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Kinetic theory of the tokamap in the diffusive regime

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Abstract

The complex structure of magnetic field configurations in various tokamaks was described in the previous work using 3/2 Hamiltonian systems and their discrete version (i.e. systems generated by area preserving maps) studied through methods specific to the dynamical systems' theory. A particular attention was paid to the study of the influence of the safety factor on the existence of internal transport barrier. It was proved that some local modifications of the safety factor (creation of a low shear zone) induce, even for monotonous safety factors, the formation of a robust transport barrier. In the present work the asymptotic radial transport in the same systems was studied using statistical methods which involve the distribution function of the magnetic field lines. The general principles of discrete-time nonequilibrium statistical mechanics were adapted for the study of the discrete system generated by tokamap.

1 Introduction

Chaotic dynamics can be considered a bridge between the regular evolution of a system and the random one because chaotic systems are deterministic, but exhibit a very complicate dynamics. The ways to study these types of dynamics are different: quasiperiodic and other regular functions in the first case and probabilistic equations in the second one.

Some deterministic chaotic systems of interest in fusion plasma physics were studied using 3/2 d.o.f. Hamiltonian systems and their discrete version, i.e. systems generate by area-preserving maps. This description was successfully used for the description of magnetic field configuration. The first application of a Hamiltonian map to the problem of magnetic field line diffusion in a tokamak in presence of a magnetic limiter appears to be due to Martin and Taylor in 1984 [1]. A global model of a specific stellarator (W VII-A) was introduced by Wobig in [2] and Punjabi studied the poloidal divertor geometry of a tokamak by means of very simple algebraic maps [3].

In [4] it was shown that the standard map and Wobig map do not satisfy a basic request of the model: they are not compatible with the toroidal geometry. In the same paper a new model was proposed, namely the tokamap. Beside the fact that this model describe accurately the main features of the magnetic configuration, an important advantage of the tokamap is that it is analytically tractable.

Many other models were proposed in the last decade [5]- [10]. Their main drawback is that they can be studied only numerically, due to their very complicate analytical expressions.

In order to use another approach for the study of some of them, the evolution of the distribution function of a dynamical system governed by the standard map was described by the methods of nonequilibrium statistical mechanics in diffusive regime [11] and in localized weak-stochasticity regime [12]. In [13] an analysis of the most appropriate kinetic approach is made and fractional kinetics is proposed. The fractional kinetics for relaxation and superdiffusion in a magnetic field was studied in [14] for a system equally generated by the standard map. However, the kinetic approach is not often applied, due to important complications related to analytical computations.

In this paper we focus on the study of the tokamap model because it is realistic (it obeys the toroidal geometry requirements) and because some results concerning the existence of the transport barriers that we previously obtained through deterministic methods can be analysed from this new point of view.

The paper is organized as follows: in Section 2 the tokamap model and its main properties is presented; the main results obtained in the study of the associated kinetic equations are presented in Section 3; conclusions and discussions can be found in Section 4.

2 The tokamap model

For the description of the magnetic field configuration a set of toroidal coordinates (r, θ, ζ) is generally used. ζ is the toroidal angle around the symmetry axis of the torus and (r, θ) are the polar coordinates in a circular poloidal section $\zeta = \zeta_0 = cst$. However, since canonicity of the coordinates is needed for the Hamiltonian description, one used the toroidal flux $\psi = r^2/2$ instead of radial coordinate [4].

The Poincare map is obtained from the intersection points of a magnetic field line starting from $(\psi, \theta\zeta_0)$ with the poloidal section $\zeta = \zeta_0$.

It is $T : [0, \infty) \times [0, 2\pi) \to [0, \infty) \times [0, 2\pi)$

$$T : \begin{cases} \overline{\psi} = \psi - \varepsilon \frac{\overline{\psi}}{1+\overline{\psi}} \sin\theta \\ \overline{\theta} = \left(\theta + 2\pi W\left(\overline{\psi}\right) - \varepsilon \frac{1}{\left(1+\overline{\psi}\right)^2} \cos\theta\right) (\mathrm{mod}\, 2\pi) \end{cases} \tag{1}$$

This map is considered to describe qualitatively the configuration of the magnetic field in various tokamaks if it involves realistic safety factors $q(\psi) = 1/W(\psi)$.

In our study we will consider a monotonous winding function

$$W(\psi) = (2 - \psi)\left(2 - 2\psi + \psi^2\right)/4.$$

which was used in [15] and obtained from magneto-hydrodynamic considerations.

In our previous studies on the tokamap model we analyzed individual long trajectories with the purpose of establishing their topological properties as revealed typically by time-correlation functions. In the globally stochastic region, such studies provided us with a very detailed picture of the motion of an individual point, with alternations of quasi-random motion and of segments sticking to islands, as well as the effect of this topology on transport.

3 Kinetic equation

Another method for studying the tokamap model involves the distribution function (or of the density profile) of a statistical ensemble, based on a kinetic equation. This global approach is complementary to the previous one. Both are exact, hence equivalent at the starting point, but the type of approximations adapted to each of them is different. Thus, in the global approach it is difficult to describe the fine structure of orbits in the neighborhood of an island. On the other hand, the local approach does not tell us anything about the shape of the density profile.

In the statistical approach the study of individual trajectories is replaced by the study of a statistical ensemble defined by the distribution function.

At the moment τ the statistical ensemble is defined by the function $F(\psi, \theta, \tau)$ which is 2π periodic in θ and is defined only for integer values of τ.

The evolution of the distribution function of the system generated by (1) can be defined using the Perron-Frobenius operator U defined bellow. It involves the inversion of (1)

$$F(\psi, \theta, \tau + 1) = (UF)(\psi, \theta, \tau) = F\left(\psi^{-1}, \theta^{-1}, \tau\right). \tag{2}$$

Severe limitations in obtaining analytical results occurs because the inverse of (1), i. e. $T^{-1}(\psi, \theta) = (\psi^{-1}, \theta^{-1})$, can be computed only numerically.

Of special interest for transport theory is the averaged distribution function, called the density profile

$$DP(\psi, \tau) = \int_0^{2\pi} F(\psi, \theta, \tau)\, d\theta.$$

The density profile is a main tool for computing the running diffusion coefficient.

We must consider also the fluctuations $G(\psi, \theta, \tau)$ defined such as

$$F(\psi, \theta, \tau) = DP(\psi, \tau) + G(\psi, \theta, \tau).$$

Direct numerical simulations were made in the weak chaotic regime ($\varepsilon << 1$) in order to enlighten the influence of some local modifications of the safety factor on transport properties in the diffusive regime (large perturbations and small values of the wave vector).

The initial density profile and the initial fluctuations considered in the simulations is

$$DP(\psi, 0) = 2\frac{Q}{\sqrt{\pi}} e^{-Q^2\psi^2} \text{ respectively } G(\psi, \theta, 0) = \begin{cases} DP(\psi, 0) & \text{if } |\theta| < \theta_0 \\ 0 & \text{elsewhere} \end{cases}. \tag{3}$$

The evolution of the distribution function is deeply related to the dynamic of the system. Its evolution is spectacular when the system has chaotic behavior.

In Figure 1 (left) is presented the phase portrait and Figure 1 (right) represents the distribution function for $\varepsilon = 0.45$. The left figure is obtained from many individual trajectories, the right figure is obtained using the evolution law (2). The blue colour in the right figure shows that $F(\psi, \theta, \tau) \approx 0$ so the spreading effect is limited, which is natural if we take into account the existence of many transport barriers, as shown in the left figure.

In Figure 2 (left) are presented the phase portrait and the distribution function in the system corresponding to $\varepsilon = 0.75$. The black zone in the left figure is a single chaotic orbit having 50000 points. The distribution function, presented in Figure 2 (right), shows similar complexity only in 498 steps. In this situation all transport barriers inside the chaotic zone were destroyed.

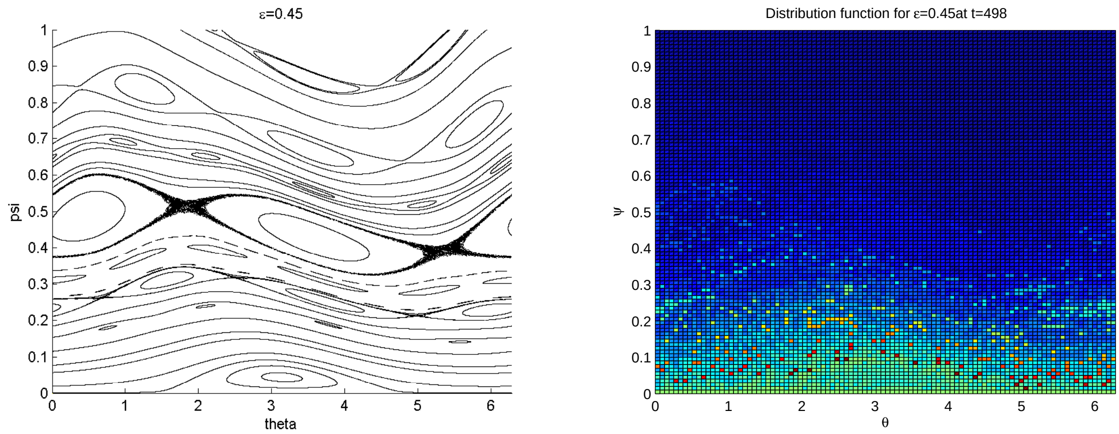

Figure 1: Phase portrait for $\varepsilon = 0.45$ (left)and Distribution function at $\tau = 0.498$ for $\varepsilon = 0.45$ (right)

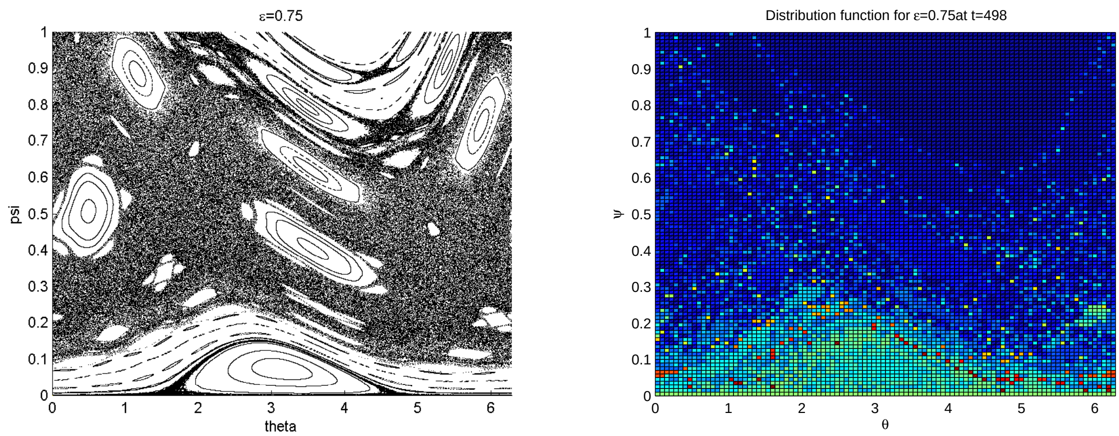

Figure 2: Phase portrait for $\varepsilon = 0.75$ (left)and Distribution function at $\tau = 0.498$ for $\varepsilon = 0.75$ (right)

It was observed that the creation of a low shear region induces the creation of a internal transport barrier. The characteristics of the radial transport were correlated with the width and the position of the transport barrier.

More information about the memory time (time when the initial fluctuations are annihilated) and the relaxation time (existence of long living fluctuations created out of density profile) can obtained from the study of the master equations.

From the spectral decomposition (i.e. the Laplace-Fourier representation) of $F(\psi, \theta, \tau)$

$$F(\psi, \theta, \tau) = \sum_{M=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{2\pi i (q\psi + M\theta)} f_M(q, \tau)\, dq \right)$$

the density profile and the fluctuations can be obtained using the relations

$$DP(\psi, \tau) = \int_{-\infty}^{\infty} e^{2\pi i q\psi} f_0(q, \tau)\, dq \overset{notation}{=} \int_{-\infty}^{\infty} e^{2\pi i q\psi} \varphi(q, \tau)\, dq$$

$$G(\psi, \theta, \tau) = \sum_{M \neq 0} \left(\int_{-\infty}^{\infty} e^{2\pi i (q\psi + M\theta)} f_M(q, \theta)\, dq \right).$$

The moments of the density profile $n(\psi, \tau)$ are simply expressed in terms of derivatives of its Fourier transform [11]:

$$\langle \psi^p(\tau) \rangle = \frac{1}{(2\pi i)^p} \frac{\partial^p \varphi}{\partial q^p}(0, \tau).$$

The aim of the study is to study the evolution of the density profile and of fluctuations using a closed master equation for the density profile and a equation expressing the fluctuations as functional of density profile.

The master equations for systems described by discrete-time iterative maps was obtained Bandtlow and Coveney (in fact there is a closed master equation for the density profile and an equation expressing the fluctuations as a functional of the density profile):

$$\varphi(\tau + 1) = \sum_{\sigma=0}^{\tau} \Psi(\sigma)\, \varphi(\tau - \sigma) + D(\tau + 1)\, g(0) \tag{4}$$

$$f_M(\psi, \tau + 1) = \sum_{\sigma=0}^{\tau} C(\sigma)\, \varphi(\tau - \sigma) \tag{5}$$

where $\Psi(t)$ is the memory kernel, $g(0)$ is the initial fluctuation, $D(t)$, $C(t)$ and $P(t)$ are related to the destruction, creation, respectively propagation fragment. These quantities can be computed only numerically because the inverse of the tokamap can not be analytically obtained. But the kinetic equation (4) and the fluctuation equation (5) describe more precisely than the direct numerical simulations the evolution of the system.

We studied the kinetic equation of the tokamap in the diffusive regime ($\varepsilon >> 1$), i.e. for large values of the stochasticity parameter, when the system has chaotic behavior. The density profile and the fluctuations (3) are initially sharply peaked around $\psi = 0$. The density profile we used is

$$n(\psi, 0) = \frac{\lambda}{2} e^{-\lambda|\psi|} \quad \text{with } \lambda = 0.01.$$

We identified two characteristic time scales: the short memory time (it is of order of 4-5 iterations) and the long relaxation time. The initial fluctuations produce during the (short) memory time a small cumulative effect in the density profile through the destruction fragment. They are replaced by long-living fluctuations created out of the density profile, which eventually decay to 0 after the (long) relaxation time. This process is characterized by an initial transient regime characterized by complex phenomena, for example the occurrence of Levy flights after long sticking in some regions of the phase space.

4 Conclusions

The tokamap model was studied through statistical methods. The results were compared with those obtained using classical methods in dynamical systems' theory and the qualitative agreement was pointed out. In the diffusive case the corresponding kinetic equation was numerically studied. In this case it is not possible to provide analytical solutions (as in standard map case) due to the fact that the inverse of tokamap can be only numerically computed. We identified two characteristic time scale (the short memory time and the long relaxation time). This results is also in agreement with that obtained through direct numerical simulations.

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