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Theoretical and numerical aspects of fractional 2D transport equation. Applications in fusion plasma theory

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Abstract

In this paper we study the 2D fractional transport equation. We present a numerical method based on matrix approach and we apply it for the study of specific transport equations which describe phenomena that occur in tokamaks.

1 Introduction

The aim of this paper is to study a generalized transport equation for particles. This model includes local effects (through Fokker-Planck equation) and non-local spatial effects (Levy flights modelled using fractional derivatives). External perturbations are introduced in the model as source term in the fractional equation.

The 1D version of the model is

$$\begin{cases} {}_t^C D_0^\alpha T - A \cdot {}_x^R D_{l_1,r_1}^\beta T - B \cdot {}_x^R D_{l_2,r_2}^\beta T = S(x,t) \\ T(0,t) = T(L,t) = 0 \qquad \forall t \geq 0 \\ T(x,0) = T0(x) \qquad \forall x \in [0,L] \end{cases}$$

where $T = T(x,t)$ is the transported scalar quantity (for example temperature or density), $A, B \in \mathbf{R}$ are parameters and $\alpha, \beta, \gamma \in (0,2]$ represent the fractional orders of partial derivatives.

The Caputo derivative

$$\left({}_t^C D_0^\alpha T\right)(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{T^{(m)}(x,\tau)}{(t-\tau)^{a-m+1}} d\tau \ if \ m-1 < \alpha \leq m$$

introduces the memory effects because it involves, for each fixed x, all values of $T(x,\tau)$ from the starting moment $\tau = 0$ to the present moment $\tau = t$.

The non-local spatial effects are introduced through the Riesz derivatives

$$\left({}_x^R D_{l,r}^\delta T\right)(x,t) = \frac{1}{\Gamma(p-\delta)} \left(\frac{\partial}{\partial x}\right)^p \left(l \int_a^x \frac{T(\xi,t)}{(x-\xi)^{b-p+1}} d\xi + r(-1)^p \int_x^b \frac{T(\xi,t)}{(x-\xi)^{b-p+1}} d\xi \right)$$

if $p - 1 < \delta \leq p$.

It is a weighted sum of left and right Riemann-Liouville (R-L) derivatives. For a fixed t it involves the all values of $T(x,t)$ for $x \in [a,b]$, if $l \neq 0$, $r \neq 0$. The coefficients l and r measures the influence of the R-L left derivative, respectively R-L right derivative: if $l = 0$ only the right R-L derivative is took into account, hence only right spatial effect are considered; if $r = 0$ only left R-L derivative are used for computing the Riesz derivative, consequently only left spatial effects are involved in the equation. The symmetric Riesz derivative, denoted by $\frac{\partial^\beta T}{\partial |x|^\beta}$, is obtained for $l = r = 1/2$. In terms of random walk schemes, the symmetric derivative corresponds to a symmetric jump probability distribution of the transported quantities. The asymmetry of the space derivative accounts a preferable direction of jumps which may occur. In many papers the relation $l + r = 1$ is considered, but it is not mandatory from strictly mathematical point of view, having only a logical argument. The order of the fractional derivative is determined by the waiting time distribution function and the order of the fractional derivative in space is determined by the algebraic asymptotic scaling of the jump distribution function.

One can remark that the classical Fokker-Plank equation is obtained for $\alpha = 1$, $\beta = 1$, $\gamma = 2$.

1D transport equations were used in order to describe radial transport in magnetically confined plasmas: $a = 0$ corresponds to the magnetic axis and $b = L$ corresponds to plasma boundary. The radial displacement of tracers was studied in [1] using a space-symmetric fractional model corresponding to $B = 0$, $S(x,t) = 0$ and $l_1 = r_1 = 1/2$. It was shown that this model reproduces, for some specific fractional orders of derivation, the shape and space-time scaling of the probability density function of the radial displacement, being in quantitative agreement with the turbulence transport calculations. The non-local radial transport was studied using non-symmetric fractional models [2] corresponding to $\gamma = 2$ and $S(x,t) = 0$. As a consequence of the spatial asymmetry some fractional pinch were observed, accompanied by the development of an uphill transport region. The propagation of cold pulses in some experiments conducted in JET were fitted using fractional diffusion models [3]. The non-local transport in the reversed field pinch was described using a fractional diffusion equation: the particles transport across the unperturbed flux surfaces is due to a spectrum of Levy flights which is associated to a fractional equation [4].

Other applications of fractional diffusion were pointed out in two dimensional (space and time) bounded domains [5] and comparisons of radial fractional transport model with tokamak experiments were performed in [6]. 1D fractional models were also used in the study of turbulence [7].

These examples show that the 1D fractional transport equation can be successfully used for explaining some features of the anomalous transport observed since long time in tokamak experiments. In these models the space coordinate is the radial coordinate and the equation deals with the probability density function of the radial displacement, which means that an implicit average on the poloidal angle is considered. More accurate models are obtained by considering a second space coordinate, i.e. 2D-transport equations. In the studies related to transport in tokamak, the second space coordinate is the poloidal angle.

The general transport equation that we will study in this paper is

$$\left\{ \begin{array}{c} {}_t^C D_0^\alpha T - \chi \left({}_x^R D_{l_1,r_1}^\beta T + {}_y^R D_{l_2,r_2}^\gamma T \right) = S(x,y,t) \\ T(x,y,0) = T0(x,y) \quad \forall (x,y) \in [0,L_x] \times [0,L_y] \\ T(0,y,t) = T(L_x,y,t) = 0 \quad \forall (y,t) \in [0,L_y] \times [0,L_t] \\ T(x,0,t) = T(x,L_y,t) = 0 \quad \forall (x,t) \in [0,L_x] \times [0,L_t] \end{array} \right. . \tag{1}$$

For the continuity of initial conditions we assume

$$T0\,(x,0) = T0\,(x,L_y) = T0\,(0,y) = T0\,(L_x,y) = 0,\ \forall\,(x,y) \in [0,L_x] \times [0,L_y] \tag{2}$$

Some already proposed models are particular cases of (1): a fractional operators was proposed in [8] for the study of the multidimensional advection and dispersion, two and three dimensional fractional Fokker-Planck equations were used for the description of relaxation and superdiffusion in a magnetic field in tokamaks [9] or for the study of the joint position-velocity probability distribution function of a single fluid particle in a turbulent flow [10].

The analytical solutions of such equations can be obtained only in simplified cases. Even in these situations the solutions are written in terms of special Mittag-Lefler functions, so they are not easy to handle in computations. For this reason a special attention was paid to their numerical study. However, while a basic framework for determining numerical solutions for ordinary fractional equations is established, relatively few methods exist for solving fractional equations with partial derivatives. More of this, many of them can be used to solve only the 1D transport equation. In this paper we will extend the matrix approach proposed in [11] and we will use it to solve some 2D transport equations.

The rest of the paper is organized as follows: in Section 2 the numerical method for solving the problem (1) is presented. In Section 3 this method is applied for solving specific problems. Conclusions and discussions are contained in Section 4 and basic reluts about fractional calculus can be found in Appendix.

2 Numerical method for solving 2D fractional transport equation

Many numerical methods for solving fractional partial differential equations have been proposed (see [12], [14], [13] and references therein).

Many toolboxes for solving such equations were developed in Matlab and are freely downloadable from Matlab Central File Exchange. We mention the toolbox CRONE, created by CRONE team [15], the Fractional State-Space toolkit [16], the Fractional Order Transfer Function Toolbox [18], the matlab code fde12.m [17], the code flmm.m [19]. These numerical methods and corresponding toolboxes can be applied only for solving 1D-fractional transport equation.

In this paper we use the matrix approach presented in [11] for solving the fractional partial differential equations. This approach, initially designed for equations having a single space-coordinate can be extended for equations with two spatial coordinates. In [20] is presented the method for solving the diffusion equation with symmetric fractional space derivatives. This method will be adapted in what follows for solving the generalized transport equations, with spatial asymmetries.

The matrix approach is based on the following observations:

1) Caputo derivative $\left({}_t^C D_0^\alpha T\right)$ and left R-L derivative ${}_t D_{0+}^\alpha T$ coincide if $T\,(x,y,0) = 0$ for all $(x,y) \in [0,L_x] \times [0,L_y]$ and $\alpha \in (0,1)$.

2) the left (respectively right) R-L derivatives and the left (respectively right) Grundwald-Letnikov derivatives (computed on the same interval) coincide. In some initial conditions (see the Appendix for details) R-L derivatives can be approximated using discretized G-L derivatives with prescribed step.

2) the discretized Grundwald-Letnikov derivatives with prescribed step can be computed using the matrix approach, which transforms the partial differential equation into a linear system of equations whose unknowns are the values of T in the grid's nodes.

The method is based on triangular strip matrix approach [21] to discretization of operators of differentiation and integration of arbitrary real order.

In usual numerical methods the solution of the equation is obtained step by step by moving from the previous moment to the next one. In the matrix approach the solution is obtained in one step in the whole time interval.

Let consider first the special case $T0\,(x,y)=0,\ \forall\,(x,t)\in[0,L_x]\times[0,L_t]$.

In order to solve numerically the problem (1) on $[0,L_x]\times[0,L_y]\times[0,L_t]$ we consider a grid with p nodes in x-direction, m-nodes in y-direction and n-nodes in t-direction. The nodes are (ih_x,jh_y,kh_t) where $h_x=\frac{L_x}{p-1}$, $h_y=\frac{L_y}{m-1}$, $h_t=\frac{L_t}{n-1}$ and $i\in\{1,2,...,p\}$, $j\in\{1,2,...,m\}$, respectively $k\in\{1,2,...,n\}$.

We denote $T\,(ih_x,jh_y,kh_t)$ by $T_{i,j,k}$.

Following the paper [20] we can say that the Caputo derivative in time at these nodes can be approximated using discretized Grundwald-Letnikov operators: for fixed (i,j) one has

$$\left[T^{(\alpha)}_{i,j,n}\ T^{(\alpha)}_{i,j,n-1}\ ...\ T^{(\alpha)}_{i,j,2}\ T^{(\alpha)}_{i,j,1}\right]=B^{(\alpha)}_n\times\left[T_{i,j,n}\ T_{i,j,n-1}\ ...\ T_{i,j,2}\ T_{i,j,1}\right]'$$

where $w^{(\delta)}_s=(-1)^s\frac{\Gamma(\delta+1)}{\Gamma(\delta-s+1)\cdot\Gamma(s+1)}$ and

$$B^{(\alpha)}_n=\frac{1}{h_t^{\alpha}}\begin{pmatrix} w^{(\alpha)}_1 & w^{(\alpha)}_2 & ... & ... & w^{(\alpha)}_{n-1} & w^{(\alpha)}_n\\ 0 & w^{(\alpha)}_1 & w^{(\alpha)}_2 & ... & ... & w^{(\alpha)}_{n-1}\\ ... & ... & ... & ... & ... & ...\\ 0 & ... & 0 & 0 & w^{(\alpha)}_1 & w^{(\alpha)}_2\\ 0 & ... & ... & 0 & 0 & w^{(\alpha)}_1\end{pmatrix}.$$

Similarly we approximate ${}^{R}_{x}D^{\beta}_{l_1,r_1}T$ (the spatial derivative in x-direction) and ${}^{R}_{y}D^{\gamma}_{l_2,r_2}T$ (the spatial derivative in y- direction) by

$$\begin{aligned}\left[T^{(\beta)}_{p,j,k}\ T^{(\beta)}_{p-1,j,k}\ ...\ T^{(\beta)}_{2,j,k}\ T^{(\beta)}_{1,j,k}\right] &= RL^{\beta}_p\times\left[T_{p,j,k}\ T_{p-1,j,k}\ ...\ T_{2,j,k}\ T_{1,j,k}\right]'\\ &\quad\text{respectively}\\ \left[T^{(\gamma)}_{i,m,k}\ T^{(\gamma)}_{i,m-1,k}\ ...\ T^{(\gamma)}_{i,2,k}\ T^{(\gamma)}_{i,1,k}\right] &= RL^{\gamma}_m\times\left[T_{i,m,k}\ T_{i,m-1,k}\ ...\ T_{i,2,k}\ T_{i,1,k}\right]'\end{aligned}$$

where $RL^{\beta}_p=\left(l_1\cdot B^{(\beta)}_p+r_1\cdot F^{(\beta)}_p\right)$, respectively $RL^{\gamma}_m=\left(l_2\cdot B^{(\gamma)}_m+r_2\cdot F^{(\gamma)}_m\right)$ and

$$B^{(\beta)}_p=\frac{1}{h_x^{\beta}}\begin{pmatrix} w^{(\beta)}_1 & w^{(\beta)}_2 & ... & ... & w^{(\beta)}_{p-1} & w^{(\beta)}_p\\ 0 & w^{(\beta)}_1 & w^{(\beta)}_2 & ... & ... & w^{(\beta)}_{p-1}\\ ... & ... & ... & ... & ... & ...\\ 0 & ... & 0 & 0 & w^{(\beta)}_1 & w^{(\beta)}_2\\ 0 & ... & ... & 0 & 0 & w^{(\beta)}_1\end{pmatrix}\ \text{and}\ F^{(\beta)}_p=\left(B^{(\beta)}_p\right)'$$

respectively

$$B^{(\gamma)}_m=\frac{1}{h_y^{\gamma}}\begin{pmatrix} w^{(\gamma)}_1 & w^{(\gamma)}_2 & ... & ... & w^{(\gamma)}_{m-1} & w^{(\gamma)}_m\\ 0 & w^{(\gamma)}_1 & w^{(\gamma)}_2 & ... & ... & w^{(\gamma)}_{m-1}\\ ... & ... & ... & ... & ... & ...\\ 0 & ... & 0 & 0 & w^{(\gamma)}_1 & w^{(\gamma)}_2\\ 0 & ... & ... & 0 & 0 & w^{(\gamma)}_1\end{pmatrix}\ \text{and}\ F^{(\gamma)}_m=\left(B^{(\gamma)}_m\right)'.$$

Before finding the matrices corresponding to the fractional time and space derivatives of $T(x,y,t)$ we arrange the function values $T_{i,j,k}$ in a 2D-structure, i.e. a matrix as follows:

$$
\begin{array}{ll}
u_{pmn} = & \\
& \begin{array}{ccccc}
[T_{p,m,n} & T_{p-1,m,n} & T_{p-2,m,n} & \cdots & T_{1,m,n} \\
T_{p,m-1,n} & T_{p-1,m-1,n} & T_{p-2,m-1,n} & \cdots & T_{1,m-1,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
T_{p,1,n} & T_{p-1,1,n} & \cdots & \cdots & T_{1,1,n} \\
T_{p,m,n-1} & T_{p-1,m,n-1} & T_{p-2,m,n-1} & \cdots & T_{1,m,n-1} \\
T_{p,m-1,n-1} & T_{p-1,m-1,n-1} & T_{p-2,m-1,n-1} & \cdots & T_{1,m-1,n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
T_{p,1,n-1} & T_{p-1,1,n-1} & T_{p-2,1,n-1} & \cdots & T_{1,1,n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
T_{p,m,1} & T_{p-1,m,1} & T_{p-2,m,1} & \cdots & T_{1,m,1} \\
T_{p,m-1,1} & T_{p-1,m-1,1} & T_{p-2,m-1,1} & \cdots & T_{1,m-1,1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
T_{p,1,1} & T_{p-1,1,1} & T_{p-2,1,1} & \cdots & T_{1,1,1}]'
\end{array}
\end{array}
$$

In a similar way we arrange the values of the perturbation term $S(x,y,t)$ in the matrix s_{pmn}.

The matrix $T^{(\alpha)}$ transforming u_{pmn} to partial fractional derivative of order α with respect to time is obtained by Kronecker product of the matrix $B_n^{(\alpha)}$ with the unit matrix E_{pm} (of order pm) as follows:

$$T^{(\alpha)} = B_n^{(\alpha)} \otimes E_{pm}.$$

Similarly the matrices $S1^{(\beta)}$ and $S2^{(\gamma)}$ transforming u_{pmn} to partial fractional derivative of order β, respectively γ, with respect to x, respectively y are also obtained using Kronecker product:

$$
\begin{aligned}
S1^{(\beta)} &= E_n \otimes RL_p^{\beta} \otimes E_m \\
S2^{(\gamma)} &= E_{np} \otimes RL_m^{\gamma}.
\end{aligned}
$$

The matrices $T^{(\alpha)}$, $S1^{(\beta)}$, $S2^{(\gamma)}$ are square matrices with $m \cdot n \cdot p$ lines (and columns).

The discrete algebraic system associated to (1) is

$$\left(T^{(\alpha)} - \chi\left(S1^{(\beta)} + S2^{(\gamma)}\right)\right) \times u_{pmn} = s_{pmn}$$

It can be solved using classical methods and one obtains

$$u_{pmn} = \left(T^{(\alpha)} - \chi\left(S1^{(\beta)} + S2^{(\gamma)}\right)\right)^{-1} \times s_{pmn}.$$

The only problem which remains is to re-arrange u_{pmn} in a 3D structure in order to find the values $T_{i,j,k}$.

In the case $T0(x,y,0) \neq 0$, for some $(x,t) \in [0,L_x] \times [0,L_t]$, the approximation of Riemann-Liouville derivative by Grunwald-Letnikov derivative is no more valid.

In this case one can make in (1) the substitution $T1(x,y,t) = T(x,y,t) - T0(x,y)$ and obtains a new equation.

We have to point out that it is important to use the derivative in time is the Caputo derivative because the Caputo derivative of a constant is 0, which is not true for Riemann-Liouville derivative.

From (2) it results that all conditions for approximation are fulfilled.

In this situation the resolution of the problem has two steps:

- through the algorithm presented above one obtain the solution $T1$ of the equation obtained by substitution;

-the solution of (1) is $T(x, y, t) = T1(x, y, t) + T0(x, y)$;

The matrix approach method is efficient from the time-consuming point of view and can be implemented in Matlab, which is a software with many toolboxes specialized in working with matrices.

3 Applications in the study of some transport equations

In order to solve some equations that can be useful in modeling transport phenomena in tokamaks we use the method presented in Section 2.

3.1 Homogenous symmetric transport equation

We consider a symmetric 2D homogenous transport equation ($S(x, y, t) = 0$, $\forall (x, y, t) \in [0, 1] \times [0, 1] \times [0, \infty)$) with sinusoidal initial condition:

$$\begin{cases} {}_{t}^{C}D_{0}^{\alpha}T - \left(\frac{\partial^{\beta}T}{\partial|x|^{\beta}} + \frac{\partial^{\gamma}T}{\partial|y|^{\gamma}}\right) = 0 \\ T(x, y, 0) = x \cdot (1 - x) \cdot \sin(3\pi y) \quad \forall (x, y) \in [0, 1] \times [0, 1] \\ T(0, y, t) = T(1, y, t) = 0 \quad \forall (y, t) \in [0, 1] \times [0, 1] \\ T(x, 0, t) = T(x, 1, t) = 0 \quad \forall (x, t) \in [0, 1] \times [0, 1] \end{cases} \tag{3}$$

The initial condition is plotted in Figure 1 A). The motivation for choosing polynomial initial condition in x-direction is that near marginal stability L-H transition models can be reduced to reaction-diffusion type systems with polynomial instabilities [1]

In order to solve numerically (3) we apply the method described in the Section 2. The discretization steps are $h_x = h_y = 0.05$ and $h_t = 0.01$. In order to observe the non-local spatial and memory effects, we considered many combinations of values for $\alpha \in (0, 1]$ (the order of the derivative in time) and $\beta = \gamma \in (1, 2]$ (the orders of spatial derivatives). For the beginning we consider symmetric spatial derivatives, i.e. $l_1 = r_1 = l_2 = r_2$.

For $\alpha = 1$, $\beta = \gamma = 2$ one obtains the classical diffusion equation, where non-local effects are not involved. The contour-plot of the solution at $t = 0.20$ are presented in Figure 1 B).

Non-local spatial effect occur if fractional spatial derivatives are considered, even if the order of derivation is close to 2. The contour plot of the solution obtained for $\alpha = 1$ and $\beta = \gamma = 1.90$ is drawn in Figure 1 C).

In order to make more precise the effect using fractional derivatives we plot in Figure 2 the solutions obtained for $\alpha = 1$ and various $\beta = \gamma \in [1.80, 2.00]$ along the line $x = 1/2$, where the variation of the initial condition is the most important.

The difference between the classical solution corresponding to $\beta = \gamma = 2$ (blue curve) and the solutions of fractional equations can be easily observed in Figure 2. As expected, the non-local effects are more pronounced when β and γ, the order of spatial derivatives, go away from 2. The symmetry observed in Figure 2 is a consequence of the symmetry of the initial condition with respect to the line $y = 0.5$, but equally of using symmetric spatial derivatives. The relation $\beta = \gamma$ does not influence the symmetry of the solution.

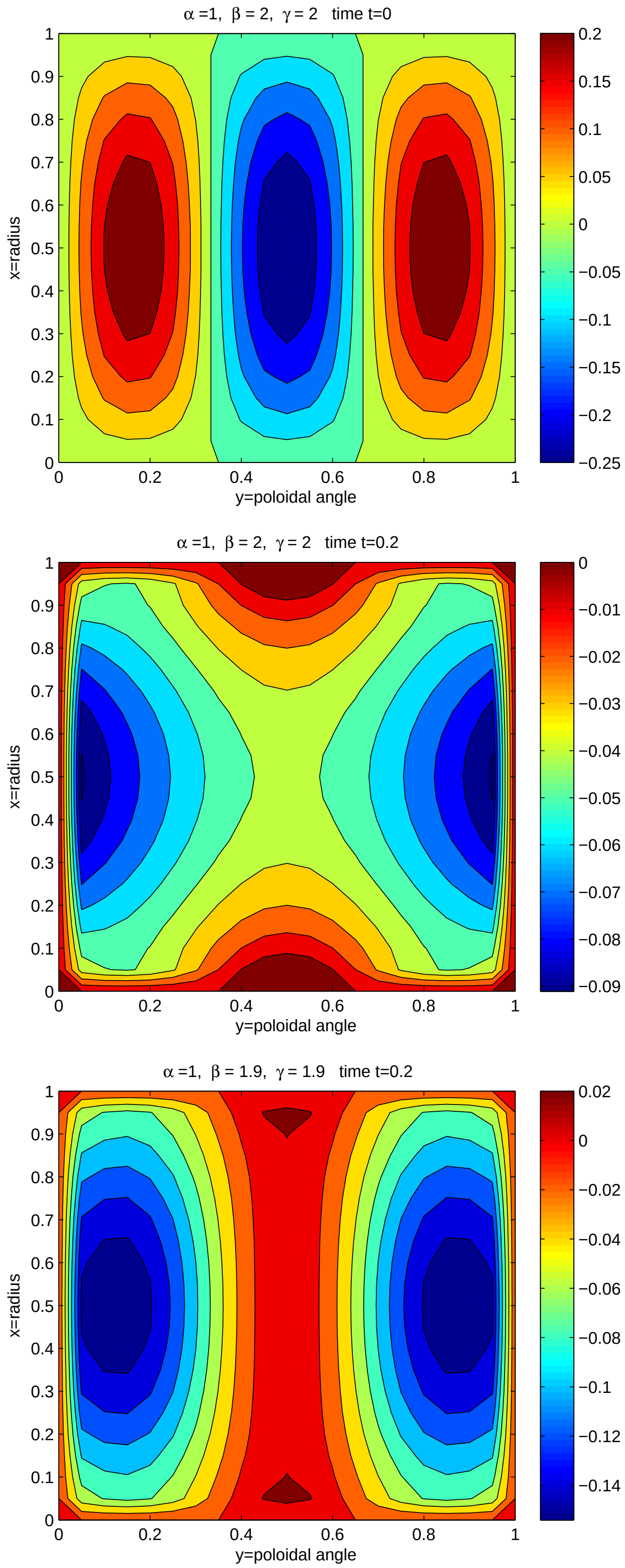

Figure 1: From top to botom: A) Initial conditions for the homogenous equation; B) Contour plot of $T(x, y, 0.2)$ for $\alpha = 1$, $\beta = \gamma = 2$; C) Contour plot of $T(x, y, 0.2)$ for $\alpha = 1$, $\beta = \gamma = 1.90$

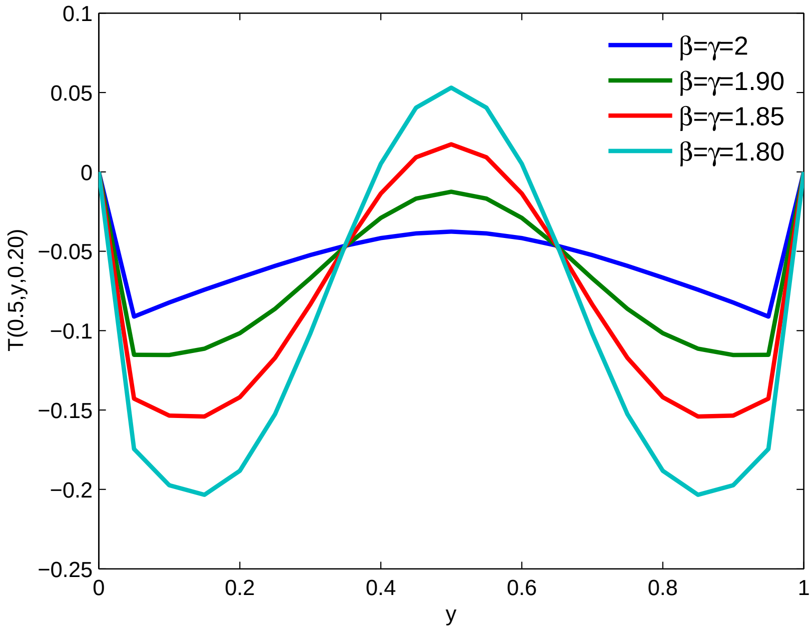

Figure 2: Solutions of fractional equations

3.2 Non-homogenous transport equation with null boundary conditions

We consider now the general non-symmetric equation

$$\left\{ \begin{array}{c} {}_{t}^{C}D_{0}^{\alpha}T - \left({}_{x}^{R}D_{l_1,r_1}^{\beta}T + {}_{y}^{R}D_{l_2,r_2}^{\gamma}T \right) = S\left(x,y,t\right) \\ T\left(x,y,0\right) = 0 \quad \forall \left(x,y\right) \in [0,1] \times [0,1] \\ T\left(0,y,t\right) = T\left(L_x,y,t\right) = 0 \quad \forall \left(y,t\right) \in [0,1] \times [0,1] \\ T\left(x,0,t\right) = T\left(x,L_y,t\right) = 0 \quad \forall \left(x,t\right) \in [0,1] \times [0,1] \end{array} \right. . \tag{4}$$

which is a 2D extension of the transport equation considered in [2]. We integrated in the system a localized source

$$S(x,y,t) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} e^{-\frac{(x-\mu_y)^2}{2\sigma_y^2}}$$

with $\sigma_x = \sigma_y = 0.05$, $\mu_x = 0.75$ and $\mu_y = 0.5$. The source, having a maximum in $(x,y) = (0.75, 0.5)$, is presented in Figure 3.

In order to observe the non-local spatial effect we choosen $\alpha = 1$ (i.e. memory effects are not present), $\beta = 1.5$ (i.e. non-local effects in radial direction are considered) and $\gamma = 2$ (i.e. there are not non-local effects in the poloidal direction).

The numerical simulations were performed in three cases:

- $l_1 = 0, \quad r_1 = 1$, when ${}_{x}^{R}D_{l_1,r_1}^{\beta}T(x,y,t)$ is computed using using the values of x in the interval $[x,1]$. In this case we speak about the right-radial-effect.

- $l_1 = r_1 = 1/2$, when ${}_{x}^{R}D_{l_1,r_1}^{\beta}T(x,y,t)$ is computed using using the values of x in the interval $[0,1]$. In this case we speak about the symmetric-radial-effect.

- $l_1 = 1, \quad r_1 = 0$, when ${}_{x}^{R}D_{l_1,r_1}^{\beta}T(x,y,t)$ is computed using using the values of x in the interval $[0,x]$. In this case we speak about the left-radial-effect.

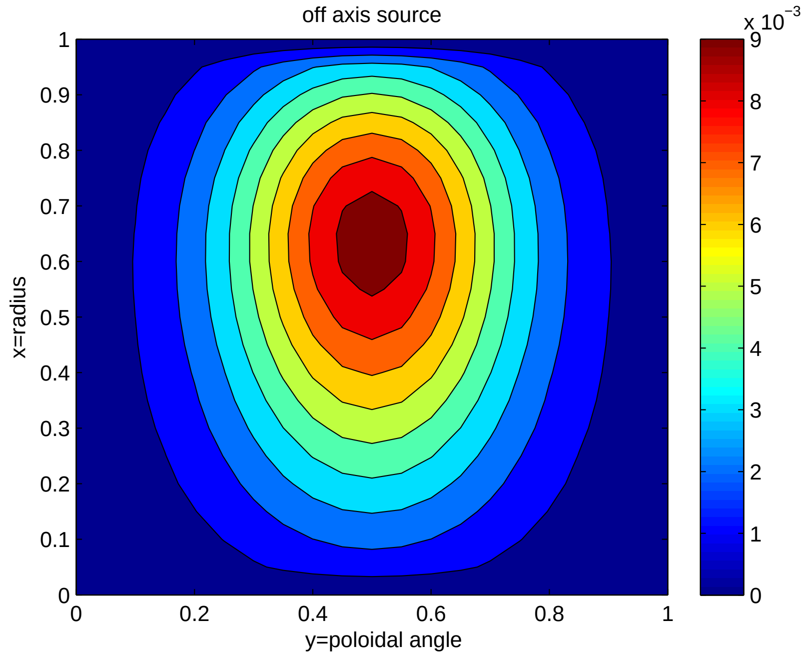

Figure 3: The off-axis source incorporated in the non-homogenous transport equation

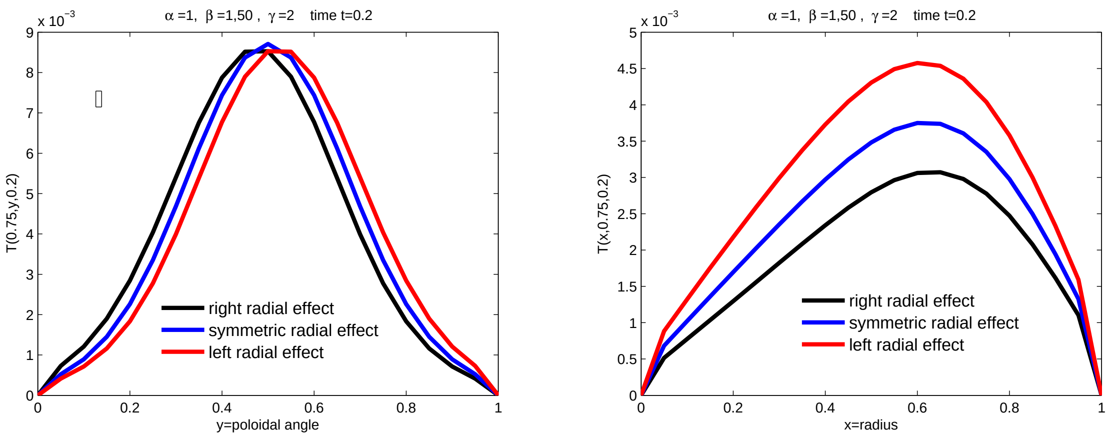

Figure 4: Non-local effects in the poloidal direction on the line $x_0 = 0.75$ (left) and non-local effects in the radial direction on the line $y_0 = 0.5$ (right)

The profiles of the solutions corresponding to $x_0 = 0.75$ (and $y \in [0,1]$, i.e. in the poloidal direction), respectively $y_0 = 0.5$ (and $x \in [0,1]$, i.e. in the radial direction) are presented in Figure 4 (left), respectively in Figure 4 (right).

The effects of using various types of fractional x-derivatives are general, they appear not only in x-direction. The effect in poloidal direction can be observed in Figure 4 (left). Symmetric x-derivatives induces symmetric solutions with respect to the line $y = 0.5$. This symmetry is natural if we take into account that the source term and the initial condition have the same symmetry. Using left or right derivatives we obtain solutions that are no more symmetric with respect to the same line, but displaced in oposite directions.

More important is the effect in x-direction: all solutions have the same shape and have a peak in the same point $x_M \approx 0.625$. The values of the solution obtained using right derivatives are smaller than those of the solutions obtained using symmetric and left derivatives.

4 Conclusions

In this paper we extended the matrix approach for solving 1D fractional transport equations to solve 2D similar equations. The reason for this is that 2D transport models are more accurate that 1D models when they are used for the study of some phenomena that occur in tokamaks. We solved two general equations (symmetric homogenous equation with non-null initial condition and non-symmetric non-homogenous quation with null initial condition) which may be applied in order to describe specific transport phenomena in tokamaks. We also pointed out some effects which are due to the non-symmetry of fractional derivatives, effect that can explain some observed features of the anomalous transport, for example the up-hill transport

5 Appendix

Many definitions for fractional derivatives are used in fractional calculus. Each of them has its own advantages. In the transport equations two of them are involved: Caputo and Riemann-Liouville derivatives.

From the classical formulas for computing the same type of ordinary derivatives (see [11] for example) one can easily obtain the formulas for computing partial fractional derivatives. If $p-1 < \delta \leq p$, the following expressions stand for computation.

Caputo fractional derivative of order δ is defined as

$$ {}_t^C D_a^{\delta} T(x,y,t) = \frac{1}{\Gamma(p-\delta)} \int_a^t \frac{T^{(p)}(x,y,\tau)}{(t-\tau)^{\delta-p+1}} d\tau $$

The left R-L fractional derivative, respectively the right R-L fractional derivative are given by

$$ \begin{aligned} \left({}_x D_{a+}^{\delta} T\right)(x,y,t) &= \frac{1}{\Gamma(p-\delta)} \left(\frac{\partial}{\partial x}\right)^p \int_a^x \frac{T(\xi,y,t)}{(x-\xi)^{b-p+1}} d\xi \\ \left({}_y D_{b-}^{\delta} T\right)(x,y,t) &= \frac{1}{\Gamma(p-\delta)} \left(-\frac{\partial}{\partial y}\right)^p \int_x^b \frac{T(x,\xi,t)}{(y-\xi)^{b-p+1}} d\xi \end{aligned} $$

The left (respectively right) Grundwald-Letnikov derivatives can be computed as

$$\begin{aligned}
\left({}_{x}^{GL}D_{a+}^{\delta}T\right)(x,y,t) &= \lim_{h\to 0}\frac{1}{h^{\delta}}\sum_{s=0}^{N_a} w_s^{(\delta)} T\left(x-s\cdot h\right) \text{ where } N_a = \left[\frac{x-a}{h}\right] \\
\left({}_{x}^{GL}D_{b-}^{\delta}T\right)(x,y,t) &= \lim_{h\to 0}\frac{1}{h^{\delta}}\sum_{s=0}^{N_b} w_s^{(\delta)} T\left(x+s\cdot h\right) \text{ where } N_a = \left[\frac{b-x}{h}\right]
\end{aligned}$$

where $w_s^{(\delta)} = (-1)^s \frac{\Gamma(\delta+1)}{\Gamma(\delta-s+1)\cdot\Gamma(s+1)} = (-1)^s \frac{\alpha(\alpha-1)\ldots(\alpha-s+1)}{s!}$.

From classical theorems in fractional calculus [11], one can obtain the corresponding relations between various types of fractional derivatives.

1) ${}_{t}^{C}D_{a}^{\delta}T(x,y,t) = \left({}_{t}D_{a+}^{\delta}T\right)(x,y,t)$ if and only if $T(x,y,a) = \frac{\partial T}{\partial t}(x,y,a) = \ldots\frac{\partial^{p-1}T}{\partial t^{p-1}}(x,y,a) = 0$

2) $\left({}_{x}^{GL}D_{a+}^{\delta}T\right)(x,y,t) = \left({}_{x}D_{a+}^{\delta}T\right)(x,y,t) + O(h) + O(T(x,y,0))$ so Grundwald-Letnikov derivative is a $O(h)$ approximation of the corresponding Riemann-Liouville derivative only if $T(x,y,0) = 0$

The previous observations are the key for imposing the initial conditions in (1).

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