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TWO SOLITON SOLUTION OF TZITZEICA EQUATION

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Abstract

Starting from a general study on how to get soliton solutions of PDE's using the dressing factor method, we will effectively obtain the 2-soliton solution for Tzitzeica equation.

1 Introduction

The soliton solutions of PDE's represent one of the most interesting type of solutions that these equations could have. Beyond of the importance of the solution itself, the concept of soliton became an important tool in mathematics and nonlinear physics, for the study of integrability of nonlinear systems.

First observed in 1834 by the ship designer John Scott Russell in the Union Channel in Scotland as a nondestructive traveling wave in a shallow water, the soliton designates a very stabile form of propagation of energy [1]. Experiments made by Russell proved that the speed of this solitary wave depends on its amplitude and that its shape remains unchanged for a long time during its evolution, no matter the interactions that might occur with other solitary waves.

At that time, there was no mathematical explanation of the phenomenon that Russell observed, and only in 1895, D.J. Korteweg and G. de Vries [2] discovered a nonlinear evolution equation (called subsequently KdV) which described the propagation of small amplitude waves in shallow and narrow channels. Thus they explained the phenomenon observed by Russel in 1834.

In a previous paper we presented in more details the above mentioned equation and its soliton solutions [3]. Now we will investigate another famous equation due to the Romanian mathematician Gheorghe Ţiţeica, known now as Tzitzeica equation [4, 5]. It was initially proposed as an equation describing special surfaces in differential geometry for which the ratio K/d^4 is constant, where K is the Gauss curvature of the surface and d is the distance from the origin to the tangent plane at the given point. Later on it turned out that the equation has wider importance, being nowadays used as an important evolutionary equation in nonlinear dynamics. In this respect, many important results on the equation have to be mentioned: it corresponds to a completely integrable Hamiltonian system [6] which accepts higher integrals of motion [7] and Lax pair can be attached for proving its integrability [8, 9] .

The explicit form of Tzitzeica equation is:

$$2\frac{\partial^2\phi}{\partial\xi\partial\eta} = e^{2\phi} - e^{-4\phi}. \tag{1}$$

The present paper proposes a study of the Tzitzeica equation using the Lax pair inspired by Mikhailov in [8, 9] and the dressing method of Zakharov-Shabat-Mikhailov [10, 9]. The paper is organized in the following sections. In section 2 we will present general facts on the Lax representation and the reduction group, as well as the concrete form of the dressing factor which can be used for Tzitzeica equation. In Section 3 the formalism will be used for building the 2-soliton solution of the considered model.

2 Lax representation of Tzitzeica equation

2.1 How to get Tzitzeica equation from Lax operators

Let us show now how the Tzitzeica equation can be obtained starting from a Lax representation. The Lax representation supposes to find two differential operators, X and T, depending on a spectral parameter λ, which satisfy the compatibility condition (identically zero with respect to λ):

$$[X, T] = 0. \tag{2}$$

We will choose, in a system of coordinates (x, t), the representation suggested in [8, 9]:

$$\begin{aligned} X\Psi(x,t,\lambda) &\equiv \left(\partial_x + V + \lambda C_1 - \frac{1}{\lambda}C_2\right)\Psi(x,t,\lambda) = 0, \\ T\Psi(x,t,\lambda) &\equiv \left(\partial_t + W + \lambda C_1 + \frac{1}{\lambda}C_2\right)\Psi(x,t,\lambda) = 0 \end{aligned} \tag{3}$$

The operators V, W, C_1, C_2 in the previous equations can be found by imposing suitable reduction conditions, and λ represents the spectral parameter of the operators. The usual reductions which are imposed are known as the $\mathbb{Z}_3$- reduction, the first $\mathbb{Z}_2$-reduction and the second $\mathbb{Z}_2$-reduction. Explicitly, the above mentioned reductions are formulated as follows:

1. $\mathbb{Z}_3$-reduction supposes:

$$Q^{-1}\Psi(x,t,\lambda)Q = \Psi(x,t,q\lambda), \qquad Q = \operatorname{diag}\left(1, q, q^2\right), \quad q = e^{2\pi i/3}; \tag{4}$$

   which restricts V, W, C_1 and C_2 by:

$$Q^{-1}VQ = V, \qquad Q^{-1}WQ = W, \qquad Q^{-1}C_kQ = q^kC_k, \tag{5}$$

   $k = 1, 2$. These conditions are satisfied identically if we chose:

$$\begin{aligned} &V = \operatorname{diag}(\phi_{1,t}, \phi_{2,t}, \phi_{3,t}), &&W = \operatorname{diag}(\phi_{1,x}, \phi_{2,x}, \phi_{3,x}), \\ &C_1 = \begin{pmatrix} 0 & 0 & c_3 \\ c_1 & 0 & 0 \\ 0 & c_2 & 0 \end{pmatrix}, &&C_2 = \begin{pmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \\ c_3 & 0 & 0 \end{pmatrix} \end{aligned} \tag{6}$$

   and

$$c_i = e^{\phi_{i+1}-\phi_i}, \quad \phi_1 + \phi_2 + \phi_3 = 0.$$

2. The first $\mathbb{Z}_2$-reduction imposes $\Psi^*(x,t,\lambda^*) = \Psi(x,t,\lambda)$, i.e. $V = V^*$, $W = W^*$, $C_k = C_k^*$. Thus the fields $\phi_k = \phi_k^*$ are real functions.

3. The second $\mathbb{Z}_2$-reduction is:

$$A_0^{-1}\Psi^\dagger(x,t,-\lambda^*)A_0 = \Psi^{-1}(x,t,\lambda), \qquad A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{7}$$

i.e.

$$A_0^{-1}V^\dagger A_0 = -V, \qquad A_0^{-1}C_k^\dagger A_0 = C_k, \qquad A_0^{-1}W^\dagger A_0 = -W. \tag{8}$$

These conditions lead to: $\phi_1 = -\phi_3 = \phi$, and $\phi_2 = 0$, and

$$\begin{aligned} V &= \frac{\partial \phi}{\partial t}\operatorname{diag}(1,0,-1), & W &= \frac{\partial \phi}{\partial x}\operatorname{diag}(1,0,-1), \\ C_1 &= -\begin{pmatrix} 0 & e^{\phi} & 0 \\ 0 & 0 & e^{\phi} \\ e^{-2\phi} & 0 & 0 \end{pmatrix} & C_2 &= \begin{pmatrix} 0 & 0 & e^{-2\phi} \\ e^{\phi} & 0 & 0 \\ 0 & e^{\phi} & 0 \end{pmatrix} \end{aligned} \tag{9}$$

Imposing the compatibility condition (2) we obtain:

$$W_x - V_t + [V,W] + 2[C_1,C_2] = 0, \tag{10}$$

$$C_{1,x} - C_{1,t} + [C_1,W] + [V,C_1] = 0 \tag{11}$$

$$C_{2,x} + C_{2,t} + [V,C_2] + [W,C_2] = 0. \tag{12}$$

Eqs. (11) and (12) become identities due to eq. (9), while eq. (10) gives the Tzitzeica equation, or one of the 2-dimensional Toda field theories [8, 9], see also [11]:

$$\phi_{xx} - \phi_{tt} + 2(e^{4\phi} - e^{-2\phi}) = 0. \tag{13}$$

Applying the change of variables $\xi = x+t$ and $\eta = x-t$, the previous equation takes the form (1) and it accepts the following Lax representation:

$$\tilde{X}\tilde{\Psi} \equiv \frac{\partial \tilde{\Psi}}{\partial \xi} + 2\phi_\xi H_0\tilde{\Psi} - \lambda\mathcal{J}\tilde{\Psi} = 0, \qquad \tilde{T}\tilde{\Psi} \equiv \frac{\partial \tilde{\Psi}}{\partial \eta} - \lambda^{-1}\tilde{V}_1(\xi,\eta)\tilde{\Psi} = 0, \tag{14}$$

where $H_0 = \operatorname{diag}(1,0,-1)$ and

$$V_1(\xi,\eta) = \begin{pmatrix} 0 & 0 & e^{-4\phi} \\ e^{2\phi} & 0 & 0 \\ 0 & e^{2\phi} & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.2 The dressing method

Let us consider (14) for the trivial solution $\phi_0 = 0$ of Tzitzeica equation:

$$\tilde{X}_0\tilde{\Psi}_0 \equiv \frac{\partial \tilde{\Psi}_0}{\partial \xi} - \lambda\mathcal{J}\tilde{\Psi}_0 = 0, \qquad \tilde{T}_0\tilde{\Psi}_0 \equiv \frac{\partial \tilde{\Psi}_0}{\partial \eta} - \lambda^{-1}\mathcal{J}^2\tilde{\Psi}_0 = 0, \tag{15}$$

Starting from (15) we intend to find a new nontrivial solution $\tilde{\Psi}$ of (1). The fundamental solution $\tilde{\Psi}_0$ of the system (15), known as the "naked" solution, can be related by $\tilde{\Psi}$ through the so called dressing factor $u(\xi,\eta,\lambda)$:

$$\tilde{\Psi}(\xi,\eta,\lambda) = u(\xi,\eta,\lambda)\tilde{\Psi}_0(\xi,\eta,\lambda). \tag{16}$$

From the equation above, taking into account that $\tilde{\Psi}$ and $\tilde{\Psi}_0$ are solutions of (14) and (15), we obtain that the dressing factor $u(\xi,\eta,\lambda)$ must satisfy:

$$\begin{aligned} \frac{\partial u}{\partial \xi} + 2\phi_\xi H_0 u(\xi,\eta,\lambda) - \lambda\,[\mathcal{J}, u(\xi,\eta,\lambda)] = 0, \\ \frac{\partial u}{\partial \eta} - \frac{1}{\lambda} V_1 u(\xi,\eta,\lambda) + \frac{1}{\lambda} u(\xi,\eta,\lambda)\mathcal{J}^2 = 0. \end{aligned} \tag{17}$$

Since both Lax pairs satisfy the three reductions, then also the dressing factor must satisfy them:

$$\begin{aligned} Q^{-1}u(\xi,\eta,\lambda)Q = u(\xi,\eta,q\lambda), \qquad u^*(\xi,\eta,\lambda^*) = u(\xi,\eta,\lambda), \\ A_0^{-1}u^\dagger(\xi,\eta,-\lambda^*)A_0 = u^{-1}(\xi,\eta,\lambda). \end{aligned} \tag{18}$$

3 Two soliton solution of Tzitzeica equation

Our anzatz for the dressing factor with simple poles in λ is:

$$u(\xi,\eta,\lambda) = \mathbb{1} + \frac{1}{3}\sum_{k=1}^{2}\left(\frac{A_k}{\lambda-\lambda_k} + \frac{Q^{-1}A_kQ}{\lambda q^2-\lambda_k} + \frac{Q^{-2}A_1Q^2}{\lambda q-\lambda_k}\right), \tag{19}$$

where $A_k(\xi,\eta)$ is a degenerate matrix of the form:

$$A_k(\xi,\eta) = |n_k(\xi,\eta)\rangle\langle m_k(\xi,\eta)|^T, \qquad (A_k)_{ij}(\xi,\eta) = n_{ki}(\xi,\eta)m_{kj}(\xi,\eta) \tag{20}$$

The first $\mathbb{Z}_2$-reduction leads to the idea that λ_k^3 may be chosen real. In addition all components of the vectors $|n_k\rangle$ and $\langle m_k|^T$ are real too.

Imposing the second $\mathbb{Z}_2$-reduction and $\lambda \to \lambda_k$ we obtain algebraic equation for $|n_k\rangle$ in terms of $\langle m_k^T|$.

Moreover, direct computation can provide $\langle m_k|^T$ in terms of ξ and η. By that, practically we express the final 2-soliton solution of Tzitzeica equation in terms of the two independent coordinates. It will have the following form:

$$\Phi = -\frac{1}{2}\ln\left|1 - \frac{1}{\lambda_1}n_{1,1}m_{1,1} - \frac{1}{\lambda_2}n_{2,1}m_{2,1}\right| \tag{21}$$

Skipping the details we write down the explicit form of the generic 1-soliton solution of first type, obtained from eq. (21) for $m_{2,1} = n_{2,1} = 0$:

$$\phi(\xi,\eta) = \frac{1}{2}\ln\frac{\mu_{02}^2 - 8|\mu_{01}|\mu_{02}F_1(\xi,\eta) + 2|\mu_{01}|^2(2F_2(\xi,\eta) - 3e^{6X_1})}{\mu_{02}^2 + 4|\mu_{01}|\mu_{02}F_1(\xi,\eta) + 4|\mu_{01}|^2F_2(\xi,\eta)} \tag{22}$$

where

$$\begin{aligned} F_1(\xi,\eta) = e^{3X_1}\cos(\Phi_1+\alpha_0), \qquad F_2(\xi,\eta) = e^{6X_1}\cos^2(\Phi_1+\alpha_0), \\ X_1(\xi,\eta) = \frac{1}{2}(\lambda_1\xi + \lambda_1^{-1}\eta), \qquad \Phi_1(\xi,\eta) = \frac{\sqrt{3}}{2}(\lambda_1\xi - \lambda_1^{-1}\eta), \end{aligned} \tag{23}$$

and μ_{0k}, α_0 are some real constants. For $\mu_{02} = 0$ eq. (22) coincides with the one obtained by Mikhailov in [8].

4 Conclusions

The aim of this paper was to make a pedagogical approach to what it means the Lax representation and how it can be used to determine soliton solutions of PDE's. Among the possible procedures we chose to use the dressing factor technique. We try to present as elementary as possible how a naked solution could be related to a new solution of the same equation through convenient anzatz for the dressing factor. To exemplify how the formalism is working, we have calculated the 2-soliton solution of Tzitzeica equation for the case of two real poles. Despite the fact that the solution looks simple in terms of the two vectors, $|n_k\rangle$ in terms and $\langle m_k|^T$ (see formula (21)), when we try to express it in terms of independent coordinates ξ and η, we get quite complicated expressions, which should be simplified by imposing specific constrains and initial conditions.The results we presented here can be simply generalized to the case of N-soliton solution, both for real and complex poles. Such solutions will be tackled on in forthcoming papers.

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