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Geometric methods for the study of dynamical systems. Symmetries and conservation laws

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Abstract

Using the geometric methods of Classical Mechanics we will study different ways to obtain symmetries and conservation laws for dynamical systems. We will extend the study from the symplectic formalism to the presymplectic formalism. Examples arising from biology and ecology will be also presented.

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1 Introduction

The problem of finding symmetries and conservation laws for different dynamical systems generated by systems of ordinary differential equations (SODE) or by systems of partial differential equations (SOPDE) is still of great interest although it was studied by many methods and many researchers. A possible description of constrained dynamical systems in the Hamiltonian formalism on Koszul type differential complexes was proposed in [1]. The geometric approach using the natural symplectic structure induced by the cotangent bundle and then the analysis of the presymplectic case ([2]-[5]) is very important for the generalizations to the polysymplectic models ([6]).

This paper is devoted to studying symmetries, conservation laws and relationship between this in the geometric framework of Classical Mechanics ([7]-[12]). More exactly we extend the study of symmetries and conservation laws from symplectic case to the presymplectic case. We will recall adapted Noether type Theorems for the presymplectic systems with global dynamic and, also, we will use the constraint algorithm of Gotay-Nester ([2]-[4]). All results remains valid for singular Lagrangian and Hamiltonian systems ([13]-[14]). We presents three very important examples from biology and ecology: Lotka-Volterra prey-predator ecological system ([15]-[18]), Bailey model for the evolution of epidemics ([19]-[20]), classical Kermack-McKendrick model of evolution of epidemics ([20]).

There is a very well-known way to obtain conservation laws for a system of differential equations given by a variational principle: the use of the Noether Theorem ([21]) which associates to every symmetry a conservation law and conversely. However, there is a method introduced by G.L. Jones ([22]) and M. Crâşmăreanu ([23]) by which new kinds of conservation laws can be obtained without applying a theorem of Noether type, only using symmetries and pseudosymmetries.

In the second and the third sections we recall the basic notions and results for the geometrical study of a dynamical system for the symplectic case. Also, we present the classical Noether Theorem ([21]) and the Theorem of Jones-Crâşmăreanu ([22]-[23]), accompanied by two examples ([23]-[25]).

In the fourth section we present a presymplectic version of the Noether theorem and, finally, we extend the results of Jones ([22]) and Crâşmăreanu ([23]) from symplectic systems to presymplectic systems, in order to obtain conservation laws.

In the fifth section we will present three important examples from biology and ecology: prey-predator ecological system, Bailey model for the evolution of epidemics, classical Kermack-McKendrick model of evolution of epidemics. For this biodynamical systems the 2-form ω_L associated to the corresponding Lagrangian is degenerate ([26]-[27]).

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

2 Basic tools

Let M be a smooth, n-dimensional manifold, $C^\infty(M)$ the ring of real-valued smooth functions, $\mathcal{X}(M)$ the Lie algebra of vector fields and $A^p(M)$ the $C^\infty(M)$-module of p-differential forms, $1 \leq p \leq n$.

Let us recall that if Δ is a distribution with a constant rank k on M and $\Phi : M \to M$ is a diffeomorfism of M, then Φ is called an *invariant transformation* or *finite symmetry* of Δ if for all $x \in M$, $T\Phi(\Delta_x) \subset \Delta_{\Phi(x)}$ ([10]).

If $\{\Phi_t^\xi\}_t$ denote the local one-parameter group of transformations of the vector field ξ on M, then ξ is called *symmetry* or *infinitesimal symmetry* or *dynamical symmetry* of Δ if for all t, $\{\Phi_t^\xi\}_t$ is an invariant transformation of Δ.

ξ is a symmetry of Δ if and only if for all $\zeta \in \Delta$, $[\xi, \zeta] \in \Delta$, or equivalently, the local flow of ξ transfer integral mappings in integral mappings and consequently, for any integral manifold Q of Δ, $\{\Phi_t^\xi\}_t(Q)$ is another integral manifold of Δ.

A function $g : U \to \mathbf{R}$ (U being an open subset of M) is called *first integral* or *conservation law* of Δ if the one-form dg belongs to Δ, i.e. $i_\xi dg = 0$, for all $\xi \in \Delta$.

If g is a first integral of Δ on U and Q is an integral manifold of Δ with integral mapping $i : Q \to U \subseteq M$, then $d(g \circ i) = 0$, that means the function g is constant along the integral manifold Q.

For $X \in \mathcal{X}(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ we consider the system of ordinary differential equations which give the flow $\{\Phi_t\}_t$ of X, locally,

$$\dot{x}^i(t) = \frac{dx^i}{dt}(t) = X^i(x^1(t), \ldots, x^n(t)). \tag{2.1}$$

A dynamical system is a couple (M, X), where M is a smooth manifold and $X \in \mathcal{X}(M)$. A dynamical system is denoted by the flow of X, $\{\Phi_t\}_t$ or by the system of differential equations (2.1).

A function $f \in C^\infty(M)$ is called *conservation law* for dynamical system (M, X) if f is constant along the every integral curves of X (solutions of (2.1)), that is

$$L_X f = 0, \tag{2.2}$$

where $L_X f$ means the Lie derivative of f with respect to X.

If $Z \in \mathcal{X}(M)$ is fixed, then $Y \in \mathcal{X}(M)$ is called *Z- pseudosymmetry* for (M, X) if there exists $f \in C^\infty(M)$ such that $L_X Y = fZ$. A X-pseudosymmetry for X is called *pseudosymmetry* for (M, X). $Y \in \mathcal{X}(M)$ is called *symmetry* for (M, X) if $L_X Y = 0$. Recall that $\omega \in A^p(M)$ is called *invariant form* for (M, X) if $L_X \omega = 0$.

Example 2.1. *The Nahm's system in the theory of static SU(2)-monopoles is presented in [25]:*

$$\frac{dx^1}{dt} = x^2x^3, \ \frac{dx^2}{dt} = x^3x^1, \ \frac{dx^3}{dt} = x^1x^2. \tag{2.3}$$

The vector field $X = x^2x^3\frac{\partial}{\partial x^1} + x^3x^1\frac{\partial}{\partial x^2} + x^1x^2\frac{\partial}{\partial x^3}$ *is homogeneous of order two, i.e.* $[Y, X] = X$*, where* $Y = \sum\limits_{i=1}^{3} x^i \frac{\partial}{\partial x^i}$. *So, Y is a pseudosymmetry for (2.3).*

The notion of pseudosymmetry defined above is a weaker notion of symmetry. This is a natural generalization of the notion of symmetry for a system of ordinary differential equations (2.1) which give the flow $\{\Phi_t\}_t$ of a vector field X with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$. Symmetries and pseudosymmetries are just infinitesimal symmetries of the distribution generated by the vector field X. Following the book of Olga Krupková ([10]), a vector field S is a symmetry of X if the transformations generated by S are symmetries of X, that means the transformations generated by S transform solutions of (2.1) into solutions of (2.1) (or, they maps integral curves of X into integral curves of X). A pseudosymmetry of X is a vector field S for which the generated transformations apply integral manifolds of (2.1) into integral manifolds of (2.1), or equivalently, the generated transformations apply integral mappings in integral mappings ([10]). So, the transformations generated by the pseudosymmetry S maps any trajectory of (2.1) into another trajectory of (2.1) (not necessarly integral curves). Given this, we can understand the geometric meaning of these concepts.

The next theorem which gives the association between pseudosymmetries and conservation laws is due to G.L. Jones ([22]) and M. Crâşmăreanu ([23]).

Theorem 2.2. *Let $X \in \mathcal{X}(M)$ be a fixed vector field and $\omega \in A^p(M)$ be a invariant p-form for X. If $Y \in \mathcal{X}(M)$ is symmetry for X and $S_1, \ldots, S_{p-1} \in \mathcal{X}(M)$ are $(p-1)$ Y-pseudosymmetry for X then*

$$\Phi = \omega(X, S_1, \ldots, S_{p-1}) \tag{2.4}$$

or, locally,

$$\Phi = S_1^{i_1} \cdots S_{p-1}^{i_{p-1}} Y^{i_p} \omega_{i_1 \ldots i_{p-1} i_p} \tag{2.5}$$

is a conservation laws for (M, X).

Particularly, if Y, $S_1, \ldots, S_{p-1}$ are symmetries for X then Φ given by (2.4) is conservation laws for (M, X).

3 The symplectic formalism

If (M, ω) is a symplectic manifold then the dynamical system (M, X) is said to be *a dynamical Hamiltonian system* (or, shortly, *Hamiltonian system*) if there exists a function $H \in C^\infty(M)$ (called *the Hamiltonian*) such that

$$i_X \omega = -dH, \tag{3.1}$$

where i_X denotes the interior product with respect to X.

It is known that the symplectic form ω is an invariant 2-form for (M, X) and the Hamiltonian H is a conservation law for (M, X).

A *Cartan symmetry* for Lagrangian L is a vector field $X \in \mathcal{X}(TM)$ characterized by $L_X\omega_L = 0$ and $L_X H = 0$, where $\omega_L = d\theta_L$ is the Cartan 2-form associated to the regular Lagrangian L, $\theta_L = J^*(dL)$, J^* being the adjoint of the natural tangent structure J on TM and $H = E_L = \frac{\partial L}{\partial y^i} y^i - L$ is the en energy of L. It is known that ([9]) that any Cartan symmetry for Lagrangian L is a symmetry for the canonical semispray S of L ([11]), that is $L_S X = 0$. For each Cartan symmetry X for (M, L) we have $dL_X\theta_L = 0$, which implies that $L_X\theta_L$ is a closed 1-form. If $L_X\theta_L$ is a exact 1-form, then we say that X is *exact Cartan symmetry* for (M, L). Obviously, the canonical semispray of L is an exact Cartan symmetry for Lagrangian L ([9], [11]).

It has been known that the Cartan symmetries induce and are induced by constants of motions (conservation laws), and these results are known as Noether Theorem and its converse ([9], [21], [22], [23], [28]).

Theorem 3.1. *(Noether Theorem) If X is an exact Cartan symmetry with $L_X\theta_L = df$, then*

$$P_X = J(X)L - f$$

is a conservation law for the Euler-Lagrange equations associated to the regular Lagrangian L.

Conversely, if F is a conservation law for the Euler-Lagrange equations assocaited to the regular Lagrangian L, then the vector field X uniquely defined by

$$i_X\omega_L = -dF$$

is an exact Cartan symmetry.

Now, we can apply theorem 2.2 to the dynamical Hamiltonian systems.

Proposition 3.2. *Let be (M, X_H) a Hamiltonian system on the symplectic manifold (M, ω), with the local coordinates (x^i, p_i). If $Y \in \mathcal{X}(M)$ is a symmetry for X_H and $Z \in \mathcal{X}(M)$ is a Y-pseudosymmetry for X_H then*

$$\Phi = \omega(Y, Z) \tag{3.2}$$

is a conservation law for the Hamiltonian system (M, X_H).

Particularly, if Y and Z are symmetries for X_H then Φ from (3.2) is a conservation law for (M, X_H).

Corollary 3.3. *If $Y \in \mathcal{X}(M)$ is a X_H-pseudosymmetry for X_H then*

$$\Phi = \omega(X_H, Y) = -L_Y H \tag{3.3}$$

or

$$\Phi = \frac{\partial H}{\partial x^k} Y^k + \frac{\partial H}{\partial p_k} \widetilde{Y}_k \tag{3.4}$$

is a conservation law for (M, X_H).

Now, if we consider the Hamiltonian system (TM, S_L) on the symplectic manifold (TM, ω_L), where S_L is the canonical semispray and ω_L the Cartan 2-form associated to a regular Lagrangian L on TM (for more details see [11]), then we have:

Corollary 3.4. *If* $Y = Y^k \frac{\partial}{\partial x^k} + \widetilde{Y}^k \frac{\partial}{\partial y^k} \in \mathcal{X}(TM)$ *is a* S_L*-pseudosymmetry for* S_L *then*

$$\Phi = \omega_L(S_L, Y) = -L_Y E_L \tag{3.5}$$

or

$$\Phi = \frac{\partial E_L}{\partial x^k} Y^k + \frac{\partial E_L}{\partial y^k} \widetilde{Y}^k \tag{3.6}$$

is a conservation law for (TM, S_L).

An immediately consequence of this last result is the following ([23]):

Corollary 3.5. *If the canonical semispray* S_L *associated to the regular Lagrangian* L *is* 2*-positive homogeneous with respect to velocity (*S_L *is a* ***spray****) and* g_{ij} *is the metric tensor of* L*, then* $\Phi = g_{ij} y^i \widetilde{Y}^j$ *is a conservation law for* (TM, S_L).

Taking into account that the canonical semispray S_L associated to the regular Lagrangian L is a spray if and only if $[S_L, C] = S_L$, that is $L_{S_L} C = S_L$, we have that the **Liouville** vector field $C = y^i \frac{\partial}{\partial y^i}$ is a pseudosymmetry for S_L and using the last corollary we obtain that $\Phi = g_{ij} y^i y^j$ is a conservation law for (TM, S_L). So, we have the conservation of the kinetic energy $\mathcal{E}(L) = \frac{1}{2} g_{ij} y^i y^j$ of the metric g_{ij} for the dynamical system given by the spray S.

Example 3.6. *([23], [24]) Let the* 2*-dimensional isotropic harmonic oscillator*

$$\begin{cases} \ddot{q}^1 + \omega^2 q^1 &= 0 \\ \ddot{q}^2 + \omega^2 q^2 &= 0 \end{cases} \tag{3.7}$$

a toy model for many methods to finding conservation laws. The Lagrangian is

$$L = \frac{1}{2}\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2\right] - \frac{\omega^2}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2\right] \tag{3.8}$$

and then applying the conservation of energy we have two conservation laws $\Phi_1 = \left(\dot{q}^1\right)^2 + \omega^2 \left(q^1\right)^2$, $\Phi_2 = \left(\dot{q}^2\right)^2 + \omega^2 \left(q^2\right)^2$.

A straightforward computation give that the complet lift of $X = q^2 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2}$ *is an exact Cartan symmetry with* $f = 0$ *and then the associated classical Noetherian conservation law is* $\Phi_3 = P_X = J(X)L = X^i \frac{\partial L}{\partial \dot{q}^i} = q^2 \dot{q}^1 - q^1 \dot{q}^2$.

But we can obtain a nonclassical conservation law with symmetries taking into account that the canonical spray of L *is* $S = \dot{q}^1 \frac{\partial}{\partial q^1} + \dot{q}^2 \frac{\partial}{\partial q^2} - \omega^2 q^1 \frac{\partial}{\partial \dot{q}^1} - \omega^2 q^2 \frac{\partial}{\partial \dot{q}^2}$ *and another computation gives that* $Y = \dot{q}^2 \frac{\partial}{\partial q^1} + \dot{q}^1 \frac{\partial}{\partial q^2} - \omega^2 q^2 \frac{\partial}{\partial \dot{q}^1} - \omega^2 q^1 \frac{\partial}{\partial \dot{q}^2}$ *is a symmetry for* S*. Also, because* S *is total* 1*-homogeneous, that means that* S *is* 1*-homogeneous with respect to all variables* $(q, \dot{q})$*, it result that* $Z = q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} + \dot{q}^1 \frac{\partial}{\partial \dot{q}^1} + \dot{q}^2 \frac{\partial}{\partial \dot{q}^2}$ *is a symmetry for* S*. Next, we have* $L_Y H = 0$*,* $L_Z H = 2H$ *and then* $\Phi = \omega_L(S, Y) = 0$*,* $\Phi = \omega_L(S, Z) = 2H$*, that means that we not have new conservation law applying theorem 2.2. But* $\Phi_4 = \omega_L(Y, Z) = \dot{q}^1 \dot{q}^2 + \omega^2 q^1 q^2$ *is a new conservation law given by theorem 2.2 or by their corollaries.*

We remark that Φ_4 *is a nonclassical conservation law, obtained by symmetries, and* Φ_4 *represent the energy of a new Lagrangian of (3.7),* $\tilde{L} = \dot{q}^1 \dot{q}^2 - \omega^2 q^1 q^2$ *([29]).*

4 The presymplectic formalism

Let M be an n-dimensional manifold, ω a closed 2-form with constant rank, and α a closed 1-form. The triple (M,ω,α) is said to be a *presymplectic system* ([5]).

The dynamics is determined by the solutions of the equation

$$i_X\omega = \alpha\,. \tag{4.1}$$

Since ω is not symplectic, (4.1) has no solution, in general, and even if it exists it will not be unique. Let $b : TM \to T^*M$ be the map defined by $b(X) = i_X\omega$. It may happen that b is not surjective. We denote by $ker\omega$ the kernel of b. Exactly, like in the symplectic case, let us remark that ω is an *invariant 2-form* for every solution ξ of (4.1), if this solution exists. It is enough to compute $L_\xi\omega = di_\xi\omega + i_\xi d\omega = 0$.

Gotay (1979) and Gotay, Nester (1979) ([2], [3], [4]) developed a constraint algorithm for presymplectic systems. They consider the points of M where (4.1) has a solution and suppose that this set, denoted by M_2, is a submanifold of M. Nevertheless, these solutions on M_2 may not be tangent to M_2. Then, we have to restrict M_2 to a submanifold where the solutions of (4.1) are tangent to M_2. Proceeding further, we obtain a sequence of submanifolds:

$$\cdots \to M_k \to \cdots \to M_2 \to M_1 = M\,.$$

Alternatively, these constraint submanifolds may be described as follows:

$$M_i = \{x \in M | \alpha(x)(v) = 0,\ \forall v \in T_xM_{i-1}^{\perp}\}$$

where

$$T_xM_{i-1}^{\perp} = \{v \in T_xM | \omega(x)(u,v) = 0,\ \forall u \in T_xM_{i-1}\}\,.$$

We call M_2 *the secondary constraint submanifold*, M_3 *the tertiary constraint submanifold*, and, in general, M_i is *the i-ary constraint submanifold*. If the algorithm stabilizes, that means there exists a positive integer k such that $M_k = M_{k+1}$ and $dimM_k \neq 0$, then we have a *final constraint submanifold* $M_f = M_k$, on which a vector field X exists such that

$$(i_X\omega = \alpha)_{|M_f}\,. \tag{4.2}$$

If ξ is a solution of (4.2), then every arbitrary solution on M_f is of the form $\xi' = \xi + Y$, where $Y \in (ker\omega \cap TM_f)$.

Next, we present the definitions of symmetries and conservation laws for the presymplectic systems which admit a global dynamics ([5], [13]). Also, the adapted Noether Theorem ([5]) is presented. We say that a presymplectic system (M,ω,α) admits a global dynamics if there exists a vector field ξ on M such that ξ satisfies (4.1). This condition is equivalent with the condition: $\alpha(ker\omega)(x) = 0,\ \forall x \in M$.

Definition 4.1. *A function $F : M \to \mathbf{R}$ is said to be a conservation law (or constant of the motion) of ξ if $\xi F = L_\xi F = 0$.*

Thus, if γ is an integral curve of ξ, then $F \circ \gamma$ is a constant function.

Definition 4.2. *A diffeomorfism $\Phi : M \to M$ is said to be a symmetry of ξ if Φ maps integral curves of ξ onto integral curves of ξ, i.e., $T\Phi(\xi) = \xi$.*

Definition 4.3. *A dynamical symmetry of ξ is a vector field X on M such that its flow consists of symmetries of ξ, or, equivalently, $[X,\xi] = L_\xi X = 0$.*

We denote by $\mathcal{X}^\omega(M)$ the set of all solutions of (4.1),

$$\mathcal{X}^\omega(M) = \{X \in \mathcal{X}(M) | i_X\omega = \alpha\}.$$

Definition 4.4. *A function $F : M \to \mathbf{R}$ is said to be a conservation law of $\mathcal{X}^\omega(M)$ if F is constant along all the integral curves of any solution of (4.1).*

That is, F satisfies $\mathcal{X}^\omega(M)F = 0$ or, equivalently, $(ker\omega)F = 0$.

Definition 4.5. *A diffeomorfism $\Phi : M \to M$ is said to be a symmetry of $\mathcal{X}^\omega(M)$ if Φ satisfies $T\Phi(\xi) \in \mathcal{X}^\omega(M)$ for all $\xi \in \mathcal{X}^\omega(M)$.*

Definition 4.6. *A dynamical symmetry of $\mathcal{X}^\omega(M)$ is a vector field X on M such that $[X, \mathcal{X}^\omega(M)] \subset ker\omega$, i.e. $[X, \xi] = L_\xi X = 0$, for all $\xi \in \mathcal{X}^\omega(M)$.*

Let us remark that if F is a constant of motion of $\mathcal{X}^\omega(M)$, then XF is also a constant of motion of $\mathcal{X}^\omega(M)$. Also, if we denote by $D(\mathcal{X}^\omega(M))$ the set of all dynamical symmetries of $\mathcal{X}^\omega(M)$, then for any $X, Y \in D(\mathcal{X}^\omega(M))$ we have $[X, Y] \in D(\mathcal{X}^\omega(M))$, i.e., $D(\mathcal{X}^\omega(M))$ is a Lie subalgebra of the Lie algebra $\mathcal{X}(M)$.

Definition 4.7. *A Cartan symmetry of the presymplectic system (M, ω, α) is a vector field X on M such that $i_X\omega = dG$, for some function $G : M \to \mathbf{R}$, and $i_X\alpha = 0$.*

This definition is a natural generalization of the exact Cartan symmetry from the symplectic case. Moreover, $L_X\alpha = di_X\alpha$, that means that in the presymplectic case the 1-form $L_X\alpha$ is always an exact form. If X is a Cartan symmetry of (M, ω, α), then X is a dynamical symmetry of $\mathcal{X}^\omega(M)$. The set $C(\omega, \alpha)$ of all Cartan symetries of (M, ω, α) is a Lie subalgebra of $\mathcal{X}(M)$ and we have $C(\omega, \alpha) \subset D(\mathcal{X}^\omega(M))$.

The presymplectic version of the Noether Theorem is the following ([5]):

Theorem 4.8. *If X is a Cartan symmetry of (M, ω, α), then the function G (as in Definition 4.7) is a conservation law of $\mathcal{X}^\omega(M)$. Conversely, if G is a conservation law of $\mathcal{X}^\omega(M)$, then there exists a vector field X on M such that $i_X\omega = dG$ and X is a Cartan symmetry of (M, ω, α). Moreover, every vector field $X + Z$, with $Z \in ker\omega$ is also a Cartan symmetry of (M, ω, α).*

Next, taking into account that the presymplectic form ω is invariant for every solution ξ of (4.1), we can use theorem 2.2 for obtain new kinds of conservation laws (non-Noetherian) for presymplectic systems which admit a global dynamics ([5], [13]). Also, the results remains valid for singular Lagrangian and Hamiltonian systems.

Definition 4.9. *Let (M, ω, α) be a presymplectic system. If we suppose that $\xi \in \mathcal{X}(M)$ is a solution of (4.1) and $Y \in \mathcal{X}(M)$, then we say that $Z \in \mathcal{X}(M)$ is a Y-dynamical pseudosymmetry of ξ if there exists a function $f \in C^\infty(M)$ such that $L_\xi Z = fY$.*
A ξ-dynamical pseudosymmetry of ξ is called dynamical pseudosymmetry of ξ.

Obviously, if $Y = 0$, a 0-dynamical pseudosymmetry of ξ is a dynamical symmetry of ξ.

Proposition 4.10. *Let (M, ω, α) be a presymplectic system such that there exists a vector field ξ on M who satisfies (4.1). If $Y \in \mathcal{X}(M)$ is a dynamical symmetry of ξ and $Z \in \mathcal{X}(M)$ is a Y-dynamical pseudosymmetry of ξ, then $F = \omega(Y, Z)$ is a conservation law for ξ.*
Particularly, if Y and Z are dynamical symmetry of ξ, then $F = \omega(Y, Z)$ is a conservation law for ξ.

Taking into account of the definition of a dynamical symmetry of $\mathcal{X}^\omega(M)$, we can say that, for a fixed $Y \in \mathcal{X}(M)$, the vector field Z on M is a Y-dynamical pseudosymmetry of $\mathcal{X}^\omega(M)$ if for every $\xi \in \mathcal{X}^\omega(M)$, there exists a function $f \in C^\infty(M)$ such that $L_\xi Z = fY$.

Proposition 4.11. *Let (M, ω, α) be a presymplectic system such that there exists at least vector field ξ on M who satisfies (4.1). If $Y \in \mathcal{X}(M)$ is a dynamical symmetry of $\mathcal{X}^\omega(M)$ and $Z \in \mathcal{X}(M)$ is a Y-dynamical pseudosymmetry of $\mathcal{X}^\omega(M)$, then $F = \omega(Y, Z)$ is a conservation law for $\mathcal{X}^\omega(M)$.*
Particularly, if Y and Z are dynamical symmetry of $\mathcal{X}^\omega(M)$, then $F = \omega(Y, Z)$ is a conservation law of $\mathcal{X}^\omega(M)$.

Example 4.12. *([5]) Let us consider the presymplectic system $(\mathbf{R}^6, \omega, \alpha)$, where*

$$\omega = dx^1 \wedge dx^4 - dx^2 \wedge dx^3 \,, \alpha = x^4 dx^4 - x^3 dx^5 - x^5 dx^3,$$

with $(x^1, \ldots, x^6)$ the standard coordinates on $\mathbf{R}^6$.
It is easy to see that $ker\omega$ is generated by $\frac{\partial}{\partial x^5}$ and $\frac{\partial}{\partial x^6}$. The only secondary constraint is $\Phi_1 = x^3 = 0$, there are not tertiary constraints and the constraints algorithm ends in M_2, i.e.

$$M_f = M_2 = \left\{(x^1, \ldots, x^6) \in \mathbf{R}^6 | x^3 = 0\right\}$$

The solution of the equation $(i_X\omega = \alpha)_{M_f}$ are

$$\mathcal{X}^\omega(M_f) = x^4 \frac{\partial}{\partial x^1} + ker\omega \,.$$

If we denote by $i : M_f \to \mathbf{R}^6$ the embedding of M_f in $\mathbf{R}^6$, then $i^*\omega = \omega_{M_f} = dx^1 \wedge dx^4$. So, $ker\omega_{M_f}$ is generated by $\frac{\partial}{\partial x^2}$, $\frac{\partial}{\partial x^5}$ and $\frac{\partial}{\partial x^6}$. The solutions of the equation $i_X\omega_{M_f} = i^*\alpha$ are

$$\mathcal{X}^{\omega_{M_f}}(M_f) = x^4 \frac{\partial}{\partial x^1} + ker\omega_{M_f} \,.$$

Thus, $\mathcal{X}^\omega(M_f)$ is strictly contained in $\mathcal{X}^{\omega_{M_f}}(M_f)$.

A function $F : M_f \to \mathbf{R}$ is a conservation law of $\mathcal{X}^\omega(M_f)$ if

$$x^4 \frac{\partial F}{\partial x^1} = 0, \ \frac{\partial F}{\partial x^5} = 0, \ \frac{\partial F}{\partial x^6} = 0.$$

In particular, each function F which depends only on x^2 and x^4 is a conservation law of $\mathcal{X}^\omega(M_f)$. For example, $F_1(x^1, \ldots, x^6) = x^2$ and $F_2(x^1, \ldots, x^6) = x^4$ are constants of the motion of $\mathcal{X}^\omega(M_f)$. A function $F : M_f \to \mathbf{R}$ is a conservation law of $\mathcal{X}^{\omega_{M_f}}(M_f)$ if

$$x^4 \frac{\partial F}{\partial x^1} = 0, \ \frac{\partial F}{\partial x^2} = 0, \ \frac{\partial F}{\partial x^5} = 0, \ \frac{\partial F}{\partial x^6} = 0.$$

Therefore, the functions F which are constants of motion of $\mathcal{X}^{\omega_{M_f}}(M_f)$ are the ones which depend only of x^4, for instance $F_2(x^1, \ldots, x^6) = x^4$.

Obviously, all the constants of motion of $\mathcal{X}^{\omega_{M_f}}(M_f)$ are also constants of motion of $\mathcal{X}^\omega(M_f)$.

The vector field $X = \frac{\partial}{\partial x^1}$ on $\mathbf{R}^6$ satisfies the conditions from the definition of Cartan symmetry, with $G(x^1, \ldots, x^6) = x^4$, and we can deduce that X is a Cartan symmetry of $\left(M_f, \omega_{M_f}, \alpha_{M_f}\right)$ and G_{M_f} is a conservation law of $\mathcal{X}^{\omega_{M_f}}(M_f)$.

If we consider the dynamical symmetries of $\xi \in \mathcal{X}^{\omega_{M_f}}(M_f)$, $Y = x^1 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^4}$, $Z = x^1 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^4}$, then we will obtain $F = \omega_{M_f}(Y, Z) = -x^1 x^4$ a conservation laws for ξ, by using the proposition 4.10.

5 Study of some biological and ecological dynamical systems

In this section we will use the geometrical results from the previous sections to make a study of the behavior of some very important examples from biology and ecology: prey-predator ecological system ([15], [16], [17], [18]), Bailey model for the evolution of epidemics ([16], [19], [20]), classical Kermack-McKendrick model of evolution of epidemics ([16], [20]). This biodynamical systems are included in the presymplectic case because the 2-form ω_L associated to the corresponding Lagrangian is degenerate.

5.1 The prey-predator ecological system

Let us consider the system of ordinary differential equations ([16]):

$$\begin{cases} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy \end{cases}, \quad a,\, b,\, c,\, d > 0. \tag{5.1}$$

This system is a complex biological system model, in which two species x and y live in a limited area, so that individuals of the species y (predator) feed only individuals of species x (prey) and they feed only resources of the area in which they live. Proportionality factors a and c are respectively increasing and decreasing prey and predator populations. If we assume that the two populations come into interaction, then the factor b is decreasing prey population x caused by this predator population y and the factor d is population growth due to this population x.

The prey-predator system (5.1) is called *Lotka–Volterra equations* and, also known as *"the predator-prey equations"*. This system is a pair of first order, nonlinear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey. x is the number of prey (for example, *rabbits*), y is the number of some predator (for example, *foxes*), $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$ represent the growth rates of the two populations over time, t represents time.

The evolution system (5.1) can be written in the form of Euler-Lagrange equations:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \end{cases} \tag{5.2}$$

where the Lagrangian L is

$$L = \frac{1}{2}\left(\frac{\ln y}{x}\dot{x} - \frac{\ln x}{y}\dot{y}\right) + c\ln x - a\ln y - dx + by \tag{5.3}$$

and the corresponding Hamiltonian H is

$$H = \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{y}}\dot{y} - L = -c\ln x + a\ln y + dx - by\,. \tag{5.4}$$

Let us remark that the total energy $E_L = H$ is *a conservation law* for prey-predator system (5.1).

If we consider the Poincaré-Cartan forms associated to L, $\theta_L = \frac{\partial L}{\partial \dot{x}}dx + \frac{\partial L}{\partial \dot{y}}dy$ and $\omega_L = -d\theta_L$, then ω_L has a constant rank, equal with 2, and so, we will obtain a presymplectic system $(T\mathbf{R}^2, \omega_L, dE_L)$.

5.2 The Bailey model for the evolution of epidemics

In Bailey model for the evolution of epidemics are considered two classes of hosts: individuals suspected of being infected, whose number is denoted by x and individuals infected carriers, whose number we denote by y.

Assume that the latency and average removal rate is zero and then remain carriers infected individuals during the entire epidemic, with no death, healing and immunity. It is proposed that, in unit time, increasing the number of individuals suspected of being infected to be proportional to the product of the number of those infected them. These considerations lead us to the evolutionary dynamical system given by the system of ordinary differential equations ([16]):

$$\begin{cases} \dot{x} &= -kxy \\ \dot{y} &= kxy \end{cases}, \quad k > 0. \tag{5.5}$$

The model is suitable for diseases known animal and plant populations and also corresponds quite well the characteristics of small populations spread runny noses, dark, people such as students of a class team.

First of all, let us remark that we have *a conservation law*, $x+y=n$. That means that n, the total number of individuals of a population, does not change during the evolution of this epidemic.

The equations (5.5) can be write as Euler-Lagrange equation, where the Lagrangian L is

$$L = \frac{1}{2}\left(\frac{\ln y}{x}\dot{x} - \frac{\ln x}{y}\dot{y}\right) + k(x+y) \tag{5.6}$$

and the corresponding Hamiltonian H is

$$H = \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{y}}\dot{y} - L = -k(x+y)\,. \tag{5.7}$$

5.3 The classical Kermack-McKendrick model of evolution of epidemics

The classical model of evolution of epidemics was formulated by Kermack (1927) and McKendrick (1932) as follows. Let us denote the numerical size of the population with n and let us divide it into three classes: the number of individuals suspected of x, the number of individuals infected carriers y, and the number of isolate infected individuals z.

For simplicity, we take zero latency period, that all individuals are simultaneously infected carriers that infect those suspected of being infected. Considering the previous example we note the rate constant k_1 of disease transmission. Changing the size of infected carriers depends on the rate k_1 and also depend on k_2, the rate that carriers are isolated. In this way, we have the system ([16]):

$$\begin{cases} \dot{x} &= -k_1xy \\ \dot{y} &= k_1xy - k_2y \\ \dot{z} &= k_2y \end{cases}, \quad k_1, k_2 > 0. \tag{5.8}$$

Let us note that $x+y+z=n$, i.e. the number of individuals of the population does not change. This *conservation law* tells us not cause deaths epidemic.

The evolution of a dynamic epidemic begins with a large population which is composed of a majority of individuals suspected of being infected and in a small number of infected individuals. Initial number of isolated infected people is considered to be zero. So, we can consider the subsystem ([16]):

$$\begin{cases} \dot{x} &= -k_1 xy \\ \dot{y} &= k_1 xy - k_2 y \end{cases}, \quad k_1, k_2 > 0. \tag{5.9}$$

The Lagrange and Hamilton functions of the system (5.9) are

$$\begin{aligned} L &= \tfrac{1}{2}\left(\tfrac{\ln y}{x}\dot{x} - \tfrac{\ln x}{y}\dot{y}\right) + k_1(x+y) - k_2 \ln x\,, \\ H &= -k_1(x+y) + k_2 \ln x\,, \end{aligned}$$

and so, we have a new *conservation law* of (5.9),
$H = E_L = -k_1(x+y) + k_2 \ln x$.

If we get back to the Kermack-McKendrick model (5.8), then we have that the Lagrangian whose Euler-Lagrange equations are really (5.8) is

$$\overline{L} = L + \frac{1}{2}(\dot{z} - k_1 y)^2\,, \tag{5.10}$$

where L is the Lagrangian of the subsystem (5.9).
The corresponding Hamiltonian is given by

$$\overline{H} = \dot{x}\frac{\partial \overline{L}}{\partial \dot{x}} + \dot{y}\frac{\partial \overline{L}}{\partial \dot{y}} + \dot{z}\frac{\partial \overline{L}}{\partial \dot{z}} - \overline{L}\,. \tag{5.11}$$

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