Chua circuit as cognitive dynamical system

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Abstract

The paper presents a specific type of nonlinear dynamical systems which is used as a toy model in the cognition sciences. The system, known as Chua circuit, represents an electronic circuit including a nonlinear element and, despite its simplicity, the attached system of nonlinear differential equations is very rich in the dynamical states, with interesting transitions between chaos to regular dynamics. A presentation of the circuit, as well as of its connection with neural networks are illustrated and problems related to symmetries, invariants and integrability are tackled out.

Keywords: Chua circuit, symmetries, invariants, integrability.

1 Introduction

Understanding the way of thinking and how the information is transmitted and shared inside the human body represents an important chalange of the XXI-st century science. Neuroscientists, psychologists, mathematicians, physicists, engineers, but philosophers too, debate and propose models for the human thinking and for the artificial intelligence. [1] Electronic memories, inteligent robots and cellular automata are nowdays used as example of cognitive systems. New concepts or scientific disciplines try to explain the complex phenomena in the field of cognitive sciences: chaos and regular dynamics, limit cycles and strange attractors, fractal geometry and fractional calculus, stochasticity and self-organization. The aim of this paper is exactly to tackle part of these notions and to show how they have been developed for modelling the transmission of neural flows, seen as a nonlinear dynamical process. The topic can be addressed from many perspectives, including biological, mathematical, and engineering ones. The main biological model of the neural flow is due to Hodgkin and Huxley [2], but many other approaches have been also proposed [3]. In what follows we will concentrate on the engineering and mathematical perspectives, illustrating the complexity of the neural flow transmission through the electronic circuit proposed by Leon Chua in 1983 [4]. The circuit produces signals quite similar with those registered for the human brain and it is able to generate a rich variety of dynamical phenomena, ranging from fixed points to cycle points, regular dynamics with multiple scrolls [5], bifurcations, sensitive dependence on initial conditions and other standard routes to chaos [6].

From engineering point of view, Chua circuit is a quite simple electric circuit containing a nonlinear element called the Chua diode. The circuit is very well known as generator of chaotic signals [7], with many practical applications, starting from telecomunication or neurosciences (modelling neural networks), till to the creation of tunes and sound effects highly appreciated in the electronic music. From mathematical point of view, the Kirchoff 's law for the circuit generate a set of three first order ordinary differential equations which, again despite their simplicity, prove to have chaotic behaviors similar with those of Lorenz type systems. The richness of the dynamical states made that the intensive studies already done to not exhaust yet the description of the circuit, various new aspects and characteristics being still emphasized in an important number of scientific contributions. We will point some of these last contributions, also proposing a generalization of the Chua system within a form which could connect Chua and Lorenz dynamics. More precisely, we will consider a system whose dynamics is described by the following equations:

$$\begin{aligned} \dot{x} &= a(y - f(x)) \\ \dot{y} &= bx + cy - g(x,z) \\ \dot{z} &= mz + h(x,y) \end{aligned} \tag{1}$$

There are three arbitrary functions and four parameters which determine the evolution of the system (1). Depending on the values of all these, the system could generate chaos or could have regular dynamics. We will not enter here in deep details concerning the possible dynamics (1) can engender, leaving this task for later investigations. We notice only that the choice $f(x) = x,\ g(x,z) = xz$ and $h(x,y) = xy$ corresponds to Lorenz case, while the Chua circuit is described by $g(x,z) = -z, b = -c = 1, m = 0, h(x,y) = -\beta y$ and arbitrary $f(x)$. The corresponding systems are respectively:

Lorenz system:

$$\begin{aligned} \dot{x} &= a(x - y) \\ \dot{y} &= bx + cy - xz \\ \dot{z} &= mz + xy \end{aligned} \tag{2}$$

Chua system:

$$\begin{aligned} \dot{x} &= \alpha(y - f(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y. \end{aligned} \tag{3}$$

We will focus our study on the system (3), but we will relate our results with those already reported in literature for Lorenz system. More concretely, we will compute the Lie symmetries and the invariants of (3) and we will identify some integrability conditions.

2 Chua circuit and the attached equations

Let us consider the circuit in Figure 1. It consists of two capacitors C_1 and C_2 and one inductance L, all these three energy-storing components being connected on three parallel branches of the circuit. The capacitors are related through a resistor with the electric resistance R or, equivalently, with the conductance $G = 1/R$. The key element of the circuit is the nonlinear resistor R_X which represents in fact a special diode with the internal structure shown in the right hand side of Figure 1.

Let us denote by V_1, respectively V_2, the potential differences on the two capacitors and by V_L, V_X those corresponding to the other two branches. Correspondingly, the current

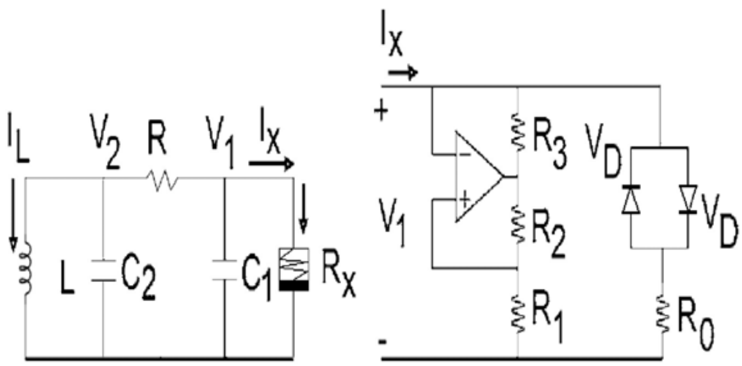

Figure 1: The Chua circuit (left) and the internal structure of the nonlinear resistor R_X (right).

intensities will be I_L, I_1, I_2, I_X. All the circuit elements have liniar characteristics, excepting the Chua diode whose charateristic $I_X = g(V_X)$ is strongly nonlinear, but chosen to be symmetric in respect with positive or negative values of the potential. As we will see, its form will be expressed in (1) through the nonlinear function $f(x)$.

The concrete form of the Chua system is obtaining by writting down the Kirchhoff's law for the circuit from below. They will have the form:

$$\begin{aligned}
\frac{dV_1}{dt} &= \frac{1}{RC_1}(V_2 - V_1) - \frac{1}{C_1}g(V_X) \\
\frac{dV_2}{dt} &= \frac{1}{RC_2}(V_1 - V_2) + \frac{1}{C_2}I_L \\
\frac{dI_L}{dt} &= -V_L \equiv V_2; V_X \equiv V_1
\end{aligned} \tag{4}$$

Many examples of nonlinear conductance $g(V_X) \equiv g(V_1)$ have been considered in literature and the usual procedure to tackle on the integrability of (1) is to apply a suitable linearization procedure. In his initial approach, Leon Chua approximated the diode's conductance by a linear function with three antisymmetrical segments:

$$g(V_1) = AV_1 + \frac{1}{2}(A - B)[|V_1 + 1| - |V_1 - 1|] \tag{5}$$

We denoted by A the slope of the central part of the linearized characteristic (passing through origin) and by B the slopes of the outer sections, connected by the central part in the breakpoints ± 1. Practically, the diode is replaced by a special limear element which changes the slope in the breakpoints and, very important, presents a "negative" electric resistance.

The main merits of the Chua circuit in its initial version were related to the simplicity, to its realistic and inexpensive elements, as well as to the possibility of transposing it in a system of differential equations suitable for numerical investigations. However, after Chua's proposal, there were many other alternative designs of equivalent circuits. For example, a single multi-tap digital potentiometer instead of the constant resistor has been used and various types of microcontrollers has been set for an automatic control of the digital potentiometer [8].

3 Chaos and symmetries for Chua system

The Lie symmetries of (1) can be studied by transforming the system in a third order differential equation. We will proceed to a direct investigation of the symmetries of (1), starting from the Lie operator:

$$X(t,x,y,z) = \partial_t + \xi\partial_x + \rho\partial_y + \phi\partial_z \tag{6}$$

As the equations are first order differential equations, we have to impose their invariance at the action of the first order prolongation of (6):

$$X^{(1)} \equiv X + \xi^t\partial_{\dot{x}} + \rho^t\partial_{\dot{y}} + \phi^t\partial_{\dot{z}} \tag{7}$$

Restricting from the beginning to the case $\varphi = const., \xi = \xi(x,t),\ \rho = \rho(t,x,y,z)$ and $\phi = \phi(t,x,y,z)$, the invariance conditions will lead to the following set of equations:

$$\xi_t + a\xi_x f(x) + a\xi\frac{\partial f}{\partial x} = 0 \tag{8}$$

$$\rho(x,y,z,t) = \xi_x(x,t)y \Longrightarrow \rho = \rho(x,y,t) \tag{9}$$

$$g(x,z)\rho_y - \xi(x,t)\frac{\partial g(x,z)}{\partial x} - \phi(x,y,z,t)\frac{\partial g(x,z)}{\partial z} = 0 \tag{10}$$

$$\rho_t + bx\rho_y - c\rho + cy\rho_y + ay\rho_x - af(x)\rho_x + b\xi = 0 \tag{11}$$

$$\phi_t + mz\phi_z + h(x,y)\phi_z + ay\phi_x - af(x)\phi_x + bx\phi_y + cy\phi_y - g(x,z)\phi_y - m\phi = 0 \tag{12}$$

$$\xi\frac{\partial h(x,y)}{\partial x} + \rho\frac{\partial h(x,y)}{\partial y} = 0 \tag{13}$$

There are 6 equations and 6 unknown functions: $f(x), g(x,z), h(x,y), \xi(x,t), \rho(x,y,z,t)$ and $\phi(x,y,z,t)$. Aparently we get a determinated system which could be solved. In reality the system can be solved by imposing some supplementary constraints only. Such constraints usually consist in concrete choices for the general functions $f(x), g(x,z)$ and $h(x,y)$.

We will not go further with the investigation of this general case, restricting ourselves to the investigation of the symmetries of Chua system (3), as a special case of the general system (1). We note than an alternative approach for investigating the Lie symmetries of the Chua system could be to transform it into a single differential equation of higer order, as it has been done for Lorenz system in [9]. Doing that, the following third-order ordinary differential equation is obtained:

$$\dddot{x} + [\dot{f}(x) + 1]\ddot{x} + \ddot{f}(x)\dot{x}^2 + [\dot{f}(x) + \beta - \alpha]\dot{x} + \beta f(x) = 0. \tag{14}$$

We will start our investigations by considering the general Lie operator of the form:

$$X(t,x) = \varphi(t,x)\partial_t + \phi(t,x)\partial_x. \tag{15}$$

The governing Eq. (14) remains invariant under the action of the symmetry generator (15) iff the following condition is satisfied:

$$X^{(3)}[\dddot{x} + [\dot{f}(x) + 1]\ddot{x} + \ddot{f}(x)\dot{x}^2 + [\dot{f}(x) + \beta - \alpha]\dot{x} + \beta f(x)] = 0. \tag{16}$$

where $X^{(3)}$ represents the third prolongation of the symmetry generator given by the expression:

$$X^{(3)}(t,x) = \varphi\partial_t + \phi\partial_x + \phi^t\partial_{\dot{x}} + \phi^{2t}\partial_{\ddot{x}} + \phi^{3t}\partial_{\dddot{x}}. \tag{17}$$

The above invariant condition (16) could be written into the equivalent form:

$$\phi[\ddot{f}(x)\ddot{x} + \dddot{f}(x)\dot{x}^2 + \ddot{f}(x)\dot{x} + \beta\dot{f}(x)] + \phi^t[2\ddot{f}(x)\dot{x} + \dot{f}(x) + \beta - \alpha] +$$
$$+\phi^{2t}[\dot{f}(x) + 1] + \phi^{3t} = 0. \tag{18}$$

Substituting into Eq. (18) the appropriate coefficients functions ϕ^t, ϕ^{2t}, ϕ^{3t} given in [10] and using Eq. (14), we can obtain a determining system which could be reduced to the following partial differential system:

$$\phi_{2x} = 0, \tag{19}$$
$$3\phi_{tx} - 3\ddot{\varphi} + (\dot{\varphi} + 1)\ \dot{f} + \phi\ \ddot{f} = 0, \tag{20}$$
$$3\phi_{t(2x)} + (\phi_x + \dot{\varphi})\ \ddot{f} + (\dot{f} + 1)\phi_{2x} + \phi\ \dddot{f} = 0, \tag{21}$$
$$3\phi_{(2t)x} + 2\phi_{tx} - \ddot{\varphi} - \dddot{\varphi} - 2(\alpha - \beta)\dot{\varphi} +$$
$$(2\dot{\varphi} + 2\phi_{tx} - \ddot{\varphi})\ \dot{f} + (2\phi_t + \phi)\ \ddot{f} = 0, \tag{22}$$
$$\phi_{3t} - (\alpha - \beta)\phi_t + \phi_{2t} + \beta(3\dot{\varphi} - \phi_x)\ f +$$
$$+(\phi_t + \phi_{2t} + \beta\phi)\ \dot{f} = 0. \tag{23}$$

with unknown functions $\varphi(t), \phi(t,x)$ and $f(x)$.

The Eq. (19) requires for ϕ the following linear form:

$$\phi(t,x) = g(t)x + h(t). \tag{24}$$

The expression (19) will be inserted into the all remaining equations of the previous system. If in addition we differentiate Eq. (20) with respect to t and correlating the result with Eq. (22), the following equivalent system is derived:

$$[g(t) + \dot{\varphi}(t)]\ \ddot{f}(x) + g(t)x\ \dddot{f}(x) + h(t)\ \dddot{f}(x) = 0,$$
$$2[\dddot{\varphi}(t) - \ddot{\varphi}(t) - (\alpha - \beta)\dot{\varphi}(t) + 2\dot{g}(t)] + 2[-\ddot{\varphi}(t) + \dot{\varphi}(t) + \dot{g}(t)]\ \dot{f}(x) +$$
$$[\dot{g}(t) + g(t)]x\ \ddot{f}(x) + [\dot{h}(t) + h(t)]\ \ddot{f}(x) = 0, \tag{25}$$
$$\dddot{h}(t) + \ddot{h}(t) - (\alpha - \beta)\dot{h}(t) + [\dddot{g}(t) + \ddot{g}(t) - (\alpha - \beta)\dot{g}(t)]x + [\ddot{g}(t) + \dot{g}(t) + \beta g(t)]x\ \dot{f}(x) +$$
$$[\ddot{h}(t) + \dot{h}(t) + \beta h(t)]\ \dot{f}(x) + [3\beta\dot{\varphi}(t) - \beta g(t)]f(x) = 0.$$

Considering the first three equations from the previous "determining system" and using Maple computational facilities, the following solutions arise for $\varphi(t), \phi(t)$ and $f(x)$.

1)

$$\begin{aligned} f(x) &= c_1 x + c_2 \\ \phi &= 0 \\ \varphi(t) &= c_3 + c_4 e^{\frac{(1+c_1)}{3}t} \end{aligned} \tag{26}$$

2)

$$f(x) = c_1 x + c_2$$

$$\begin{aligned}
\phi &= F_1(t)x + F_2(t) \\
\varphi(t) &= \int e^{\frac{(1+c_1)}{3}t}\left[\int \frac{dF_1(t)}{dt} e^{\frac{-(1+c_1)}{3}t} dt + c_3\right] dt + c_4
\end{aligned} \tag{27}$$

3)

$$\begin{aligned}
f(x) &= \frac{c_4}{2}x^2 + (c_1c_4 - 1 + 6c_2)x + c_5 \\
\phi &= c_3 e^{c_2 t}(x + c_1) \\
\varphi(t) &= -\frac{c_3}{c_2}e^{c_2 t} + c_6
\end{aligned} \tag{28}$$

4)

$$\begin{aligned}
f(x) &= -\frac{3}{c_2}x^2 - (1 + \frac{6c_1}{c_2})x + c_3 \\
\phi &= c_2 \frac{dF_1(t)}{dt} + F_1(t)(x + c_1) \\
\varphi(t) &= -\int - F_1(t)dt + c_4
\end{aligned} \tag{29}$$

5)

$$\begin{aligned}
f(x) &= -x + \ln(c_1 + x)c_2 + \ln(c_1 + x)c_1 + c_3 \\
\phi &= F_1(t)(x + c_1) \\
\varphi(t) &= \int F_1(t)dt + c_4
\end{aligned} \tag{30}$$

6)

$$\begin{aligned}
f(x) &= \frac{e^{c_4 c_5}(x + c_1)^{(1-c_4)}}{c_4(1 - c_4)} - \left(1 + \frac{3c_2}{c_4} - 3c_2\right)x + c_6 \\
\phi &= c_3 e^{c_2 t}(x + c_1) \\
\varphi(t) &= \frac{c_3 c_4 e^{c_2 t}}{c_2} + c_7
\end{aligned} \tag{31}$$

These solutions represent cases in which nontrivial Lie symmetries exist and invariant solutions of the Chua system can be generated.

Two particular cases of the previous equations will be considered below. The first case is $f(x) = x$. It can be checked that the Lie symmetry has in this case the form:

$$X = \left(-\frac{3}{2}t + c_4\right)\partial_t + \left(tx + c_1 + c_2 t + c_3 t^2\right)\partial_x \tag{32}$$

for the following values of the parameters:

$$\alpha = -\frac{1}{27},\ \beta = \frac{8}{27} \tag{33}$$

Another choice, very often discussed in literature as an interesting case of chaotic behavior, is $f(x) = x^3$. In this case, equating with zero the time dependent coefficient functions of terms with various powers of x in the system (25), we get:

$$h(t) = 0 \tag{34}$$

and the following differential system:

$$\begin{aligned}
\dot{\varphi}(t)+2g(t) &= 0,\\
\ddot{g}(t)+\dot{g}(t)-(\alpha-\beta)g(t) &= 0,\\
3\ddot{g}(t)+3\dot{g}(t)-4\beta g(t) &= 0,\\
4\dot{g}(t)-g(t) &= 0,\\
5\dot{g}(t)+4(\alpha-\beta)g(t) &= 0.
\end{aligned} \tag{35}$$

The solution admitted by the previous system is:

$$g(t)=0,\ \varphi(t)=c=const. \tag{36}$$

It corresponds to the Lie operator $X = c\partial_t$ which generates the time translations.

The master equation (14) admits Lie symmetries corresponding to $\phi \neq 0$ only in the case for which the compatibility condition for the system (35) is verified:

$$\alpha=\frac{7\beta}{3},\ \beta=-\frac{15}{64}. \tag{37}$$

The associated symmetry operator is:

$$X=-8e^{\frac{t}{4}}\partial_t+e^{\frac{t}{4}}x\partial_x \tag{38}$$

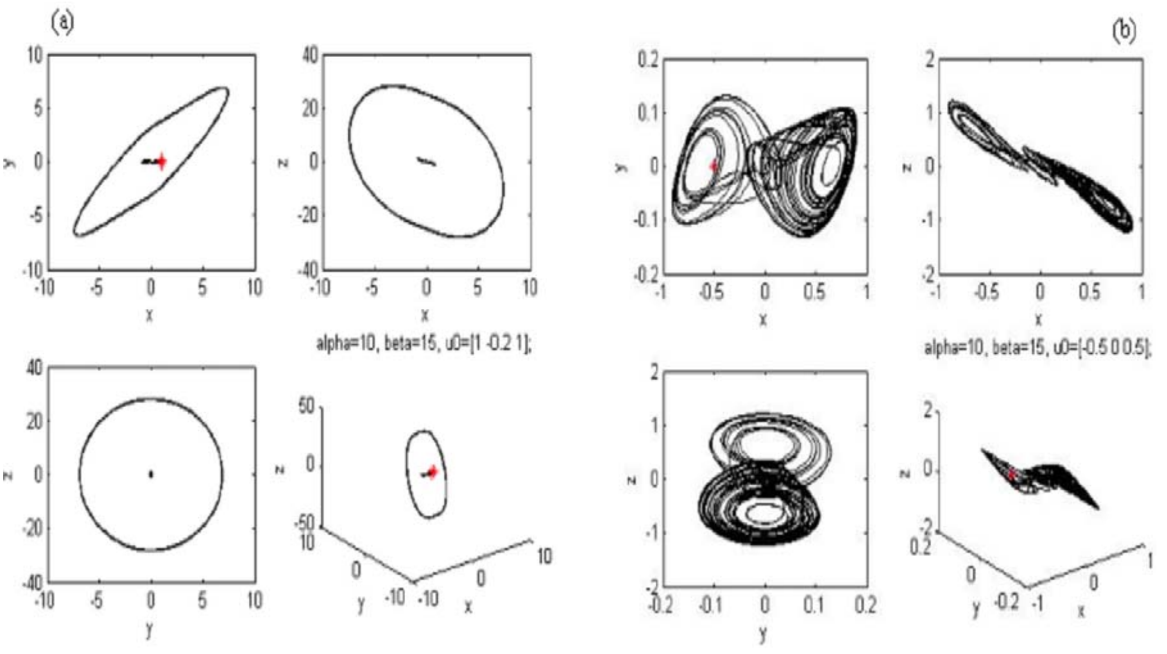

Figure 2: Regular and chaotic behavior of Chua system.

As far as the dynamics of the systems, some simulations we made, show that, depending on the values of the parameters α and β, by numerical investigation we will notice a dual dynamical behavior, chaotic and regular orbits, as it is presented in the Figure 2. No Hopf bifurcation and no limit cycle appear because there are not pure imaginary roots.

4 Conclusions

We investigated Chua type circuits, well-known for their rich dynamical behavior and extensively used in the description of phenomena specific for cognitive systems. The presentation focused on the aspects related to the stability and integrability of the underlying system of differential equations. The investigation has been done using the Lie symmetry method [11] and the invariant theory. For determining the Lie symmetries of the system. the Chua system (3) was previously transformed in a third order differential equation. The procedure allowed to determine all possible nonlinearities $f(x)$ for which nontrivial Lie symmetries could appear. The special cases $f(x) = x$ and $f(x) = x^3$, very often considered in the studies of the Chua circuit, were also tackled out and we found that, for special choices of the parameters α and β interesting invariant solutions can be also generated.

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References

[1] W. Kinsner and all, Proc. 2012 Canadian Engineering Education Association (CEEA12) Conf., Paper 119.

[2] A.L.Hodgkin, A.F.Huxley, J.Physiol.(London) Vol. **117** (1952), 500.

[3] J. Malmivuo, R. Plonsey, Bioelectromagnetism - Principles and Applications of Bioelectric and Biomagnetic Fields, Oxford University Press, 1995.

[4] T. Matsumoto, IEEE Trans. CAS-31, **12** (1984), 1055-1058.

[5] L.O.Chua, M.Komuro, and T.Matsumoto, IEEE Transactions on Circuits and Systems, **33** (1986), 1072-1118.

[6] Leon O. Chua, Intern. Journal of Circuit Theory and Applications, **22** (2006), 279–305.

[7] G. Q. Zhong, F. Ayrom, Int. J. Circuit Theory Appl. **13** (1985), 93-98.

[8] A. Byagowi, W. Kinsner, Proc. 2012 Canadian Engineering Education Association (CEEA12) Conf., Paper 096.

[9] M.C.Nucci, J.Math.Phys. **44** (2004), 4107. M.C. Nucci, Math. Comput.Modelling **25** (1997), 181.

[10] P. J. Olver, Application of Lie Groups to Differential Equations, Springer, New York, 1986.

[11] R. Cimpoiasu, R. Constantinescu, Nonlinear Analysis Series A: Theory, Methods & Applications , vol.**68** (2008), 2261-2268.