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Cohomological properties of the massless tensor field with the mixed symmetry $(k, 1)$. I. Results on the cohomology of the exterior longitudinal differential

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Abstract

In this former part of a paper dedicated to the computation of local BRST cohomology for a free massless tensor field with the mixed symmetry $(k, 1)$ $(k \geq 4)$ we focus on the main cohomological properties of the exterior longitudinal differential.

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1 Introduction

Real tensor fields transforming according to exotic representations of $GL(D, \mathbb{R})$ corresponding to two-column Young diagrams with $(k+1)$ cells and k lines ("hook" diagrams) or, briefly, tensor fields with the mixed symmetry $(k, 1)$ have been studied starting more than two decades ago in [1]–[5] and more recently (inclusively within the BRST setting) in [6]–[9]. Such fields are present for instance in the bosonic sector of Chern–Simons gravities in odd dimensions [10]–[12] due to the fact that the free limit of their massless version describes one of the dual formulations of linearized gravity in $k+3$ spacetime dimensions. The limit $k = 1$ provides the linearized Einstein–Hilbert action without cosmological terms, known as the Pauli–Fierz model [13, 14].

The aim of this paper is to analyze the main properties of the local BRST cohomology for the free theory describing a massless tensor field with the mixed symmetry $(k, 1)$ for $k \geq 4$. The case $k = 2$ is covered in [15] and $k = 3$ respectively in [16]. To this end we rely on the general BRST cohomological results for gauge field theories with a well-defined Cauchy order [17]–[20] completed by specific techniques and results from [15, 16] and [21]–[26]. In this context the findings on some BRST cohomological aspects related to a massless tensor field corresponding to a two-column non-rectangular Young tableau [27] are also interesting. More precisely, in this former part we will evaluate the cohomology of the exterior longitudinal differential and its local version. The latter part [28] will be dedicated to the computation of the local cohomology of the Koszul–Tate differential and of its invariant version and finally to the study of the core properties of the local cohomology of the BRST differential in maximum form degree. We use the conventions, notations, and results from [29] on the Lagrangian formulation and BRST symmetry of a single massless tensor field $(k, 1)$.

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2 Lagrangian formulation. BRST symmetry

Let $t_{\mu_1\ldots\mu_k|\alpha}$ be a real tensor field with the mixed symmetry $(k,1)$ on a D-dimensional Minkowski space $\mathcal{M}$, meaning it is antisymmetric with respect to its first k indices and satisfies the identity $t_{[\mu_1\ldots\mu_k|\alpha]}\equiv 0$, where $[\mu\ldots\nu]$ stands for full antisymmetry. We assume that $\mathcal{M}$ is endowed with the metric $\sigma_{\mu\nu}=\sigma^{\mu\nu}=(-+\ldots+)$. The trace of this field, $t_{\mu_1\ldots\mu_{k-1}}=t_{\mu_1\ldots\mu_k|\alpha}\sigma^{\mu_k\alpha}$, defines an antisymmetric tensor of order $(k-1)$.

The Lagrangian formulation of a free, massless tensor field $(k,1)$ $(k\geq 4)$ relies on the general principle of gauge invariance in terms of a generating set of gauge symmetries

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} t_{\mu_1\ldots\mu_k|\alpha}=\partial_{[\mu_1}\overset{(1)}{\theta}_{\mu_2\ldots\mu_k]|\alpha}+\partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_k\alpha]}+(-)^{k+1}(k+1)\partial_\alpha\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}, \tag{1}$$

which renders in the limits $k=2$ and $k=3$ the well-known results [15, 16]. The gauge parameters $\overset{(1)}{\theta}$ display the mixed symmetry $(k-1,1)$, so they are antisymmetric in their first $(k-1)$ indices and fulfill the identity $\overset{(1)}{\theta}_{[\mu_1\ldots\mu_{k-1}|\alpha]}\equiv 0$, while $\overset{(1)}{\epsilon}$ are fully antisymmetric. It has been shown in [29] that the corresponding Lagrangian reads as

$$S_0^{\mathrm{t}}[t_{\mu_1\ldots\mu_k|\alpha}]=-\tfrac{1}{2\cdot(k+1)!}\int\left[F_{\mu_1\ldots\mu_{k+1}|\alpha}F^{\mu_1\ldots\mu_{k+1}|\alpha}-(k+1)F_{\mu_1\ldots\mu_k}F^{\mu_1\ldots\mu_k}\right]d^Dx, \tag{2}$$

where $D\geq k+2$ in order to ensure a non-negative number of physical degrees of freedom. The tensor $F_{\mu_1\ldots\mu_{k+1}|\alpha}$ is linear in the first-order derivatives of field components

$$F_{\mu_1\ldots\mu_{k+1}|\alpha}=\partial_{[\mu_1}t_{\mu_2\ldots\mu_{k+1}]|\alpha}, \tag{3}$$

exhibits the mixed symmetry $(k+1,1)$, and possesses the gauge transformation

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} F_{\mu_1\ldots\mu_{k+1}|\alpha}=(-)^{k+1}k\partial_\alpha\partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_{k+1}]}. \tag{4}$$

Its trace, $F_{\mu_1\ldots\mu_k}=F_{\mu_1\ldots\mu_{k+1}|\alpha}\sigma^{\mu_{k+1}\alpha}$, is completely antisymmetric

$$F_{\mu_1\ldots\mu_k}=\partial_{[\mu_1}t_{\mu_2\ldots\mu_k]}+(-)^k\partial^\alpha t_{\mu_1\ldots\mu_k|\alpha}, \tag{5}$$

and presents the gauge variation $\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}} F_{\mu_1\ldots\mu_k}=-k\partial^\alpha\partial_{[\alpha}\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k]}$. The generating set of gauge transformations of action (2), given by (1), has been shown in [29] to be Abelian and off-shell reducible of order $(k-1)$.

The field equations

$$\frac{\delta S_0^{\mathrm{t}}}{\delta t_{\nu_1\ldots\nu_k|\alpha}}\equiv\tfrac{1}{k!}T^{\nu_1\ldots\nu_k|\alpha}\approx 0, \tag{6}$$

are expressed in terms of the tensor $T^{\nu_1\ldots\nu_k|\alpha}$, linear in the field components $t_{\mu_1\ldots\mu_k|\beta}$, first-order in its derivatives, and with the mixed symmetry $(k,1)$

$$\begin{aligned}T^{\nu_1\ldots\nu_k|\alpha} &= \Box t^{\nu_1\ldots\nu_k|\alpha}+\partial_\mu\big((-)^k\partial^{[\nu_1}t^{\nu_2\ldots\nu_k]\mu|\alpha}-\partial^\alpha t^{\nu_1\ldots\nu_k|\mu}\big)+(-)^{k+1}\partial^\alpha\partial^{[\nu_1}t^{\nu_2\ldots\nu_k]}\\ &\quad+\sigma^{\alpha[\nu_1}\big[(-)^k\Box t^{\nu_2\ldots\nu_k]}+\partial_\mu\big((-)^{k+1}\partial_\beta t^{\nu_2\ldots\nu_k]\mu|\beta}-\partial^{\nu_2}t^{\nu_3\ldots\nu_k]\mu}\big)\big].\end{aligned} \tag{7}$$

It is useful to write $T^{\nu_1\ldots\nu_k|\alpha}$ in terms of the tensor $F_{\mu_1\ldots\mu_{k+1}|\beta}$ introduced in (3)

$$T^{\nu_1\ldots\nu_k|\alpha}=\partial_\mu F^{\mu\nu_1\ldots\nu_k|\alpha}-\sigma^{\alpha[\nu_1}\partial_\mu F^{\nu_2\ldots\nu_k\mu]}. \tag{8}$$

The most general gauge-invariant quantities constructed out of $t_{\mu_1\ldots\mu_k|\alpha}$ and its space-time derivatives are given by the components of the "curvature tensor"

$$K_{\mu_1\ldots\mu_{k+1}|\alpha\beta} = \partial_\alpha F_{\mu_1\ldots\mu_{k+1}|\beta} - F_{\mu_1\ldots\mu_{k+1}|\alpha} \equiv \partial_{[\mu_1} t_{\mu_2\ldots\mu_{k+1}]|[\beta,\alpha]}, \tag{9}$$

together with their derivatives. The tensor $K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}$ is linear in the original field, second-order in its derivatives, and displays the mixed symmetry $(k+1,2)$, so it is separately antisymmetric in its first $(k+1)$ indices and in the last two ones and satisfies the first Bianchi identity $K_{[\mu_1\ldots\mu_{k+1}|\alpha]\beta} \equiv 0$. Moreover, it satisfies the second Bianchi identity

$$\partial_{[\mu_1} K_{\mu_2\ldots\mu_{k+2}]|\alpha\beta} \equiv 0, \quad K_{\mu_1\ldots\mu_{k+1}|[\alpha\beta,\gamma]} \equiv 0. \tag{10}$$

The invariance of action (2) with respect to the gauge transformations (1) is equivalent to the Noether identities $\partial_{\nu_1} T^{\nu_1\ldots\nu_k|\alpha} \equiv 0$, $\partial_\alpha T^{\nu_1\ldots\nu_k|\alpha} \equiv 0$, while the reducibility of this generating set of gauge symmetries shows that not all Noether identities are independent. The free theory of a massless $(k,1)$ tensor field satisfies the general regularity conditions [30] and generates a linear gauge theory with the Cauchy order equal to $(k+1)$.

Next, we briefly review the antibracket-antifield BRST symmetry of this free theory, exposed in [29]. The first step of this procedure requires the identification of the algebra on which the BRST differential s acts. The BRST generators are of two kinds: fields/ghosts and antifields. The ghost spectrum is composed of the tensor fields

$$\Big\{ \big\{ \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} \big\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}_\mu \Big\}, \tag{11}$$

where $\overset{(1)}{C}$ and $\overset{(1)}{\eta}$ are respectively associated with the gauge parameters $\overset{(1)}{\theta}$ and $\overset{(1)}{\epsilon}$ from (1), while the other ghost fields correspond to the reducibility parameters detailed in [29]. We ask that $\overset{(m)}{C}$ with $m = \overline{1,k-1}$ possess the mixed symmetry $(k-m,1)$, and therefore are antisymmetric in their first $(k-m)$ (where applicable) and fulfill the identities $\overset{(m)}{C}_{[\mu_1\ldots\mu_{k-m}|\alpha]} \equiv 0$, $m = \overline{1,k-1}$, while $\overset{(m)}{\eta}$ with $m = \overline{1,k-1}$ remain antisymmetric. For further purposes we make the compact notation

$$\Phi^A \equiv \Big\{ t_{\mu_1\ldots\mu_k|\alpha}, \big\{ \overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} \big\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}_\mu \Big\}. \tag{12}$$

The antifield spectrum corresponds to the original fields and to the newly added ghosts, being structured into

$$\Phi^*_A \equiv \Big\{ t^{*\mu_1\ldots\mu_k|\alpha}, \big\{ \overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}|\alpha}, \overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} \big\}_{m=\overline{1,k-1}}, \overset{(k)}{\eta}{}^{*\mu} \Big\}. \tag{13}$$

The mixed symmetry/antisymmetry properties of the antifields are the same with those of the corresponding fields/ghosts, so in particular $t^{*[\mu_1\ldots\mu_k|\alpha]} \equiv 0$, $\overset{(m)}{C}{}^{*[\mu_1\ldots\mu_{k-m}|\alpha]} \equiv 0$, $m = \overline{1,k-1}$.

The BRST differential of this free model splits into

$$s = \delta + \gamma, \quad s^2 = 0 \Leftrightarrow (\delta^2 = 0,\, \gamma^2 = 0,\, \delta\gamma + \gamma\delta = 0), \tag{14}$$

with δ the Koszul–Tate differential, $\mathbb{N}$-graded in terms of the antighost number agh ($\mathrm{agh}(\delta) = -1$) and γ the exterior longitudinal derivative, which is a true differential here, anticommuting with δ and $\mathbb{N}$-graded according to the pure ghost number pgh (pgh$(\gamma) = 1$). These two degrees are independent (agh$(\gamma) = 0$, pgh$(\delta) = 0$). The overall degree of the BRST differential is the ghost number (gh), defined as the difference between pgh and agh, such that gh$(s) = $ gh$(\delta) = $ gh$(\gamma) = 1$. Consequently, the BRST differential is $\mathbb{Z}$-graded in terms of gh. The standard rules of the antibracket-antifield formalism endow the generators of the BRST complex with the gradings collected in Table 1, where ε denotes the Grassmann parity.

BRST generator	pgh	agh	gh	ε
$t_{\mu_1\ldots\mu_k\vert\alpha}$	0	0	0	0
$\left\{\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha}\right\}_{m=\overline{1,k-1}}$	m	0	m	m mod 2
$\left\{\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}}\right\}_{m=\overline{1,k}}$	m	0	m	m mod 2
$t^{*\mu_1\ldots\mu_k\vert\alpha}$	0	1	-1	1
$\left\{\overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}\vert\alpha}\right\}_{m=\overline{1,k-1}}$	0	$m+1$	$-(m+1)$	$(m+1)$ mod 2
$\left\{\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}}\right\}_{m=\overline{1,k}}$	0	$m+1$	$-(m+1)$	$(m+1)$ mod 2

Table 1: Gradings of BRST generators.

The actions of the operators δ and γ on the BRST generators (assuming they act like right derivations) that comply with all the BRST requirements are expressed by

$$\gamma t_{\mu_1\ldots\mu_k\vert\alpha} = \partial_{[\mu_1}\overset{(1)}{C}_{\mu_2\ldots\mu_k]\vert\alpha} + \partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\partial_\alpha\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}, \tag{15}$$

$$\begin{aligned}\gamma\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha} &= \partial_{[\mu_1}\overset{(m+1)}{C}_{\mu_2\ldots\mu_{k-m}]\vert\alpha} + \partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m}\alpha]}\\ &+(-)^{k-m+1}(k-m+1)\partial_\alpha\overset{(m+1)}{\eta}_{\mu_1\ldots\mu_{k-m}},\quad m=\overline{1,k-2},\end{aligned} \tag{16}$$

$$\gamma\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}} = \tfrac{k-m}{k-m+2}\partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\ldots\mu_{k-m+1}]},\quad m=\overline{1,k-1}, \tag{17}$$

$$\gamma\overset{(k-1)}{C}_{\mu_1\vert\alpha} = \partial_{(\mu_1}\overset{(k)}{\eta}_{\alpha)},\quad \gamma\overset{(k)}{\eta}_\mu = 0,\quad \gamma\Phi^*_A = 0, \tag{18}$$

$$\delta\Phi^A = 0,\quad \delta t^{*\mu_1\ldots\mu_k\vert\alpha} = -\tfrac{1}{k!}T^{\mu_1\ldots\mu_k\vert\alpha}, \tag{19}$$

$$\delta\overset{(1)}{C}{}^{*\mu_1\ldots\mu_{k-1}\vert\alpha} = -\partial_\mu\left(kt^{*\mu\mu_1\ldots\mu_{k-1}\vert\alpha} + (-)^k t^{*\mu_1\ldots\mu_{k-1}\alpha\vert\mu}\right), \tag{20}$$

$$\begin{aligned}\delta\overset{(m)}{C}{}^{*\mu_1\ldots\mu_{k-m}\vert\alpha} &= (-)^m\partial_\mu\Bigg((k-m+1)\overset{(m-1)}{C}{}^{*\mu\mu_1\ldots\mu_{k-m}\vert\alpha}\\ &+(-)^{k-m+1}\overset{(m-1)}{C}{}^{*\mu_1\ldots\mu_{k-m}\alpha\vert\mu}\Bigg),\quad m=\overline{2,k-2},\end{aligned} \tag{21}$$

$$\delta\overset{(k-1)}{C}{}^{*\mu_1\vert\alpha} = (-)^{k-1}\partial_\mu\overset{(k-2)}{C}{}^{*\mu(\mu_1\vert\alpha)},\quad \delta\overset{(1)}{\eta}{}^{*\mu_1\ldots\mu_k} = (-)^k(k+1)\partial_\alpha t^{*\mu_1\ldots\mu_k\vert\alpha}, \tag{22}$$

$$\begin{aligned}\delta\overset{(m)}{\eta}{}^{*\mu_1\ldots\mu_{k-m+1}} &= (-)^k(k-m+2)\partial_\alpha\overset{(m-1)}{C}{}^{*\mu_1\ldots\mu_{k-m+1}\vert\alpha}\\ &+\tfrac{(-)^m(k-m+2)(k-m+1)}{k-m+3}\partial_\mu\overset{(m-1)}{\eta}{}^{*\mu\mu_1\ldots\mu_{k-m+1}},\quad m=\overline{2,k},\end{aligned} \tag{23}$$

with $T^{\mu_1 \cdots \mu_k|\alpha}$ like in (7). These definitions may be written more compactly if we perform some linear transformations on the ghosts/antifields without affecting their homogeneity with respect to the various gradings

$$\overset{(m)}{C'}_{\mu_1 \ldots \mu_{k-m}||\alpha} \equiv \overset{(m)}{C}_{\mu_1 \ldots \mu_{k-m}|\alpha} + (k-m+2) \overset{(m)}{\eta}_{\mu_1 \ldots \mu_{k-m}\alpha}, \tag{24}$$

$$\overset{(m)}{C'}{}^{*\mu_1 \ldots \mu_{k-m}||\alpha} \equiv \overset{(m)}{C}{}^{*\mu_1 \ldots \mu_{k-m}|\alpha} + \tfrac{1}{k-m+2} \overset{(m)}{\eta}{}^{*\mu_1 \ldots \mu_{k-m}\alpha}, \tag{25}$$

with $m = \overline{1, k-1}$. The double bar "$||$" means full antisymmetry with respect to the indices placed before (if applicable) without further identities. The redefined variables are useful at various computations since for every $m = \overline{1, k-1}$ the independent components of the ghost tensor $\overset{(m)}{C'}$ are given by the union between the independent components of all ghosts of pure ghost number m, namely $\overset{(m)}{C}$ and $\overset{(m)}{\eta}$. Similarly, $\overset{(m)}{C'}{}^{*}$ gather all the independent components of the antifields with the antighost number equal to $(m+1)$. Now, some of formulas (15)–(23) take a simpler form

$$\gamma t_{\mu_1 \ldots \mu_k|\alpha} = \partial_{[\mu_1} \overset{(1)}{C'}_{\mu_2 \ldots \mu_k]||\alpha} - \tfrac{1}{k+1} \partial_{[\mu_1} \overset{(1)}{C'}_{\mu_2 \ldots \mu_k||\alpha]}, \tag{26}$$

$$\gamma \overset{(m)}{C'}_{\mu_1 \ldots \mu_{k-m}||\alpha} = \partial_{[\mu_1} \overset{(m+1)}{C'}_{\mu_2 \ldots \mu_{k-m}]||\alpha}, \;\; m = \overline{1, k-2}, \tag{27}$$

$$\gamma \overset{(k-1)}{C'}_{\mu_1||\alpha} = 2\partial_{\mu_1} \overset{(k)}{\eta}_{\alpha}, \delta \overset{(1)}{C'}{}^{*\mu_1 \ldots \mu_{k-1}||\alpha} = -k\partial_\mu t^{*\mu\mu_1 \ldots \mu_{k-1}|\alpha}, \delta \overset{(k)}{\eta}{}^{*\alpha} = (-)^k 2\partial_{\mu_1} \overset{(k-1)}{C'}{}^{*\mu_1||\alpha}, \tag{28}$$

$$\delta \overset{(m)}{C'}{}^{*\mu_1 \ldots \mu_{k-m}||\alpha} = (-)^m (k-m+1) \partial_\mu \overset{(m-1)}{C'}{}^{*\mu\mu_1 \ldots \mu_{k-m}||\alpha}, \;\; m = \overline{2, k-1}. \tag{29}$$

3 Local BRST cohomology. Generalities

All cohomological computations will be carried out on the algebra of local differential forms with coefficients from the BRST algebra without explicit dependence on the spacetime coordinates x^μ, to be denoted by Λ. In other words, the form coefficients are elements of the BRST algebra $\mathcal{A}$ of local "functions" that do not explicitly depend on the global coordinates of the Minkowski spacetime $\mathcal{M}$, and therefore polynomials in ghosts, antifields, and their spacetime derivatives up to a finite order, 'smooth' in the original tensor field with the mixed symmetry $(k, 1)$ and also polynomials in its derivatives up to a finite order. Consequently, the algebra Λ will inherit the four gradings of the BRST algebra [the $\mathbb{Z}_2$-grading in terms of the Grassmann parity ε, the $\mathbb{Z}$-grading according to gh as well as the two $\mathbb{N}$-gradings involving agh and pgh] introduced via Table 1, accompanied by

$$\varepsilon(dx^\mu) = 1, \quad \varepsilon(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}) = p \bmod 2, \tag{30}$$

$$\mathrm{agh}(dx^\mu) = 0, \quad \mathrm{pgh}(dx^\mu) = 0, \quad \mathrm{gh}(dx^\mu) = 0, \tag{31}$$

where $\wedge$ is the symbol for wedge product. In addition, Λ is endowed with a supplementary $\mathbb{N}$-grading in terms of the form degree deg

$$\Lambda = \bigoplus_{p \in \mathbb{N}} \overset{[p]}{\Lambda}, \quad \deg(\overset{[p]}{\omega}) = p \Leftrightarrow \overset{[p]}{\omega} \in \overset{[p]}{\Lambda}, \tag{32}$$

$$\overset{[p]}{\omega} = \tfrac{1}{p!} a_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \;\; a_{\mu_1 \ldots \mu_p} \in \mathcal{A}. \tag{33}$$

Since the dimension of $\mathcal{M}$ is by hypothesis finite and denoted by D, the decomposition (32) stops at D; all forms that are homogeneous with respect to deg like in (33) and with the form degree $p > D$ vanish. The operators δ, γ, and s are extended to the algebra Λ via relations (15)–(23) together with

$$\delta(dx^{\mu}) = 0, \quad \gamma(dx^{\mu}) = 0, \quad s(dx^{\mu}) = 0 \tag{34}$$

and by assuming their actions as right derivatives on Λ with respect to the wedge product. In this context we recall that for any element of the form (33) with $a_{\mu_1 \ldots \mu_p} \in \mathcal{A}$ of well-defined Grassmann parity, $\varepsilon(a)$, we have that

$$\varepsilon(\overset{[p]}{\omega}) = [\varepsilon(a) + p] \bmod 2. \tag{35}$$

In this way all the properties of the operators δ, γ, and $s = \delta + \gamma$ are transferred from the BRST algebra $\mathcal{A}$ to the algebra of local forms Λ. In particular, these operators remain differentials and δ still anticommutes with γ. Also, δ continues to be acyclic on Λ in strictly positive values of the antighost number agh, $H(\delta) \equiv H_0(\delta)$, and it makes sense to compute the cohomology algebras $H(\gamma|H_0(\delta))$ and $H(s)$. Moreover, the isomorphisms $H^j(s) \simeq H^0_{-j}(\delta)$ for $j < 0$ and $H^l(s) \simeq H^l(\gamma|H_0(\delta))$ for $l \geq 0$ still hold. Regarding the last relations, j from $H^j(s)$ stands for the ghost number, $(-j)$ from $H^0_{-j}(\delta)$ represents the antighost number, and the superscript 0 refers to the value equal to zero of pgh; l from $H^l(s)$ means the ghost number, while l from $H^l(\gamma|H_0(\delta))$ signifies the pure ghost number. From (34) we notice that the form degree of δ, γ, and s is equal to zero

$$\deg(\delta) = \deg(\gamma) = \deg(s) = 0. \tag{36}$$

We define a linear operator $d : \Lambda \to \Lambda$ as an odd, right derivation

$$da = \partial_{\mu} a \, dx^{\mu}, \;\; a \in \mathcal{A}, \quad d(dx^{\mu}) = 0, \tag{37}$$

$$d(\omega_1 \wedge \omega_2) = \omega_1 \wedge d\omega_2 + (-)^{\varepsilon(\omega_2)} (d\omega_1) \wedge \omega_2, \;\; \omega_{1,2} \in \Lambda, \tag{38}$$

where it was assumed that $\omega_{1,2}$ possess well-defined Grassmann parities. The operator d becomes a differential on Λ with respect to deg, traditionally known as the exterior spacetime differential: $\varepsilon(d) = 1$, $\deg(d) = +1$, $d^2 = 0$. From (31) and (37) it follows that

$$\mathrm{agh}(d) = 0, \quad \mathrm{pgh}(d) = 0, \quad \mathrm{gh}(d) = 0. \tag{39}$$

The operators δ, γ, and s are also differentials that anticommute with d on Λ

$$O^2 = 0 = d^2, \quad Od + dO = 0, \quad O = \delta, \gamma, s. \tag{40}$$

Their gradings to not interfere

$$\mathrm{grad}(d) = 0, \quad \deg(O) = 0, \quad \mathrm{grad} = \mathrm{agh}, \mathrm{pgh}, \mathrm{gh}, \tag{41}$$

so it makes sense to compute the local cohomologies $H(O|d)$ in Λ. These are standardly defined like the set of equivalence classes of local forms O-closed modulo d, $O\omega + dj = 0$, modulo the local forms that are O-exact modulo d, $\omega' = Ow + dm$. We highlight that there is a strict correspondence between O and grad, namely: $O = \delta \leftrightarrow \mathrm{grad} = \mathrm{agh}$, $O = \gamma \leftrightarrow \mathrm{grad} = \mathrm{pgh}$, $O = s \leftrightarrow \mathrm{grad} = \mathrm{gh}$. These means that whenever $O = s$ the local BRST cohomology $H(s|d)$ is a vector space simultaneously $\mathbb{Z}_2$-graded (according to the

Grassmann parity) and $\mathbb{Z}$-graded in terms of gh, $H(s|d) = \bigoplus_{g\in\mathbb{Z}} H^g(s|d)$, where for every $g \in \mathbb{Z}$ the space $H^g(s|d)$ is in turn $\mathbb{N}$-graded according to the form degree, $H^g(s|d) = \bigoplus_{p=0}^{D} H^{g,p}(s|d)$. The subspace $H^{g,p}(s|d)$ is called local BRST cohomology in ghost number g and form degree p. If $O = \delta$, then the local cohomology of the Koszul–Tate differential $H(\delta|d)$ is a vector space $\mathbb{Z}_2$-graded and meanwhile $\mathbb{N}$-graded in terms of agh, $H(\delta|d) = \bigoplus_{j\in\mathbb{N}} H_j(\delta|d)$, where for every $j \in \mathbb{N}$ the space $H_j(\delta|d)$ is again $\mathbb{N}$-graded according to deg, $H_j(\delta|d) = \bigoplus_{p=0}^{D} H_j^p(\delta|d)$. The subspace $H_j^p(\delta|d)$ is known as the local cohomology of the Koszul–Tate differential in antighost number j and form degree p. Finally, if $O = \gamma$, then the local cohomology of the exterior longitudinal differential $H(\gamma|d)$ is a vector space $\mathbb{Z}_2$-graded, but also $\mathbb{N}$-graded in terms of pgh, $H(\gamma|d) = \bigoplus_{l\in\mathbb{N}} H^l(\gamma|d)$, where for every $l \in \mathbb{N}$ the space $H^l(\gamma|d)$ is $\mathbb{N}$-graded according to deg, $H^l(\gamma|d) = \bigoplus_{p=0}^{D} H^{l,p}(\gamma|d)$. The subspace $H^{l,p}(\gamma|d)$ means the local cohomology of the exterior longitudinal differential in pure ghost number l and form degree p.

The study of the local BRST cohomology is an essential step in view of constructing consistent interactions involving a massless tensor field with the mixed symmetry $(k,1)$ by means of the deformation of the solution to the master equation [31]–[34]. This deformation method requires the computation of the local BRST cohomology in ghost number 0 and in maximum form degree. From this perspective in what follows we approach the main cohomological ingredients related to the spaces $H(\gamma)$ and $H(\gamma|d)$.

4 $H(\gamma)$ and $H(\gamma|d)$

In the sequel we evaluate the cohomology algebra $H(\gamma)$ in the algebra of local forms Λ, defined like the set of equivalence classes of γ-closed local forms modulo γ-exact ones. Due to the second relation in (34) it is enough to compute $H(\gamma)$ in the BRST algebra of local "functions" $\mathcal{A}$, defined as the set of equivalence classes of γ-closed elements from $\mathcal{A}$ modulo γ-exact ones. The computation of the cohomology $H(\gamma)$ in $\mathcal{A}$ or in Λ makes sense since the operator γ is a true differential on both algebras in this case, with $\mathrm{pgh}(\gamma) = +1$, $\gamma^2 = 0$. We recall that $H(\gamma)$ defines a supercommutative algebra ($\mathbb{Z}_2$-graded), $\mathbb{N}$-graded in terms of pgh, $H(\gamma) = \bigoplus_{l\in\mathbb{N}} H^l(\gamma)$. Moreover, if we work on Λ, then for every $l \in \mathbb{N}$ the space $H^l(\gamma)$ is also $\mathbb{N}$-graded with respect to the form degree

$$H^l(\gamma) = \bigoplus_{p=0}^{D} H^{l,p}(\gamma), \quad l \in \mathbb{N}. \tag{42}$$

We rely on definitions (15)–(18) and approach the construction gradually, according to the increasing values of pgh.

From Table 1 we observe that there are no BRST generators with negative pure ghost numbers, such that in pgh = 0 the cohomology $H^0(\gamma)$ coincides with the kernel of γ, $H^0(\gamma) = (\mathrm{Ker}(\gamma))^0$ and, due to the additive behavior of pgh with respect to the multiplication operation on $\mathcal{A}$, it will actually be an algebra. Table 1 helps us to identify the BRST generators of pure ghost number equal to 0 being given by the antifields Φ_A^* introduced in (13) and their spacetime derivatives up to a finite order together with the field $t_{\mu_1\ldots\mu_k|\alpha}$ and its derivatives up to a finite order. The last definition from (18) implies that the (polynomial) dependence on $[\Phi_A^*]$ produces elements belonging to $(\mathrm{Ker}(\gamma))^0$, and thus implicitly to $H^0(\gamma)$, where the generic notation $f[\varphi]$ means that f depends on φ and its derivatives up to a finite order. Relation (15) compared with (1) shows that the action of γ on the field with the mixed symmetry $(k,1)$ follows from its gauge transformation by replacing the gauge parameters $\left\{\overset{(1)}{\theta},\overset{(1)}{\epsilon}\right\}$ respectively with the ghosts $\left\{\overset{(1)}{C},\overset{(1)}{\eta}\right\}$. Since the most general gauge-invariant quantities constructed out of the field with the mixed symmetry $(k,1)$ and its derivatives are given by the curvature tensor (9) together with its derivatives, we obtain that the entire dependence on $t_{\mu_1\ldots\mu_k|\alpha}$ of the elements from $H^0(\gamma)$ is represented by polynomials (in order to ensure the spacetime locality) in $[K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}]$. In conclusion, $H^0(\gamma)$ computed in the BRST algebra of local "functions" $\mathcal{A}$ is precisely the algebra of invariant polynomials (local "functions" with pgh $=0$ that are γ-invariant and therefore true polynomials in $[\Phi^*_A]$ and $[K_{\mu_1\ldots\mu_{k+1}|\alpha\beta}]$ since they are not allowed to depend on the undifferentiated components of the field t)

$$H^0(\gamma)\text{ in }\mathcal{A}=\{\text{algebra of invariant polynomials}\}\equiv\{\alpha([\Phi^*_A],[K])\}. \tag{43}$$

Consequently, $H^0(\gamma)$ computed in the algebra of local differential forms Λ will also be an algebra (where the function multiplication must be replaced with the wedge product among the forms) allowing for a decomposition of the form (42) with $l=0$, where the elements of each space $H^{0,p}(\gamma)$ are p-forms whose coefficients are invariant polynomials

$$H^0(\gamma)=\bigoplus_{p=0}^{D}H^{0,p}(\gamma),\quad H^{0,p}(\gamma)\ni\overset{[p]}{\alpha}=\tfrac{1}{p!}\alpha_{\mu_1\ldots\mu_p}([\Phi^*_A],[K])dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}. \tag{44}$$

In the next step, from Table 1 we identify the BRST generators of pure ghost number 1 being expressed by the ghosts $\overset{(1)}{C}$ and $\overset{(1)}{\eta}$ together with their derivatives up to a finite order and further use definitions (16) and (17) for $m=1$. Equivalently, from (24) for $m=1$ we get that the BRST generators of pure ghost number 1 are given by linear combinations of the ghosts $\overset{(1)}{C}{}'$ and of their derivatives up to a finite order. From the action of γ on the latter generators, given by (27) for $m=1$, we deduce the most general γ-closed quantities (so from $\mathrm{Ker}(\gamma)$ at pgh $=1$) linear in $\overset{(1)}{C}{}'$ and their derivatives up to a finite order under the form

$$\left\{\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha},\partial_{\rho_1}\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha},\cdots,\partial_{\rho_1\ldots\rho_n}\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha}\right\}\in(\mathrm{Ker}(\gamma))^1.$$

It is more convenient to introduce the notations

$$\partial_{[\mu_1}\overset{(1)}{C}{}'_{\mu_2\ldots\mu_k]||\alpha}\equiv\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}, \tag{45}$$

in terms of which the previous relation becomes

$$\left\{\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha},\partial_{\rho_1}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha},\partial_{\rho_1\rho_2}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha},\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}\right\}\in(\mathrm{Ker}(\gamma))^1. \tag{46}$$

With the help of formula (26) it can be shown that

$$\partial_{\rho_1}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}=\gamma(\partial_{\rho_1}t_{\mu_1\ldots\mu_k|\alpha}+(-)^{k+1}\partial_{[\mu_1}t_{\mu_2\ldots\mu_k\alpha]|\rho_1}), \tag{47}$$

such that all the elements from (46) excepting the first one are γ-exact (or, in other words, trivial in $H(\gamma)$)

$$\left\{\partial_{\rho_1}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha},\partial_{\rho_1\rho_2}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha},\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}\right\}\in(\mathrm{Im}(\gamma))^1. \tag{48}$$

The proof of the result that the undifferentiated ghost $\overset{(1)}{\mathcal{T}}$ is not γ-exact can be done by direct computation via reductio ad absurdum [16]. The last observation shows that the only nontrivial quantities from $H(\gamma)$ linear in the ghosts of pure ghost number 1 and in their derivatives are represented by the components of the tensor $\overset{(1)}{\mathcal{T}}$ introduced in (45) (obviously not symmetrized with respect to the last index, placed after the double bar, and to any other index before the double bar).

On the other hand, definition (17) for $k=1$ leads to the fact that the only γ-invariant quantities, linear in the pure ghost number 1 ghost $\overset{(1)}{\eta}$ and its derivatives are

$$\left\{\partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_{k+1}]},\partial_{\rho_1}\partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_{k+1}]},\cdots,\partial_{\rho_1\ldots\rho_n}\partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_{k+1}]}\right\}\in(\mathrm{Ker}(\gamma))^1.$$

Equivalently, by means of the notation

$$\partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\ldots\mu_{k+1}]}\equiv\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}},\quad\varepsilon\left(\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\right)=1,\quad\mathrm{pgh}\left(\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\right)=1\tag{49}$$

we have that

$$\left\{\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}},\partial_{\rho_1}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}},\partial_{\rho_1\rho_2}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}},\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\right\}\in(\mathrm{Ker}(\gamma))^1.\tag{50}$$

Formula (4) translated in terms of the longitudinal exterior differential (replacing the gauge variation with γ and the gauge parameters $\overset{(1)}{\epsilon}$ with the ghosts $\overset{(1)}{\eta}$) becomes

$$\partial_{\rho_1}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}=\gamma\left(\tfrac{(-)^{k+1}}{k}F_{\mu_1\ldots\mu_{k+1}|\rho_1}\right)\tag{51}$$

and shows that all the quantities from (50) excepting the first one are γ-exact

$$\left\{\partial_{\rho_1}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}},\partial_{\rho_1\rho_2}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}},\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}\right\}\in(\mathrm{Im}(\gamma))^1.\tag{52}$$

Applying a technique similar to that described in the above it can be checked that $\overset{(1)}{\mathcal{F}}$ defined in (49) is not a trivial (γ-exact) element from $H^1(\gamma)$.

In this manner we inferred two apparently different results, namely, that the most general nontrivial quantity from $H^1(\gamma)$, linear in the ghosts of pure ghost number equal to 1 is given by (45) as well as by (49). These two elements are clearly independent since the former depends on the ghost $\overset{(1)}{C}{}'$, so it effectively involves both ghosts $\overset{(1)}{C}$ and $\overset{(1)}{\eta}$, while the latter depends only on $\overset{(1)}{\eta}$. This statement is not contradictory and will be explained in what follows. The element (45) does not display the mixed symmetry $(k,1)$; it is only antisymmetric with respect to its first k indices. So, $\overset{(1)}{\mathcal{T}}$ does not transform according to an irreducible representation of the group $GL(D,\mathbb{R})$, but allows for a unique decomposition, being written as the sum between a completely antisymmetric tensor and another with the mixed symmetry $(k,1)$ (each of these ones therefore irreducible)

$$\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}=\overset{(1)}{\mathcal{T}'}_{\mu_1\ldots\mu_k\alpha}+\overset{(1)}{\mathcal{T}''}_{\mu_1\ldots\mu_k|\alpha},\tag{53}$$

where

$$\overset{(1)}{\mathcal{T}'}_{\mu_1\ldots\mu_k\alpha}=\tfrac{1}{k+1}\overset{(1)}{\mathcal{T}}_{[\mu_1\ldots\mu_k||\alpha]},\qquad\overset{(1)}{\mathcal{T}''}_{\mu_1\ldots\mu_k|\alpha}=\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha}-\tfrac{1}{k+1}\overset{(1)}{\mathcal{T}}_{[\mu_1\ldots\mu_k||\alpha]}.\tag{54}$$

Thus, even if $\overset{(1)}{\mathcal{T}}{}'$ as well as $\overset{(1)}{\mathcal{T}}{}''$ from (53) are γ-invariant, only the former is nontrivial, while

$$\overset{(1)}{\mathcal{T}}{}''_{\mu_1\ldots\mu_k|\alpha} = \gamma t_{\mu_1\ldots\mu_k|\alpha}, \tag{55}$$

such that $\overset{(1)}{\mathcal{T}}{}'$ belongs to the same (nontrivial) equivalence class from $H(\gamma)$ like $\overset{(1)}{\mathcal{T}}$

$$\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha} = \overset{(1)}{\mathcal{T}}{}'_{\mu_1\ldots\mu_k\alpha} + \gamma t_{\mu_1\ldots\mu_k|\alpha}, \quad \overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha} \sim \overset{(1)}{\mathcal{T}}{}'_{\mu_1\ldots\mu_k\alpha}. \tag{56}$$

At this moment we remark that the former relation in (54) together with (45) and with formula (24) for $m = 1$ allow us to correlate the tensor $\overset{(1)}{\mathcal{T}}{}'$ with the tensor $\overset{(1)}{\mathcal{F}}$ defined in (49)

$$\overset{(1)}{\mathcal{T}}{}'_{\mu_1\ldots\mu_{k+1}} = k\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}. \tag{57}$$

Combining the last two relations, we finally managed to show that

$$\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha} = k\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_k\alpha} + \gamma t_{\mu_1\ldots\mu_k|\alpha}, \quad \overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_k||\alpha} \sim k\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_k\alpha}, \tag{58}$$

so the tensor $\overset{(1)}{\mathcal{T}}$ pertains to the same cohomological class from $H(\gamma)$ like $k\overset{(1)}{\mathcal{F}}$ and we can choose any of these as nontrivial representative of $H^1(\gamma)$. In what follows we will work with $\overset{(1)}{\mathcal{F}}$ because on one hand it is irreducible and on the other hand contains no trivial components. The conclusion of the above discussion is that the most general, nontrivial quantities from $H(\gamma)$ that are linear in the ghosts of pure ghost number equal to 1 of the free theory associated with a massless tensor field with the mixed symmetry $(k, 1)$ are precisely the components of the tensor (49).

As a side comment we emphasize that $\overset{(1)}{\mathcal{F}}$ is nothing but the nontrivial part of $\overset{(1)}{\mathcal{T}}$ with respect to $H(\gamma)$ in the sense that there is no linear combination among the components of $\overset{(1)}{\mathcal{F}}$ yielding a γ-exact element. Nevertheless, this statement does not apply to $\overset{(1)}{\mathcal{T}}$ since its part symmetrized with respect to the last two indices is γ-trivial

$$\overset{(1)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-1}(\mu_k||\alpha)} = \gamma t_{\mu_1\ldots\mu_{k-1}(\mu_k|\alpha)}. \tag{59}$$

On the other hand, if we are simply interested in computing $(\mathrm{Ker}(\gamma))^1$ (without inquiring its factorization to $(\mathrm{Im}(\gamma))^1$), then (46) are the most general objects linear in the ghosts pertaining to this space, and not (50). Indeed, (46) depend on both $\overset{(1)}{\eta}$ and $\overset{(1)}{C}$, while (50) only on $\overset{(1)}{\eta}$.

Next, we come back to Table 1 and notice that the BRST generators of pure ghost number m, for each fixed value of m within the range $\overline{2, k-1}$, are precisely the ghosts $\overset{(m)}{C}$, $\overset{(m)}{\eta}$, and their spacetime derivatives up to a finite order. Equivalently, from (24) we observe that all the independent components of these ghosts are expressed via the transformed ghosts $\overset{(m)}{C}{}'$ (for each $m = \overline{2, k-1}$) and their derivatives up to a finite order. Invoking definitions (27) for $m = \overline{2, k-2}$ and the first relation from (28) (in pure ghost number $(k-1)$) it can be shown that for each value of the pure ghost number $m=\overline{2,k-1}$ the most general γ-closed quantities, linear in $\overset{(m)}{C'}$ and its derivatives read as

$$\left\{\partial_{[\mu_1}\overset{(m)}{C'}_{\mu_2\ldots\mu_{k-m+1}]||\alpha},\partial_{\rho_1}\partial_{[\mu_1}\overset{(m)}{C'}_{\mu_2\ldots\mu_{k-m+1}]||\alpha},\cdots,\partial_{\rho_1\ldots\rho_n}\partial_{[\mu_1}\overset{(m)}{C'}_{\mu_2\ldots\mu_{k-m+1}]||\alpha}\right\}\in(\mathrm{Ker}(\gamma))^m, \tag{60}$$

and can be equivalently written with the aid of the notations

$$\partial_{[\mu_1}\overset{(m)}{C'}_{\mu_2\ldots\mu_{k-m+1}]||\alpha}\equiv\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha},\ \ m=\overline{2,k-1}, \tag{61}$$

in the form

$$\left\{\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha},\partial_{\rho_1}\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha},\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha}\right\}\in(\mathrm{Ker}(\gamma))^m, \tag{62}$$

with $m=\overline{2,k-1}$. All the objects present in (62) are nevertheless trivial in $H(\gamma)$ on account of the relations

$$\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha}=\gamma\overset{(m-1)}{C'}_{\mu_1\ldots\mu_{k-m+1}||\alpha},\ \ m=\overline{2,k-1}, \tag{63}$$

so we can state that

$$\left\{\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha},\partial_{\rho_1}\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha},\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(m)}{\mathcal{T}}_{\mu_1\ldots\mu_{k-m+1}||\alpha}\right\}\in(\mathrm{Im}(\gamma))^m, \tag{64}$$

for $m=\overline{2,k-1}$. In conclusion, the nontrivial representatives of the cohomology $H(\gamma)$ (computed in any of the algebras $\mathcal{A}$ or Λ) do not depend on any of the ghosts with the pure ghost number within the range $m=\overline{2,k-1}$ or on their derivatives.

Finally, using one more time Table 1 we observe that the only ghost with maximum pure ghost number, equal to k, is $\overset{(k)}{\eta}$. The second definition from (18) furnishes the most general γ-invariant elements that are linear in this ghost and its derivatives up to a finite order in the form

$$\left\{\overset{(k)}{\eta}_\alpha,\partial_{\rho_1}\overset{(k)}{\eta}_\alpha,\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(k)}{\eta}_\alpha\right\}\in(\mathrm{Ker}(\gamma))^k. \tag{65}$$

Due to the first relation in (28) all the elements from (65) excepting the first one fall in $(\mathrm{Im}(\gamma))^k$

$$\partial_{\rho_1}\overset{(k)}{\eta}_\alpha=\gamma\left(\tfrac{1}{2}\overset{(k-1)}{C'}_{\rho_1||\alpha}\right)\Rightarrow\left\{\partial_{\rho_1}\overset{(k)}{\eta}_\alpha,\cdots,\partial_{\rho_1\ldots\rho_n}\overset{(k)}{\eta}_\alpha\right\}\in(\mathrm{Im}(\gamma))^k, \tag{66}$$

and therefore we conclude that the most general nontrivial elements of the cohomology $H(\gamma)$ that are linear in the ghost with the pure ghost number equal to k and in its derivatives are solely the components of the undifferentiated ghost itself, $\overset{(k)}{\eta}_\alpha$.

Putting together all the results deduced so far we are able to organize the dependence of the nontrivial representatives of the cohomology algebra $H(\gamma)$ evaluated in the BRST algebra of local functions $\mathcal{A}$ on the BRST generators together with their spacetime derivatives corresponding to a massless tensor field with the mixed symmetry $(k,1)$ like in Table 2. Consequently, we can claim that the general expression of a nontrivial representative belonging to the cohomology algebra of the exterior longitudinal differential $H(\gamma)$ computed in the algebra $\mathcal{A}$ exhibiting well-defined values of both pure ghost and antighost numbers, i.e. the general solution to the equation

$$\gamma a=0,\quad a\in\mathcal{A},\quad \mathrm{pgh}(a)=l\geq 0,\quad \mathrm{agh}(a)=j\geq 0, \tag{67}$$

BRST generator	Nontrivial representatives	pgh
$[t_{\mu_1\ldots\mu_k\vert\alpha}]$	$[K_{\mu_1\ldots\mu_{k+1}\vert\alpha\beta}]$	0
$[\Phi^*_A]$	$[\Phi^*_A]$	0
$\big[\overset{(1)}{\eta}_{\mu_1\ldots\mu_k}\big], \big[\overset{(1)}{C}_{\mu_1\ldots\mu_{k-1}\vert\alpha}\big]$	$\overset{(1)}{\mathcal{F}}_{\mu_1\ldots\mu_{k+1}}$	1
$\big[\overset{(m)}{\eta}_{\mu_1\ldots\mu_{k-m+1}}\big], \big[\overset{(m)}{C}_{\mu_1\ldots\mu_{k-m}\vert\alpha}\big]$	—	$m,\ m=\overline{2,k-1}$
$\big[\overset{(k)}{\eta}_{\alpha}\big]$	$\overset{(k)}{\eta}_{\alpha}$	k

Table 2: Nontrivial representatives of the cohomology $H(\gamma)$ computed in the algebra $\mathcal{A}$.

is given (up to trivial contributions) by

$$a = \sum_J \alpha_J\big([\Phi^*_A],[K]\big)e^J\Big(\overset{(1)}{\mathcal{F}},\overset{(k)}{\eta}\Big),\quad \mathrm{agh}(\alpha_J)=j\geq 0,\quad \mathrm{pgh}(e^J)=l\geq 0. \tag{68}$$

The object α_J denotes an invariant polynomial (of pure ghost number equal to 0) and the notation $e^J\Big(\overset{(1)}{\mathcal{F}},\overset{(k)}{\eta}\Big)$ signifies the elements with the pure ghost number equal to l (and obviously of antighost number 0) of a basis in the space of polynomials in the tensor $\overset{(1)}{\mathcal{F}}$ defined by (49) and in the (undifferentiated) vector ghost $\overset{(k)}{\eta}$. With the help of the results from Table 2, of the concluding remarks stated so far, and of the fact that the action of the differential γ on dx^μ is vanishing, we have gathered all the ingredients necessary at the evaluation of the cohomology algebra of the exterior longitudinal differential $H(\gamma)$ in the algebra of local forms Λ. More precisely, the general expression of a nontrivial representative from $H(\gamma)$ computed in Λ displaying well-defined values of the form degree, pure ghost number, and antighost number

$$\gamma\varpi=0,\quad \varpi\in\Lambda,\quad \deg(\varpi)=p\leq D,\quad \mathrm{pgh}(\varpi)=l\geq 0,\quad \mathrm{agh}(\varpi)=j\geq 0, \tag{69}$$

can be written, up to the addition of trivial terms, in the form

$$\varpi = \sum_J \overset{[p]}{\alpha}_J\big([\Phi^*_A],[K]\big)e^J\Big(\overset{(1)}{\mathcal{F}},\overset{(k)}{\eta}\Big), \tag{70}$$

where $\deg(\overset{[p]}{\alpha}_J)=p\leq D$, $\mathrm{agh}(\overset{[p]}{\alpha}_J)=j\geq 0$, $\mathrm{pgh}(e^J)=l\geq 0$. In the last formula each form $\overset{[p]}{\alpha}_J$ is constructed out of invariant polynomials like in (44), so it provides an element of the space $H^{0,p}(\gamma)$. By abuse of terminology from now on we will call the algebra $H^0(\gamma)$ computed in Λ also the algebra of invariant polynomials and invoke its elements as invariant polynomials. The pure ghost number of each invariant polynomial $\overset{[p]}{\alpha}_J$ is equal to 0. The notation e^J has precisely the same meaning like in (68), so for every J its antighost number is equal to 0 and it does not depend on dx^μ. The entire dependence on dx^μ is assumed to enter the invariant polynomial only ($\deg(\varpi)=p=\deg(\overset{[p]}{\alpha})$).

The operator d remains a differential also on the algebra of invariant polynomials $H^0(\gamma)\equiv(\mathrm{Ker}(\gamma))^0$ evaluated in Λ, such that the computation of the cohomology of the exterior spacetime differential in the algebra of invariant polynomials still makes sense. The next important result, related precisely to the last cohomology, can be shown to hold.

Theorem 1 *The cohomology of the exterior spacetime differential computed in the algebra of invariant polynomials corresponding to the free theory of a massless $(k,1)$ tensor field is trivial in form degrees strictly less than D and in strictly positive antighost numbers*

$$\Big[\, d\overset{[p]}{\alpha}([\Phi_A^*],[K]) = 0,\ p < D,\ \mathrm{agh}(\overset{[p]}{\alpha}) > 0\Big] \Rightarrow \overset{[p]}{\alpha} = d\overset{[p-1]}{\beta}, \tag{71}$$

where $\overset{[p-1]}{\beta}$ can be taken to be an invariant polynomial, $\overset{[p-1]}{\beta}([\Phi_A^*],[K])$.

In form degree D the previous result must be rephrased as: if the invariant polynomial $\overset{[D]}{\alpha}([\Phi_A^*],[K])$ of strictly positive antighost number is d-exact, $\overset{[D]}{\alpha} = d\overset{[D-1]}{\beta}$, then the $(D-1)$-form $\overset{[D-1]}{\beta}$ can be taken to be an invariant polynomial.

The proof of this theorem goes along the same line with the similar result from [16, 25]. The key points are given by the decomposition of the differential d into two anticommuting differentials $d = d_0 + d_1$ (where d_0 acts nontrivially only on $[K]$ and d_1 only on $[\Phi_A^*]$) accompanied by the triviality of the cohomology of d_1 in the algebra of invariant polynomials in form degree strictly less than D [19]. The second part of the theorem may be reformulated in dual language in terms of invariant polynomials with the form degree equal to zero: if $\alpha([\Phi_A^*],[K])$ with $\mathrm{agh}(\alpha) > 0$ is an invariant polynomial with the form degree equal to 0 displaying vanishing Euler–Lagrange derivatives, $\alpha = \partial_\mu j^\mu$, then the current j^μ can be taken to be an invariant polynomial. The previous theorem can be generalized to the cohomology of the exterior spacetime differential computed in the entire cohomology $H(\gamma)$ (the latter evaluated in Λ). The computation of this cohomology also makes sense since d induces a differential in $H(\gamma)$ evaluated in Λ, to be denoted also by d.

Theorem 2 *The cohomology of the exterior spacetime differential computed in $H(\gamma)$ for a free massless tensor field $(k,1)$ is trivial in form degree strictly less than D and in strictly positive antighost numbers*

$$H_j^{g,p}(d, H(\gamma)) = 0, \quad p < D, \quad j > 0, \tag{72}$$

where p denotes the form degree, j the antighost number, and g the ghost number.

The proof of the last theorem is carried out similarly to the corresponding results from [16, 25]. We specify that an object from $H_j^{g,p}(d, H(\gamma))$ is a local p-form of ghost number g and antighost number j (both fixed) that is γ-closed and d-closed modulo γ: $\gamma\overset{[p]}{a} = 0$, $d\overset{[p]}{a} = \gamma\overset{[p+1]}{b}$. The above theorem states that if $p < D$ and $j > 0$, then $\overset{[p]}{a} = d\overset{[p-1]}{c} + \gamma\overset{[p]}{e}$, with $\gamma\overset{[p-1]}{c} = 0$. Theorem 2 allows for an extremely important corollary.

Corollary 3 *In strictly positive values of the antighost number the local cohomology of the exterior longitudinal differential $H(\gamma|d)$ computed in the algebra Λ of local forms corresponding to the free theory of a massless $(k,1)$ tensor field can be replaced by the cohomology $H(\gamma)$ evaluated in the same algebra.*

The above corollary states that if $\overset{[p]}{a} \in \Lambda$, with $\mathrm{agh}(\overset{[p]}{a}) = j > 0$, $\mathrm{pgh}(\overset{[p]}{a}) = l \geq 0$, and $\deg(\overset{[p]}{a}) = p \leq D$ satisfies the equation $\gamma\overset{[p]}{a} + d\overset{[p-1]}{b} = 0$, where $\mathrm{agh}(\overset{[p-1]}{b}) = j > 0$,

$\mathrm{pgh}(\overset{[p-1]}{b}) = l+1 > 0$, and $\deg(\overset{[p-1]}{b}) = p-1 < D$, then one can always redefine $\overset{[p]}{a}$ by $\overset{[p]}{a} \to \overset{[p]}{a'} = \overset{[p]}{a} + d\overset{[p-1]}{c}$ such that $\gamma\overset{[p]}{a'} = 0$. It is important to stress that $\overset{[p-1]}{b}$ as well as $\overset{[p-1]}{c}$ are not necessarily γ-invariant elements. In other words, in strictly positive antighost numbers an element of the local cohomology $H^{l,p}(\gamma|d)$ can always be replaced by an element of the cohomology $H^{l,p}(\gamma)$ computed in Λ. This corollary is especially important since it simplifies enormously the computation of the local cohomology of the exterior longitudinal cohomology in strictly positive antighost numbers by reduction to the computation of the longitudinal exterior differential only, which has already been accomplished (see formula (70)). The proof of this corollary is done similarly with the corresponding results from [16, 20, 23, 25]. The connection of this result to the general context of BRST quantization method is highlighted in [24].

5 Conclusions

In conclusion, in this paper we succeeded in elucidating the cohomological aspects related to the exterior longitudinal differential associated with the free theory of a massless tensor field with the mixed symmetry $(k,1)$ for $k \geq 4$. These results represent the first step toward the computation of the local BRST cohomology for this model and will be continued by further cohomological investigations regarding the Koszul–Tate differential [28].

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