat{m}_Q^2, \hat{m}_{Q+1}^2(g,h,m^2), \ldots, \hat{m}_M^2(g,h,m^2); \mu \;]
\end{aligned} \tag{18}$$

has the same renormalization group flow as the original one.

The requirement for the reduced theory to be perturbative renormalizable means that the functions $\hat{g}_i$, f_a^b, e_α^β and k_α^{ab}, defined in eqs. (9-11), should have a power series expansion in the primary coupling g:

$$\begin{aligned}
\hat{g}_i &= g \sum_{n=0}^{\infty} \rho_i^{(n)} g^n \; , \; f_a^b = g \sum_{n=0}^{\infty} \eta_a^{b \; (n)} g^n \; , \\
e_\alpha^\beta &= \sum_{n=0}^{\infty} \xi_\alpha^{\beta \; (n)} g^n \; , \; k_\alpha^{ab} = \sum_{n=0}^{\infty} \chi_\alpha^{ab \; (n)} g^n \; .
\end{aligned} \tag{19}$$

To obtain the expansion coefficients, we insert the power series ansatz above into eqs. (12,15–17) and require that the equations are satisfied at each order in g. Note that the existence of a unique power series solution is a non-trivial matter: it depends on the theory as well as on the choice of the set of independent parameters.

2.2 Finiteness in N=1 Supersymmetric Gauge Theories

Let us consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W \;=\; \frac{1}{2} \, m_{ij} \, \phi_i \, \phi_j + \frac{1}{6} \, C_{ijk} \, \phi_i \, \phi_j \, \phi_k \; , \tag{20}$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the matter field ϕ_i transforms according to the irreducible representation R_i of the gauge group G. The renormalization constants associated with the superpotential (20), assuming that SUSY is preserved, are

$$\phi_i^0 = (Z_i^j)^{(1/2)}\,\phi_j \;, \tag{21}$$
$$m_{ij}^0 = Z_{ij}^{i'j'}\,m_{i'j'} \;, \tag{22}$$
$$C_{ijk}^0 = Z_{ijk}^{i'j'k'}\,C_{i'j'k'} \;. \tag{23}$$

The $N = 1$ non-renormalization theorem [37–39] ensures that there are no mass and cubic-interaction-term infinities and therefore

$$\begin{aligned} Z_{ijk}^{i'j'k'}\,Z_{i'}^{1/2\,i''}\,Z_{j'}^{1/2\,j''}\,Z_{k'}^{1/2\,k''} &= \delta_{(i}^{i''}\,\delta_j^{j''}\,\delta_{k)}^{k''} \;, \\ Z_{ij}^{i'j'}\,Z_{i'}^{1/2\,i''}\,Z_{j'}^{1/2\,j''} &= \delta_{(i}^{i''}\,\delta_{j)}^{j''} \;. \end{aligned} \tag{24}$$

As a result the only surviving possible infinities are the wave-function renormalization constants Z_i^j, i.e., one infinity for each field. The one -loop β-function of the gauge coupling g is given by [40]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\,[\sum_i l(R_i) - 3\,C_2(G)\,] \;, \tag{25}$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C_{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_{ij} of the matter fields ϕ_i as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} \;=\; C_{ijl}\,\gamma_k^l + C_{ikl}\,\gamma_j^l + C_{jkl}\,\gamma_i^l \;. \tag{26}$$

At one-loop level γ_{ij} is [40]

$$\gamma_j^{i(1)} = \frac{1}{32\pi^2}\,[\,C^{ikl}\,C_{jkl} - 2\,g^2\,C_2(R_i)\delta_j^1\,], \tag{27}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C_{ijk}^*$. Since dimensional coupling parameters such as masses and couplings of cubic scalar field terms do not influence the asymptotic properties of a theory on which we are interested here, it is sufficient to take into account only the dimensionless supersymmetric couplings such as g and C_{ijk}. So we neglect the existence of dimensional parameters, and assume furthermore that C_{ijk} are real so that C_{ijk}^2 always are positive numbers.

As one can see from Eqs. (25) and (27), all the one-loop β-functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma_{ij}^{(1)}$ vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G)\,, \tag{28}$$

$$C^{ikl}C_{jkl} = 2\delta_j^i g^2 C_2(R_i)\,, \tag{29}$$

The conditions for finiteness for $N = 1$ field theories with $SU(N)$ gauge symmetry are discussed in [41], and the analysis of the anomaly-free and no-charge renormalization requirements for these theories can be found in [42]. A very interesting result is that the conditions (28,29) are necessary and sufficient for finiteness at the two-loop level [40, 43–46].

In case SUSY is broken by soft terms, the requirement of finiteness in the one-loop soft breaking terms imposes further constraints among themselves [47]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of the theory two-loop finite [48].

The one- and two-loop finiteness conditions (28,29) restrict considerably the possible choices of the irreps. R_i for a given group G as well as the Yukawa couplings in the superpotential (20). Note in particular that the finiteness conditions cannot be applied to the minimal supersymmetric standard model (MSSM), since the presence of a $U(1)$ gauge group is incompatible with the condition (28), due to $C_2[U(1)] = 0$. This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the MSSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that SUSY (most probably) can only be broken due to the soft breaking terms. Indeed, due to the unacceptability of gauge singlets, F-type spontaneous symmetry breaking [49] terms are incompatible with finiteness, as well as D-type [50] spontaneous breaking which requires the existence of a $U(1)$ gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [27, 51] which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As we have seen, the necessary and also sufficient, condition for this to happen is to require that such relations are solutions to the REs

$$\beta_g \frac{dC_{ijk}}{dg} = \beta_{ijk} \tag{30}$$

and hold at all orders. Remarkably, the existence of all-order power series solutions to (30) can be decided at one-loop level, as already mentioned.

Then the all order finiteness theorem states that an N=1 supersymmetric theory [27, 51], states that an N=1 supersymmetric theory can become finite to all orders in the sense of vanishing β-functions, that is of physical scale invariance. It is based on (a) the structure of the supercurrent in $N = 1$ supersymmetric gauge theory [52–54], and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [27, 28, 51, 55]. Details on the proof can be found in refs. [27, 51] and further discussion in Refs. [28, 29, 55–57].

2.3 Sum rule for SB terms in $N = 1$ Supersymmetric and Finite theories: All-loop results

As we have seen in Sect. 2.1, the method of reducing the dimensionless couplings has been extended [21], to the soft SUSY breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [58] that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule. Here we will describe first how the use of the available two-loop RG functions and the requirement of finiteness of the SSB parameters up to this order leads to the soft scalar-mass sum rule [59].

Consider the superpotential given by (20) along with the Lagrangian for SSB terms

$$\begin{aligned} -\mathcal{L}_{\rm SB} &= \frac{1}{6}\, h^{ijk}\, \phi_i \phi_j \phi_k + \frac{1}{2}\, b^{ij}\, \phi_i \phi_j \\ &+ \frac{1}{2}\, (m^2)^j_i\, \phi^{*\, i} \phi_j + \frac{1}{2}\, M\, \lambda\lambda + \text{h.c.}, \end{aligned} \tag{31}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i , λ are the gauginos and M their unified mass. Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form

$$C^{ijk} \;=\; g \sum_n \rho^{ijk}_{(n)} g^{2n} \; . \tag{32}$$

According to the finiteness theorem of Refs. [27, 51], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved by [48]

$$h^{ijk} \;=\; -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)}\, g + O(g^5) \; , \tag{33}$$

where ... stand for higher order terms.

Now, to obtain the two-loop sum rule for soft scalar masses, we assume that the lowest order coefficients $\rho^{ijk}_{(0)}$ and also $(m^2)^i_j$ satisfy the diagonality relations

$$\rho_{ipq(0)}\rho^{jpq}_{(0)} \;\propto\; \delta_i^j \text{ for all } p \text{ and } q \;\text{ and }\; (m^2)^i_j = m^2_j \delta^i_j \; , \tag{34}$$

respectively. Then we find the following soft scalar-mass sum rule [23, 59, 60]

$$(\, m_i^2 + m_j^2 + m_k^2 \,)/MM^{\dagger} \;=\; 1 + \frac{g^2}{16\pi^2}\, \Delta^{(2)} + O(g^4) \tag{35}$$

for i, j, k with $\rho^{ijk}_{(0)} \neq 0$, where $\Delta^{(2)}$ is the two-loop correction

$$\Delta^{(2)} \;=\; -2 \sum_l [(m_l^2/MM^{\dagger}) - (1/3)]\; T(R_l), \tag{36}$$

which vanishes for the universal choice in accordance with the previous findings of Ref. [48].

If we know higher-loop β-functions explicitly, we can follow the same procedure and find higher-loop RGI relations among SSB terms. However, the β-functions of the soft scalar masses are explicitly known only up to two loops. In order to obtain higher-loop results some relations among β-functions are needed.

Making use of the spurion technique [39, 61–64], it is possible to find the following all-loop relations among SSB β-functions, [65–70]

$$\beta_M \;=\; 2\mathcal{O}\left(\frac{\beta_g}{g}\right) \; , \tag{37}$$

$$\begin{aligned} \beta_h^{ijk} &= \gamma^i{}_l h^{ljk} + \gamma^j{}_l h^{ilk} + \gamma^k{}_l h^{ijl} \\ &\quad -2\gamma^i_1{}_l C^{ljk} - 2\gamma^j_1{}_l C^{ilk} - 2\gamma^k_1{}_l C^{ijl} \; , \end{aligned} \tag{38}$$

$$(\beta_{m^2})^i{}_j \;=\; \left[\Delta + X \frac{\partial}{\partial g}\right] \gamma^i{}_j \; , \tag{39}$$

$$\mathcal{O} \;=\; \left(Mg^2 \frac{\partial}{\partial g^2} - h^{lmn} \frac{\partial}{\partial C^{lmn}}\right) \; , \tag{40}$$

$$\Delta \;=\; 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} + \tilde{C}_{lmn} \frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn} \frac{\partial}{\partial C^{lmn}} \; , \tag{41}$$

where $(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j$, $C_{lmn} = (C^{lmn})^*$, and

$$\tilde{C}^{ijk} = (m^2)^i{}_l C^{ljk} + (m^2)^j{}_l C^{ilk} + (m^2)^k{}_l C^{ijl} \ . \tag{42}$$

It was also found [66] that the relation

$$h^{ijk} = -M(C^{ijk})' \equiv -M\frac{dC^{ijk}(g)}{d\ln g} \ , \tag{43}$$

among couplings is all-loop RGI. Furthermore, using the all-loop gauge β-function of Novikov *et al.* [71–73] given by

$$\beta_g^{\mathrm{NSVZ}} = \frac{g^3}{16\pi^2}\left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C(G)}{1 - g^2C(G)/8\pi^2}\right] \ , \tag{44}$$

it was found the all-loop RGI sum rule [74],

$$\begin{aligned} m_i^2 + m_j^2 + m_k^2 &= |M|^2\{ \frac{1}{1 - g^2C(G)/(8\pi^2)}\frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2}\frac{d^2\ln C^{ijk}}{d(\ln g)^2} \} \\ &+ \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2}\frac{d\ln C^{ijk}}{d\ln g} \ . \end{aligned} \tag{45}$$

In addition the exact-β-function for m^2 in the NSVZ scheme has been obtained [74] for the first time and is given by

$$\begin{aligned} \beta_{m_i^2}^{\mathrm{NSVZ}} &= \left[|M|^2\{ \frac{1}{1 - g^2C(G)/(8\pi^2)}\frac{d}{d\ln g} + \frac{1}{2}\frac{d^2}{d(\ln g)^2} \} \right. \\ &\left. + \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2}\frac{d}{d\ln g} \right] \gamma_i^{\mathrm{NSVZ}} \ . \end{aligned} \tag{46}$$

Surprisingly enough, the all-loop result (45) coincides with the superstring result for the finite case in a certain class of orbifold models [59] if $d\ln C^{ijk}/d\ln g = 1$.

The all-loop results on the SSB β-functions lead to all-loop RGI relations (see e.g. [75]). If we assume:
(a) the existence of a RGI surfaces on which $C = C(g)$, or equivalently that

$$\frac{dC^{ijk}}{dg} = \frac{\beta_C^{ijk}}{\beta_g} \tag{47}$$

holds, i.e. reduction of couplings is possible, and
(b) the existence of a RGI surface on which

$$h^{ijk} = -M\frac{dC(g)^{ijk}}{d\ln g} \tag{48}$$

holds too in all-orders, then one can prove that the following relations are RGI to all-loops [76, 77] (note that in the above assumptions (a) and (b) we do not rely on specific solutions to these equations)

$$M = M_0 \ \frac{\beta_g}{g}, \tag{49}$$

$$h^{ijk} = -M_0 \ \beta_C^{ijk}, \tag{50}$$

$$b^{ij} = -M_0 \ \beta_\mu^{ij}, \tag{51}$$

$$(m^2)^i{}_j = \frac{1}{2} \ |M_0|^2 \ \mu\frac{d\gamma^i{}_j}{d\mu}, \tag{52}$$

where M_0 is an arbitrary reference mass scale to be specified shortly.

Finally we would like to emphasize that under the same assumptions (a) and (b) the sum rule given in Eq.(45) has been proven [74] to be all-loop RGI, which gives us a generalization of Eq.(52) to be applied in considerations of non-universal soft scalar masses, which are necessary in many cases including the MSSM.

As it was emphasized in ref [76] the set of the all-loop RGI relations (49)-(52) is the one obtained in the *Anomaly Mediated SB Scenario* [78, 79], by fixing the M_0 to be $m_{3/2}$, which is the natural scale in the supergravity framework. A final remark concerns the resolution of the fatal problem of the anomaly induced scenario in the supergravity framework, which is here solved thanks to the sum rule (45). Other solutions have been provided by introducing Fayet-Iliopoulos terms [80].

3 Applications of the Reduction of Couplings Method

In this section we show how to apply the reduction of couplings method in two scenarios, the MSSM and Finite Unified Theories (see also Refs. [81, 82]). We will apply it only to the third generation of fermions and in the soft supersymmetry breaking terms. After the reduction of couplings takes place, we are left with relations at the unification scale for the Yukawa couplings of the quarks in terms of the gauge coupling according to Eq. (32), for the trlininear terms in terms of the Yukawa couplings and the unified gaugino mass Eq. (48), and a sum rule for the soft scalar masses also proportional to the unified gaugino mass Eq. (45), as applied in each model.

3.1 RE in the MSSM

We will examine here the reduction of couplings method applied to the MSSM, which is defined by the superpotential,

$$W = Y_t H_2 Q t^c + Y_b H_1 Q b^c + Y_\tau H_1 L \tau^c + \mu H_1 H_2, \tag{53}$$

with soft breaking terms,

$$\begin{aligned} -\mathcal{L}_{SSB} = \sum_\phi m_\phi^2 \phi^* \phi + \left[m_3^2 H_1 H_2 + \sum_{i=1}^3 \frac{1}{2} M_i \lambda_i \lambda_i + \text{h.c} \right] \\ + \left[h_t H_2 Q t^c + h_b H_1 Q b^c + h_\tau H_1 L \tau^c + \text{h.c.} \right], \end{aligned} \tag{54}$$

where the last line refers to the scalar components of the corresponding superfield. In general $Y_{t,b,\tau}$ and $h_{t,b,\tau}$ are 3×3 matrices, but we work throughout in the approximation that the matrices are diagonal, and neglect the couplings of the first two generations.

Assuming perturbative expansion of all three Yukawa couplings in favour of g_3 satisfying the reduction equations we find that the coefficients of the Y_τ coupling turn imaginary. Therefore, we take Y_τ at the GUT scale to be an independent variable. Thus, in the application of the reduction of couplings in the MSSM that we examine here, in the first stage we neglect the Yukawa couplings of the first two generations, while we keep Y_τ and the gauge couplings g_2 and g_1, which cannot be reduced consistently, as corrections. This "reduced" system holds at all scales, and thus serve as boundary conditions of the RGEs of the MSSM at the unification scale, where we assume that the gauge couplings meet [75].

In that case, the coefficients of the expansions (again at the unification scale)

$$\frac{Y_t^2}{4\pi} = c_1 \frac{g_3^2}{4\pi} + c_2 \left(\frac{g_3^2}{4\pi}\right)^2 \quad ; \quad \frac{Y_b^2}{4\pi} = p_1 \frac{g_3^2}{4\pi} + p_2 \left(\frac{g_3^2}{4\pi}\right)^2 \tag{55}$$

are given by

$$\begin{aligned}
c_1 &= \frac{157}{175} + \frac{1}{35} K_\tau = 0.897 + 0.029 K_\tau \\
p_1 &= \frac{143}{175} - \frac{6}{35} K_\tau = 0.817 - 0.171 K_\tau \\
c_2 &= \frac{1}{4\pi} \frac{1457.55 - 84.491 K_\tau - 9.66181 K_\tau^2 - 0.174927 K_\tau^3}{818.943 - 89.2143 K_\tau - 2.14286 K_\tau^2} \\
p_2 &= \frac{1}{4\pi} \frac{1402.52 - 223.777 K_\tau - 13.9475 K_\tau^2 - 0.174927 K_\tau^3}{818.943 - 89.2143 K_\tau - 2.14286 K_\tau^2}
\end{aligned} \tag{56}$$

where

$$K_\tau = Y_\tau^2 / g_3^2 \tag{57}$$

The couplings Y_t,Y_b and g_3 are not only reduced, but they provide predictions consistent with the observed experimental values, as we will show in subsection 4.2. According to the analysis presented in Section 2 the RGI relations in the SSB sector hold, assuming the existence of RGI surfaces where Eqs.(47) and (48) are valid.

Since all gauge couplings in the MSSM meet at the unification point, we are led to the following boundary conditions at the unification scale:

$$Y_t^2 = c_1 g_U^2 + c_2 g_U^4/(4\pi) \quad \text{and} \quad Y_b^2 = p_1 g_U^2 + p_2 g_U^4/(4\pi) \tag{58}$$

$$h_{t,b} = -M_U Y_{t,b}, \tag{59}$$

$$m_3^2 = -M_U \mu, \tag{60}$$

where M_U is the unification scale, $c_{1,2}$ and $p_{1,2}$ are the solutions of the algebraic system of the two reduction equations taken at the unification scale (while keeping only the first term[1] of the perturbative expansion of the Yukawas in favour of g_3 for Eqs.(59) and (60)), and a set of equations resulting from the application of the sum rule

$$m_{H_2}^2 + m_Q^2 + m_{t^c}^2 = M_U^2, \quad m_{H_1}^2 + m_Q^2 + m_{b^c}^2 = M_U^2, \tag{61}$$

noting that the sum rule introduces four free parameters.

3.2 Predictions of the Reduced MSSM

With these boundary conditions we run the MSSM RGEs down to the SUSY scale, which we take to be the geometrical average of the stop masses, and then run the SM RGEs down to the electroweak scale (M_Z), where we compare with the experimental values of the third generation quark masses. The RGEs are taken at two-loops for the gauge and Yukawa couplings and at one-loop for the soft breaking parameters. We let M_U and $|\mu|$ at the unification scale to vary between ~ 1 TeV ~ 11 TeV, for the two possible signs of μ. In evaluating the τ and bottom masses we have taken into account the one-loop radiative corrections that come from the SUSY breaking [83, 84]; in particular for large $\tan\beta$ they can give sizeable contributions to the bottom quark mass.

[1]The second term can be determined once the first term is known.

Recall that Y_τ is not reduced and is a free parameter in this analysis. The parameter K_τ, related to Y_τ through Eq. (57) is further constrained by allowing only the values that are also compatible with the top and bottom quark masses simultaneously within 1 and 2σ of their central experimental value. In the case that $\text{sign}(\mu) = +$, there is no value for K_τ where both the top and the bottom quark masses agree simultaneously with their experimental value, therefore we only consider the negative sign of μ from now on. We use the experimental value of the top quark pole mass as [85][2]

$$m_t^{\text{exp}} = (173.2 \pm 0.9) \text{ GeV} . \tag{62}$$

The bottom mass is calculated at M_Z to avoid uncertainties that come from running down to the pole mass and, as previously mentioned, the SUSY radiative corrections both to the tau and the bottom quark masses have been taken into account [87]

$$m_b(M_Z) = (2.83 \pm 0.10) \text{ GeV}. \tag{63}$$

The variation of K_τ is in the range $\sim 0.33 \sim 0.5$ if the agreement with both top and bottom masses is at the 2σ level.

Finally, assuming the validity of Eq.(48) for the corresponding couplings to those that have been reduced before, we have calculated the Higgs mass as well as the whole Higgs and sparticle spectrum using Eqs.(58)-(61). The Higgs mass was calculated using a "mixed-scale" one-loop RG approach, which is known to approximate the leading two-loop corrections as evaluated by the full diagrammatic calculation [88,89]. Since this evaluation further higher-order corrections became available [90], which change the results for larger values of K_τ. Pending a re-evaluation we only give a rough description of our results.

The spectacular discovery of a Higgs boson at ATLAS and CMS, as announced in July 2012 [1, 3] can be interpreted as the discovery of the light CP-even Higgs boson of the MSSM Higgs spectrum, see, e.g., Ref. [91]. Here we take as experimental average for the (SM) Higgs boson mass the value

$$M_H^{\text{exp}} = 125.6 \pm 0.3 \text{ GeV} . \tag{64}$$

Adding a 3 (2) GeV theory uncertainty [92] for the Higgs boson mass calculation in the MSSM we arrive at

$$M_h = 125.6 \pm 3.1\,(2.1) \text{ GeV} \tag{65}$$

as our allowed range.

The prediction in this model for M_h falls naturally into the range of Eq. (65), leading to restrictions on K_τ and thus on the obtained values for the MSSM spectrum. Without a dedicated re-evaluation of M_h, see above, one can say that a relatively heavy spectrum is obtained. Only the lowest mass values of the colored particles could be reachable at the LHC (even including the HL phase). Similarly, only the lower part of the electroweak spectrum could be accessible at a future e^+e^- linear collider (ILC or CLIC), even going to a center-of-mass energy of $\sqrt{s} \sim 3$ TeV. The light Higgs boson, being in the range of Eq. (65) and the CP-odd mass scale above the 1 TeV range result in a light Higgs boson being in excellent agreement with the experimental results of ATLAS and CMS [1–4].

[2] We did not include the latest LHC/Tevatron combination, leading to $m_t^{\text{exp}} = (173.34 \pm 0.76)$ GeV [86], which would have a negligible impact on our analysis.

3.3 An $SU(5)$ Finite Unified Theory

We examine an all-loop Finite Unified theory with $SU(5)$ as gauge group, where the reduction of couplings has been applied to the third generation of quarks and leptons. The particle content of the model we will study, which we denote **FUT** consists of the following supermultiplets: three $(\overline{\mathbf{5}} + \mathbf{10})$, needed for each of the three generations of quarks and leptons, four $(\overline{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$ considered as Higgs supermultiplets. When the gauge group of the finite GUT is broken the theory is no longer finite, and we will assume that we are left with the MSSM [15, 17–19, 22].

A predictive Gauge-Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.
2. Three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, which obviously should not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

After the reduction of couplings the symmetry is enhanced, leading to the following superpotential [93]

$$
\begin{aligned}
W \;=\; & \sum_{i=1}^{3} [\, \frac{1}{2} g_i^u \,\mathbf{10}_i \mathbf{10}_i H_i + g_i^d \,\mathbf{10}_i \overline{\mathbf{5}}_i \,\overline{H}_i \,] + g_{23}^u \,\mathbf{10}_2 \mathbf{10}_3 H_4 \\
& + g_{23}^d \,\mathbf{10}_2 \overline{\mathbf{5}}_3 \,\overline{H}_4 + g_{32}^d \,\mathbf{10}_3 \overline{\mathbf{5}}_2 \,\overline{H}_4 + g_2^f \, H_2 \,\mathbf{24}\, \overline{H}_2 + g_3^f \, H_3 \,\mathbf{24}\, \overline{H}_3 + \frac{g^\lambda}{3} (\mathbf{24})^3 \; .
\end{aligned} \tag{66}
$$

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ give us:

$$
\begin{aligned}
& (g_1^u)^2 = \frac{8}{5}\, g^2 \;,\; (g_1^d)^2 = \frac{6}{5}\, g^2 \;,\; (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5}\, g^2 \;, \\
& (g_2^d)^2 = (g_3^d)^2 = \frac{3}{5}\, g^2 \;,\; (g_{23}^u)^2 = \frac{4}{5}\, g^2 \;,\; (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5}\, g^2 \;, \\
& (g^\lambda)^2 = \frac{15}{7} g^2 \;,\; (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2}\, g^2 \;,\; (g_1^f)^2 = 0 \;,\; (g_4^f)^2 = 0 \;,
\end{aligned} \tag{67}
$$

and from the sum rule we obtain:

$$
m^2_{H_u} + 2 m^2_{\mathbf{10}} = M^2 \;,\; m^2_{H_d} - 2 m^2_{\mathbf{10}} = -\frac{M^2}{3} \;,\; m^2_{\overline{\mathbf{5}}} + 3 m^2_{\mathbf{10}} = \frac{4M^2}{3} \;, \tag{68}
$$

i.e., in this case we have only two free parameters $m_{\mathbf{10}}$ and M for the dimensionful sector.

As already mentioned, after the $SU(5)$ gauge symmetry breaking we assume we have the MSSM, i.e. only two Higgs doublets. This can be achieved by introducing appropriate mass terms that allow to perform a rotation of the Higgs sector [15–19, 94–96], in such a way that only one pair of Higgs doublets, coupled mostly to the third family, remains light and acquire vacuum expectation values. To avoid fast proton decay the usual fine tuning to achieve doublet-triplet splitting is performed, although the mechanism is not identical to minimal $SU(5)$, since we have an extended Higgs sector.

Thus, after the gauge symmetry of the GUT theory is broken we are left with the MSSM, with the boundary conditions for the third family given by the finiteness conditions, while the other two families are not restricted.

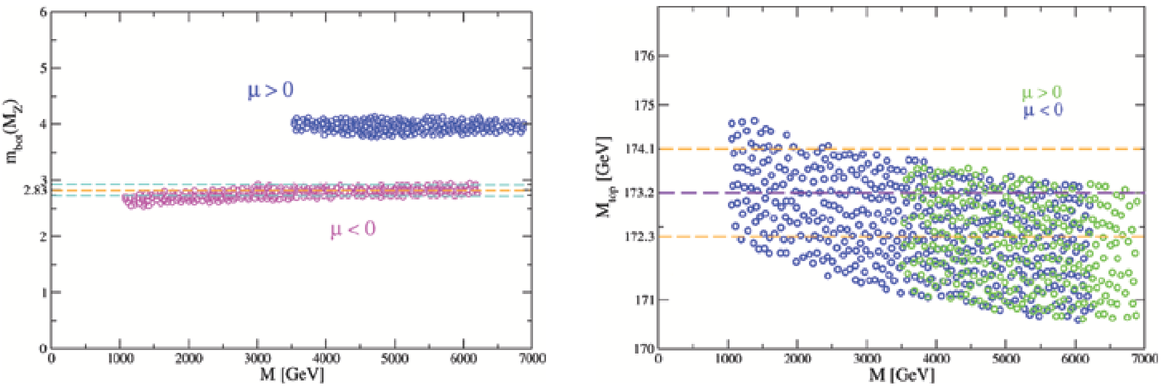

Figure 1: The bottom quark mass at the Z boson scale (left) and top quark pole mass (right) are shown as function of M, the unified gaugino mass.

3.4 Predictions of the Finite Model

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (67), the $h = -MC$ (33) relation, and the soft scalar-mass sum rule at $M_{\rm GUT}$. The analysis follows along the same lines as in the MSSM case.

In Fig.1 we show the **FUT** predictions for m_t and $m_b(M_Z)$ as a function of the unified gaugino mass M, for the two cases $\mu < 0$ and $\mu > 0$. The bounds on the $m_b(M_Z)$ and the m_t mass clearly single out $\mu < 0$, as the solution most compatible with these experimental constraints [97, 98].

We now analyze the impact of further low-energy observables on the model **FUT** with $\mu < 0$. As additional constraints we consider the flavour observables ${\rm BR}(b \to s\gamma)$ and ${\rm BR}(B_s \to \mu^+\mu^-)$.

For the branching ratio ${\rm BR}(b \to s\gamma)$, we take the value given by the Heavy Flavour Averaging Group (HFAG) is [99]

$$ {\rm BR}(b \to s\gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}. \tag{69} $$

For the branching ratio ${\rm BR}(B_s \to \mu^+\mu^-)$, the SM prediction is at the level of 10^{-9}, while we employ an upper limit of

$$ {\rm BR}(B_s \to \mu^+\mu^-) \lesssim 4.5 \times 10^{-9} \tag{70} $$

at the 95% C.L. [100]. This is in relatively good agreement with the recent direct measurement of this quantity by CMS and LHCb [101]. As we do not expect a sizable impact of the new measurement on our results, we stick for our analysis to the simple upper limit.

For the lightest Higgs mass prediction we used the code `FeynHiggs` [90, 92, 102–104]. The evaluation of Higgs boson masses within `FeynHiggs` is based on the Feynman-diagrammatic calculation as discussed above. `FeynHiggs` has recently been updated to version `2.10.0`, where the principal focus of the improvements has been to attain greater accuracy for large stop masses. The new version, `FeynHiggs 2.10.0` [90] contains a resummation of the leading and next-to-leading logarithms of type $\log(m_{\tilde t}/m_t)$ in all orders of perturbation theory, which yields reliable results for $m_{\tilde t}, M_A \gg M_Z$. To this end the two-loop Renormalization-Group Equations (RGEs) [105, 106] have been solved, taking into account the one-loop threshold corrections to the quartic coupling at the SUSY scale: see [107] and references therein. In this way at n-loop order the terms

$$\sim \log^n(m_{\tilde{t}}/m_t), \quad \sim \log^{n-1}(m_{\tilde{t}}/m_t) \tag{71}$$

are taken into account. As we shall see, `FeynHiggs 2.10.0` yields a larger estimate of M_h for stop masses in the multi-TeV range (as we find in our evaluations), and a correspondingly improved estimate of the theoretical uncertainty, as discussed in [90,108] (and indicated in Eq. (65)).

The prediction for M_h of **FUT** is shown in Fig. 2 [109] as a function of M in the range 1 TeV $\lesssim M \lesssim$ 8 TeV. All points fulfill the quark mass requirements, while the blue points in addition also fulfill the B-physics constraints. The lightest Higgs mass ranges in

$$M_h \sim 124 - 133 \text{ GeV} , \tag{72}$$

where larger masses could reached for larger values of M. The main uncertainty for fixed M comes from the variation of the other soft scalar masses. As discussed above, to this value one has to add at least ± 2 GeV coming from unkonwn higher order corrections [90, 108]. We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5% of the FUT boundary conditions. Overall, M_h is found at somewhat higher values in comparison with our previous analyses [97, 110–114]. This is clearly due to the newly included resummed logarithmic corrections.

The horizontal lines in Fig. 2 show the central value of the experimental measurement (solid), the ± 2.1 GeV uncertainty (dashed) and the ± 3.1 GeV uncertainty (dot-dashed). The requirement to obtain a light Higgs boson mass value in the correct range yields an upper limit on M of about 3 (4.5) TeV for $M_h = 125.6 \pm 2.1\,(3.1)$ GeV. Naturally this also sets an upper limit on the low-energy SUSY masses as will be reviewed in the next section.

The full particle spectrum of model **FUT**, compliant with quark mass constraints and the B-physics observables is shown in Fig. 3. In the upper (lower) plot we impose $M_h = 125.6 \pm 3.1\,(2.1)$ GeV. Including the Higgs mass constraints in general favors the lower part of the SUSY particle mass spectra (as compared to previous evaluations [113–117]). The "old" uncertainty estimate of ± 3.1 GeV permits SUSY masses in the multi-TeV range, which would remain unobservable at the LHC, the ILC or CLIC. Even the mass of the LSP (the lightest neutralino) could be above 2 TeV. On the other hand, using the "improved" theory estimate of ± 2.1 GeV[3] results in substantially lower upper limits of the SUSY mass spectrum. In this case the LSP ranges from about 0.6 TeV to about 1.5 TeV, so that it could be produced at CLIC (with $\sqrt{s} = 3$ TeV) via the process $e^+e^- \to \tilde{\chi}_1^0\tilde{\chi}_1^0\alpha$. Also the second lightest neutralino as well as the two scalar taus could be in a mass range either accessible at the ILC (depending on the final center-of-mass energy) or at CLIC. The lightest scalar tau always turns out to be the Lightest Observable SUSY particle (LOSP). Similarly, the light chargino mass is found between ~ 1.2 TeV and ~ 2.6 TeV, with the second chargino mass slightly higher. The colored spectrum (scalar tops and bottoms, as well as the gluino) all have masses well above 1.7 TeV and are bounded from above by about $4 - 6$ TeV. Only for the lighter part of the spectrum a discovery at the LHC might be possible. At the HL-LHC larger parts of the spectrum can be covered, but still part of the spectrum remains out of reach. The heavy Higgs boson masses range between ~ 1.2 TeV and ~ 5 TeV. The lower part could be covered at the LHC (in particular for

[3] A more precise estimate requires a re-analysis of all sources of missing higher-order corrections.

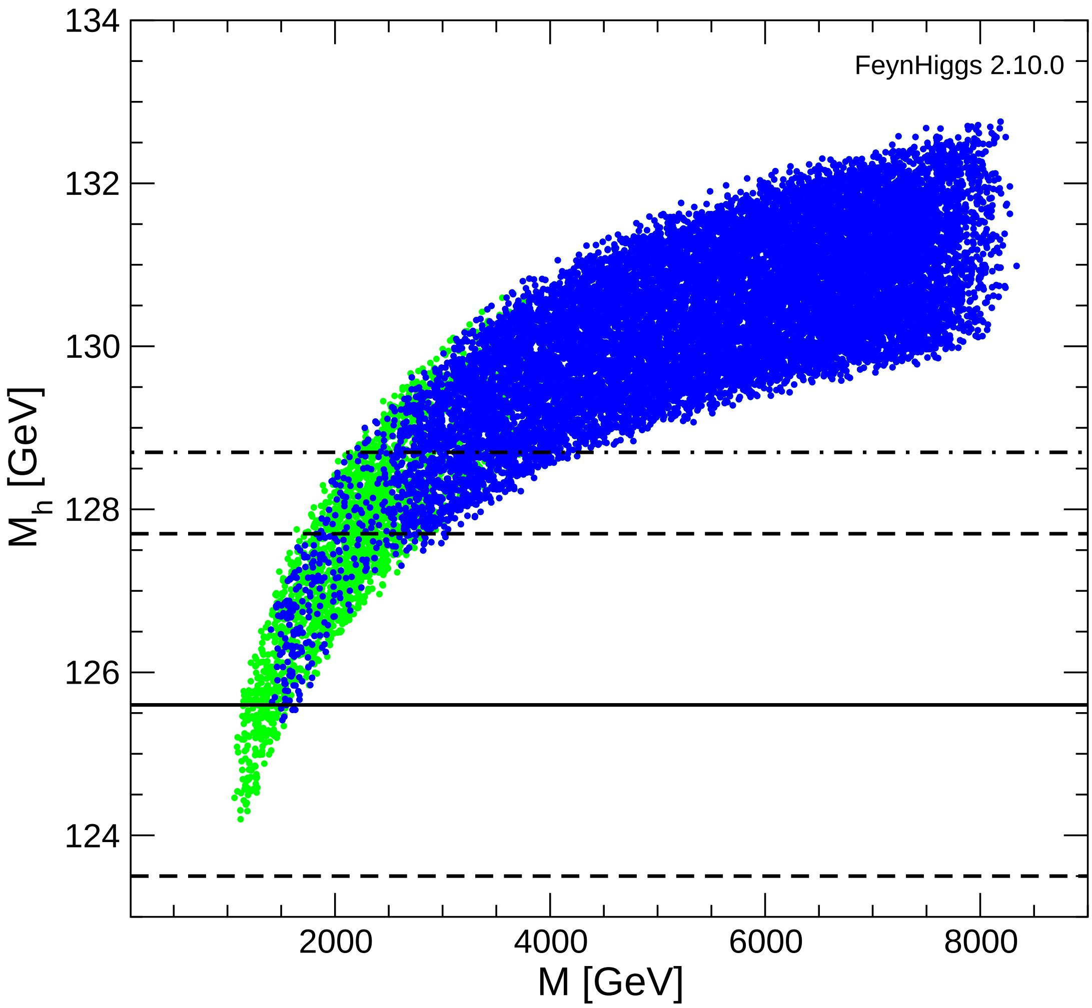

Figure 2: The lightest Higgs boson mass, M_h, as a function of M in the model **FUT**. The blue points fulfill the B-physics constraints (see text).

the high $\tan\beta$ values found in our analysis) or later at CLIC, whereas the higher part could escape all current and planned collider experiments. The mass gap found for the masses of the heavy Higgs bosons stems from the fact that for intermediate values too *low* values of $\mathrm{BR}(B_s \to \mu^+\mu^-)$ are found, whereas in the very high-mass regime the SM value is recovered.

Overall, the discovery of a Higgs boson, interpreted as the lightest MSSM Higgs boson, together with the refined M_h calculation allows to put substantially improved limits on the allowed particle spectrum. While in the older evaluations always large parts of the parameter space where out of reach for the LHC, the ILC and CLIC, the improved analysis nearly guarantees the discovery of one or more particles at the LHC or future e^+e^- colliders.

4 Conclusions

The serious problem of the appearance of many free parameters in the SM of Elementary Particle Physics, takes dramatic dimensions in the MSSM, where the free parameters are proliferated by at least hundred more, while it is considered as the best candidate for

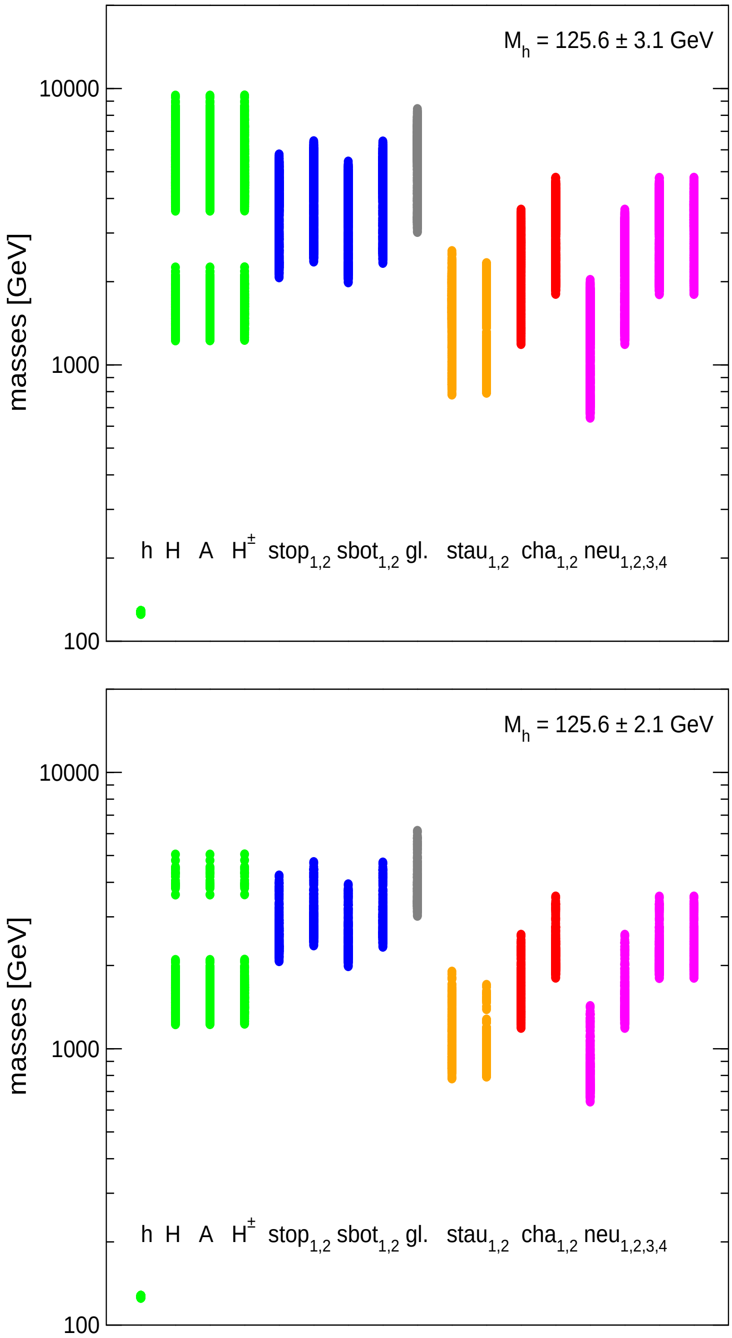

Figure 3: The upper (lower) plot shows the spectrum of **FUT** (with $\mu < 0$) after imposing the constraint $M_h = 125.6 \pm 3.1\,(2.1)$ GeV. The points shown are in agreement with the quark mass constraints and the B-physics observables. The light (green) points on the left are the various Higgs boson masses. The dark (blue) points following are the two scalar top and bottom masses, followed by the lighter (gray) gluino mass. Next come the lighter (beige) scalar tau masses. The darker (red) points to the right are the two chargino masses followed by the lighter shaded (pink) points indicating the neutralino masses.

Physics Beyond the SM. The idea that the Theory of Particle Physics is more symmetric at high scales, which is broken but has remnant predictions in the much lower scales of the SM, found its best realisation in the framework of the MSSM assuming further a GUT beyond the scale of the unification of couplings. However, the unification idea, although successful, seems to have exhausted its potential to reduce further the free parameters of the SM.

A new interesting possibility towards reducing the free parameters of a theory has been put forward in refs. [25, 26] which consists on a systematic search on the RGI relations among couplings. This method might lead to further symmetry, however its scope is much wider. After several trials it seems that the basic idea found very nice realisations in a Finite Unified Theory and the MSSM. In the first case one is searching for RGI relations among couplings holding beyond the unification scale, which morever guarantee finiteness to all-orders in perturbation theory. In the second, the search of RGI relations among couplings is concentrated within the MSSM itself and the assumption of GUT is not necessarily required. The results in both cases are indeed impressive as we have discussed. Certainly one can add some more comments on the Finite Unified Theories. These are related to some fundamental developments in Theoretical Particle Physics based on reconsiderations of the problem of divergencies and serious attempts to solve it. They include the motivation and construction of string and noncommutative theories, as well as $N = 4$ supersymmetric field theories [118, 119], $N = 8$ supergravity [120–124] and the AdS/CFT correspondence [125]. It is a thoroughly fascinating fact that many interesting ideas that have survived various theoretical and phenomenological tests, as well as the solution to the UV divergencies problem, find a common ground in the framework of $N = 1$ Finite Unified Theories, which have been discussed here. From the theoretical side they solve the problem of UV divergencies in a minimal way. On the phenomenological side in both cases of reduction of couplings discussed here the celebrated success of predicting the top-quark mass [15, 17] is now extended to the predictions of the Higgs masses and the supersymmetric spectrum of the MSSM, which so far have been confronted very successfully with the findings and bounds at the LHC.

The various scenarios will be refined/scrutinized in various ways in the upcoming years. Important improvements in the analysis are expected from progress on the theory side, in particular in an improved calculation of the light Higgs boson mass. The corrections introduced in [90] not only introduce a shift in M_h (which should to some extent be covered by the estimate of theory uncertainties). They will also reduce the theory uncertainties, see [90, 108], and in this way refine the selected model points, leading to a sharper prediction of the allowed spectrum. One can hope that with even more higher-order corrections included in the M_h calculation an uncertainty below the 0.5 GeV level can be reached.

The other important improvements in the future will be the continuing searches for SUSY particles at collider experiments. The LHC will re-start in 2015 with an increased center-of-mass enery of $\sqrt{s} \sim 13$ TeV, largely extending its SUSY search reach. The lower parts of the currently allowed/predicted colored SUSY spectra will be tested in this way. For the electroweak particles, on the other hand, e^+e^- colliders might be the better option. The ILC, operating at $\sqrt{s} \lesssim 1$ TeV, has only a limited potential for our model spectra. Going to higher energies, $\sqrt{s} \lesssim 3$ TeV, that might be realized at CLIC, large parts of the predicted electroweak model spectra can be covered.

All spectra, however, (at least with the current calculation of M_h and its corresponding uncertainty), contain parameter regions that will escape the searches at the LHC, the ILC and CLIC. In this case we would remain with a light Higgs boson in the decoupling limit, i.e. would be undistinguishable from a SM Higgs boson. The only hope to overcome this situation is that an improved M_h calculation would cut away such high spectra.

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