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On the diffusion of inhomogeneous sheared stochastic magnetic field lines in tokamak plasma

M. Negrea [1], I. Petrisor[1], D. Constantinescu[2]
Asocciation EURATOM-MEdC Romania,
[1]*Department of Physics, University of Craiova,*
[2]*Department of Applied Mathematics, University of Craiova*
13 A.I. Cuza Str., Craiova 200585, Romania

Abstract

In our paper we have developed the tools necesary to calculate the mean squared displacements for magnetic field lines for various dimensionless Kubo numbers: the magnetic Kubo number K_m, the dimensionless inhomogeneous magnetic Kubo number K_B and the magnetic shear Kubo number K_s. The model developed in our paper considers a stochastic magnetic field that contains two linear deterministic terms representing the gradient of the magnetic field and the shear term and also two fluctuating terms that are described by the dimensionless functions $b_i(X, Y, Z),\ i = (x, y)$, taken to be Gaussian processes and that are perpendicular to the main magnetic field B_0.

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1 Introduction

One of the most important feature in plasma physics is the study of the diffusion of magnetic field lines in tokamak. The simplified model used in our paper is a Taylor expansion of the stochastic magnetic field that contains two linear deterministic terms representing the gradient of the magnetic field and the magnetic shear term and also two fluctuating terms that are described by the dimensionless functions $b_i(X, Y, Z),\ i = (x, y)$, taken to be Gaussian processes and that are perpendicular to the main magnetic field B_0. The magnetic fluctuations, whose intensity is measured by the dimensionless magnetic Kubo number K_m (to be defined below), even when small, can destroy the nested magnetic and thus enhancing the radial transport. By the other hand, the presence of an inhomogeneity of the magnetic field (given by the existence of the magnetic field gradient and measured by the dimensionless inhomogeneous magnetic Kubo number K_B) and of the magnetic shear (measured by the dimensionless shear Kubo number K_s), can be important keys in order to explain the observed increase of the poloidal flow which is equivalent to the appearance of a transport barrier. In our paper we developed the main tools of the DCT method whose idea concerns in the study of the Langevin system not in the whole space of the realizations of the potential fluctuations; the whole space is subdivided into *subensembles* S, characterized by given values of the potential and of the fluctuating field components at the starting point of the trajectories. The exact expression of the Lagrangian correlation can be written in the form of a superposition of Lagrangian correlations in various subensembles. The validity of the approximation involved in DCT method can be assessed by *a posteriori* comparison with experiment and simulations, as is done in all theories of strong turbulence.

2 The magnetic field model and the related Langevin equations

A local expansion of the stochastic magnetic field that considers the perpendicular variations is:

$$\mathbf{B}(X,Y,Z) = B_0 \left\{ \left[1 + XL_B^{-1}\right] \mathbf{e}_z + \beta b_x\left(X,Y,Z\right)\mathbf{e}_x + \left[\beta b_y\left(X,Y,Z\right) + XL_s^{-1}\right]\mathbf{e}_y \right\} \quad (1)$$

where β is a dimensionless parameter measuring the amplitude of the magnetic field fluctuations relative to the main magnetic field B_0. There are two linear terms depending on X in the right hand side of eq.(1): the shear term XL_s^{-1} where L_s is the shear length and the nonhomogeneous term XL_B^{-1} where L_B is the gradient scale length, which are the radial distance on Ox axis over the magnitude of the magnetic field would double in this linear model. We will define the term $B_0\left[1 + XL_B^{-1}\right]\mathbf{e}_z$ as the gradient $\mathbf{B}$ term. Because the model expression given in (1) represents a Taylor series expansion of the magnetic field it is only valid for small distances from the origin, i.e. are valid the following approximations:

$$XL_B^{-1} << 1 \quad XL_S^{-1} << 1 \quad (2)$$

The magnetic field lines corresponding to the definition given in (1) are:

$$\frac{dX}{\beta B_0 b_x} = \frac{dY}{B_0\left(\beta b_y + XL_s^{-1}\right)} = \frac{dZ}{B_0\left(1 + XL_B^{-1}\right)} \quad (3)$$

Using (2), i.e. considering that

$$\frac{1}{1 + XL_B^{-1}} \simeq 1 - XL_B^{-1}$$

and neglecting the terms quadratic in X we obtain the dimensional Langevin system of equations

$$\frac{dX}{dZ} = \beta b_x\left(\mathbf{X};\ Z\right)\left(1 - XL_B^{-1}\right) = \beta b_x - \beta b_x XL_B^{-1} \quad (4)$$

$$\frac{dY}{dZ} = \beta b_y + XL_s^{-1} - \beta b_y XL_B^{-1} \quad (5)$$

where the coordinate Z plays the role of time.

We introduce the dimensionless coordinates $\mathbf{x} = (x,\ y,\ z)$ which are related to the dimensional ones by the relations:

$$x = \frac{X}{\lambda_\perp},\quad y = \frac{Y}{\lambda_\perp},\quad z = \frac{Z}{\lambda_\parallel} \quad (6)$$

The magnetic field given in Eq.(1) satisfies the zero-divergence constraint $\nabla \cdot \mathbf{B} = 0$ that is imposed by Maxwell's equations. This condition is automatically fulfilled if we consider that the fluctuating magnetic field derives from the following vector potential which has only the z - component:

$$\mathbf{A}(\mathbf{X}; Z) = B_0 \lambda_\perp \beta \ \psi(\mathbf{x}; z) \ \mathbf{e}_z \tag{7}$$

The dimensionless system equivalent to the one given in (4, 5) is:

$$\frac{dx}{dz} = (1 - x K_m K_B)\, b_x \equiv K(x)\, b_x \tag{8}$$

$$\frac{dy}{dz} = x K_s + (1 - x K_m K_B)\, b_y \equiv x K_s + K(x)\, b_y \tag{9}$$

We have the following relations between the fluctuating parts of the magnetic field and the magnetic potential $\psi(\mathbf{x}; z)$:

$$b_x[\mathbf{x}(z); z] = \left. \frac{\partial \psi(\mathbf{x}(z); z)}{\partial y} \right|_{\mathbf{x}=\mathbf{x}(z)}$$

and

$$b_y[\mathbf{x}(z); z] = - \left. \frac{\partial \psi(\mathbf{x}(z); z)}{\partial x} \right|_{\mathbf{x}=\mathbf{x}(z)}$$

In the system (8-9) the following Kubo numbers are introduced:

1. The magnetic Kubo number $K_m = \beta \frac{\lambda_\parallel}{\lambda_\perp}$ (10)
2. The shear Kubo number $K_s = \frac{\lambda_\perp}{L_s}$ (11)
3. The inhomogeneous Kubo number $K_B = \frac{\lambda_\parallel}{L_B}$ (12)

The Langevin Eqs.(9) will be used in order to calculate the running and asymptotic diffusion coefficient of the magnetic field lines for different values of the Kubo numbers. The Lagrangian correlation (which is the main tool for determining the running and asymptotic diffusion coefficient) of the directly fluctuating quantities $b_i[\mathbf{x}(z);\ z]$ is defined as:

$$L_{ij}(z) = \langle b_i(\mathbf{x}(0); 0) b_j\, [\mathbf{x}(z); z] \rangle \tag{13}$$

where $\langle ... \rangle$ denotes the ensemble average over the realizations of the fluctuating magnetic field components. The running diffusion coefficient matrix is calculated using (13) as:

$$D_{ij}(z) = \int_0^z d\zeta L_{ij}(\zeta) \tag{14}$$

provided that the stochastic field is "stationary"; the corresponding asymptotic diffusion coefficient matrix is then:

$$D_{ij}^{as} = \lim_{z \to \infty} D_{ij}(z) \tag{15}$$

An important simplification of the calculus can be done if a relation between the Lagrangian correlation and the corresponding Eulerian one can be established. Unfortunately, until now, does not exist a general exact relation between these two types of correlations, valid for both weak and strong turbulence regime. For a weak magnetic turbulence regime ($K_m < 1$) an approximate formula which relates the two types of correlations already exists: this is the celebrated Corrsin approximation [7] which includes the quasilinear and the Bohm approximations. We write here, for convenience, the Corrsin's relation between the Lagrangian and the Eulerian correlations:

$$L_{ij}(z) = \int d\mathbf{x} \left\langle b_i \left[\mathbf{x}(0);\ 0\right] b_j \left[\mathbf{x};\ z\right] \delta \left[\mathbf{x} - \mathbf{x}(z)\right] \right\rangle \simeq$$

$$\overset{Corrs}{\simeq} \int d\mathbf{x} \left\langle b_i \left[\mathbf{x}(0);\ 0\right] b_j \left[\mathbf{x};\ z\right] \right\rangle \left\langle \delta \left[\mathbf{x} - \mathbf{x}(z)\right] \right\rangle \tag{16}$$

where at least in some asymptotic sense, the exact propagator $\delta\left[\mathbf{x} - \mathbf{x}(z)\right]$ is approximated by its ensemble average.

3 The DCT tools for our magnetic field model

In our paper we closely follow the results obtained in [4]. The DCT method main idea concerns in the study of the Langevin system (8-9) not in the whole space of the realizations of the potential fluctuations; the whole space is subdivided into *subensembles* S, characterized by given values of the potential and of the fluctuating field components at the starting point of the trajectories. The exact expression of the Lagrangian correlation can be written in the form of a superposition of Lagrangian correlations in various subensembles. The validity of the approximation involved in DCT method can be assessed by *a posteriori* comparison with experiment and simulations, as is done in all theories of strong turbulence.

The DCT method is now systematically developed for the present problem. We first define a set of subensembles S of the realizations of the stochastic sheared magnetic field that are defined by given values of the potential ψ and magnetic field fluctuation $\mathbf{b}$ in the point $\mathbf{x} = 0$ at the "moment" $z = 0$:

$$\psi(\mathbf{0};\ 0) = \psi^0, \quad b_i(\mathbf{0};\ 0) = b_i^0,\ i = (x,\ y) \tag{17}$$

The correlation of the Lagrangian fluctuating components of the magnetic field can be represented as a sum over the subensembles S of the correlations $L_{ij}^S(z)$ calculated in each subensemble:

$$L_{ij}(z) = \int d\psi^0 d\mathbf{b}^0 P(\mathbf{b}^0,\ \psi^0) \left\langle b_i(\mathbf{0};\ 0) b_j \left[\mathbf{x}(z);\ z\right] \right\rangle^S \tag{18}$$

where

$$P(\mathbf{b}^0,\ \psi^0) = P(b_x^0)P(b_y^0)P(\psi^0) = (2\pi)^{-3/2} \exp\left[-\frac{(\psi^0)^2 + (b_x^0)^2 + \left(b_y^0\right)^2}{2}\right] \tag{19}$$

is the probability density of $(\mathbf{b},\ \psi)$ having the values $(\mathbf{b}^0, \psi^0)$ at $\mathbf{x} = 0$ and at the "moment" $z = 0$.

Since the initial fluctuating fields in the subensemble S are $b_i(\mathbf{0};\ 0) = b_i^0$ for all trajectories, the subensemble average defined in (18) is:

$$\left\langle b_i(\mathbf{0};\ 0) b_j \left[\mathbf{x}(z);\ z\right] \right\rangle^S = b_i^0 \left\langle b_j \left[\mathbf{x}(z);\ z\right] \right\rangle^S \tag{20}$$

and thus the Lagrangian correlation $L_{ij}(z)$ is simply the weighted average Lagrangian of the fluctuating field in all subensembles. We need first to calculate the average Eulerian fields b_i in the subensemble S:

$$b_i^S(\mathbf{x};\ z) \equiv \left\langle b_i\left(\mathbf{x};\ z\right) \right\rangle^S \quad , i = (x,\ y) \tag{21}$$

The next step in the DCT method is to define a deterministic trajectory in each subensemble as a solution of the system (8-9) that becomes

$$\frac{dx^S(z)}{dz} = \left(1 - x^S K_m K_B\right) b_x^S \left[\mathbf{x}^S(z);\ z\right]$$

$$\frac{dy^S(z)}{dz} = x^S K_s + \left(1 - x^S K_m K_B\right) b_y^S \left[\mathbf{x}^S(z);\ z\right]$$

in which the right hand sides are replaced by the average fields b_j^S in the subensemble. It can be seen also that the Kubo number $K(x) = (1 - xK_m K_B)$ differs from a subensemble to another. We made the following assumption: in a subensemble S we factorize the average of the terms of the form

$$\langle x b_j (\mathbf{x};\ z)\rangle^S \equiv x^S b_j^S \left(\mathbf{x}^S;\ z\right), \quad j = x, y$$

The system used in the DCT calculations can be formally written in the following compact form

$$\frac{dx^S}{dz} = K\left(x^S\right) b_x^S \tag{22}$$

$$\frac{dy^S}{dz} = x^S K_s + K\left(x^S\right) b_y^S \tag{23}$$

We need first to calculate the average Eulerian fields b_i in the subensemble S:

$$b_i^S(\mathbf{x};\ z) \equiv \langle b_i (\mathbf{x};\ z)\rangle^S \quad , i = (x,\ y) \tag{24}$$

The DCT method is now systematically developed for the present problem. We first define a set of subensembles S of the realizations of the stochastic sheared magnetic field that are defined by given values of the potential ψ and magnetic field fluctuation $\mathbf{b}$ in the point $\mathbf{x} = 0$ at the "moment" $z = 0$:

$$\psi(\mathbf{0};\ 0) = \psi^0, \ \ b_i(\mathbf{0};\ 0) = b_i^0, \ i = (x,\ y) \tag{25}$$

The fluctuating magnetic potential $\psi(\mathbf{x}; z)$ is assumed to be a Gaussian stochastic process with zero average. The second order moment of $\psi(\mathbf{x}; z)$, *i.e.,* its Eulerian autocorrelation function $M(\mathbf{x}; z)$ is assumed to have the following factorized form:

$$M(\mathbf{x}; z) = \langle \psi(\mathbf{0}; 0)\psi(\mathbf{x}; z)\rangle = M_1(\mathbf{x}) M_2(z) \tag{26}$$

where:

$$M_1(\mathbf{x}) = \exp(-\frac{\mathbf{x}^2}{2}), \quad M_2(z) = \exp(-\frac{z^2}{2}) \tag{27}$$

The mixed Eulerian correlations between the potential and the fluctuating magnetic field components are defined in [5] as:

$$M_{\psi n}(\mathbf{x}; z) = \langle \psi(\mathbf{0}; 0) b_n(\mathbf{x}; z)\rangle$$

$$M_{n\psi}(\mathbf{x}; z) = \langle b_n(\mathbf{0}; 0)\psi(\mathbf{x}; z)\rangle\,,\ n = (x,\ y) \tag{28}$$

and the following relations between these correlations hold [5]:

$$M_{\psi x}(\mathbf{x}; z) = -M_{x\psi}(\mathbf{x}; z) = \frac{\partial M(\mathbf{x}; z)}{\partial y} = -yM(\mathbf{x}; z)$$

and

$$M_{\psi y}(\mathbf{x};z) = -M_{y\psi}(\mathbf{x};z) = -\frac{\partial M(\mathbf{x};z)}{\partial x} = xM(\mathbf{x};z) \tag{29}$$

The dimensionless fluctuating magnetic field autocorrelation tensor components are derived from $M(\mathbf{x};z)$ as [5], [6]:

$$\begin{aligned} M_{xx}(\mathbf{x};z) &= -\frac{\partial^2 M(\mathbf{x};z)}{\partial y^2} = \left(1-y^2\right)M(\mathbf{x};z) \\ M_{yy}(\mathbf{x};z) &= -\frac{\partial^2 M(\mathbf{x};z)}{\partial x^2} = \left(1-x^2\right)M(\mathbf{x};z) \end{aligned}$$

$$M_{xy}(\mathbf{x};z) = M_{yx}(\mathbf{x};z) = \frac{\partial^2 M(\mathbf{x};z)}{\partial x \partial y} = xyM(\mathbf{x};z) \tag{30}$$

The Eulerian averages of the fluctuations are:

$$\begin{aligned} b_x^S\left(\mathbf{x}^S;\ z\right) &= \psi^0 M_{\psi x}\left(\mathbf{x}^S;\ z\right) + b_x^0 M_{xx}\left(\mathbf{x}^S;\ z\right) + b_y^0 M_{yx}\left(\mathbf{x}^S;\ z\right) \equiv \\ &\equiv \left[-\psi^0 y^S + b_y^0 x^S y^S + b_x^0\left(1-y^{S\ 2}\right)\right] M(\mathbf{x}^S;z) \end{aligned} \tag{31}$$

$$\begin{aligned} b_y^S\left(\mathbf{x}^S;\ z\right) &= \psi^0 M_{\psi y}\left(\mathbf{x}^S;\ z\right) + b_x^0 M_{xy}\left(\mathbf{x}^S;\ z\right) + b_y^0 M_{yy}\left(\mathbf{x}^S;\ z\right) \equiv \\ &\equiv \left[\psi^0 x^S + b_x^0 x^S y^S + b_y^0\left(1-x^{S\ 2}\right)\right] M(\mathbf{x}^S;z) \end{aligned} \tag{32}$$

The Lagrangian correlation tensor has the following components:

$$L_{ij}(z) = \int_{-\infty}^{\infty} d\psi^0 \int_{-\infty}^{\infty} db_y^0 \int_{-\infty}^{\infty} db_x^0 P(\mathbf{b}^0,\ \psi^0) b_i^0 b_j^S \left[\mathbf{x}^S(z);\ z\right] \tag{33}$$

4 Conclusion

In this paper we have prepared the framework for the calculation of the diffusion coefficients for the stochastic inhomogeneous and sheared magnetic field lines. Different Kubo numbers involved in the model will influence the diffusion. Some results were obtained in [4] for the anisotropic case but not in the inhomogeneous case. In a future paper we will calculate the diffusion coefficients and we will compare them with the anisotropic homogeneous case.

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