

# The Rényi entropy of Lévy distribution

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## Abstract

The equivalence between non-extensive C. Tsallis entropy and the extensive entropy introduced by Alfréd Rényi is discussed. The Rényi entropy is studied from the perspective of the geometry of the Lebesgue and generalised, exotic Lebesgue spaces. A duality principle is established. The Rényi entropy for the Lévy distribution, in the domain when the numerical methods fails, is approximated by asymptotic expansion for the large values of the Rényi parameter.

## 1 Introduction

The independent discovery of the equivalent generalizations of the Shannon-Boltzmann entropy by A. Rényi [1], [2], by pure axiomatic reasoning respectively by C. Tsallis [3], [4], from physical considerations has a great influence both on statistical physics as well as on mathematical methods in statistics, engineering and informatics. In the case of systems with small degree of freedom the entropy of C. Tsallis and A. Rényi are completely equivalent, in the sense that the Tsallis entropy can be computed easily when the Rényi entropy is known and conversely. The axioms that uniquely determines the Rényi entropy can be exactly reformulated for the Tsallis entropy, despite the Rényi formulation is closely related to the formalism of topological metric vector spaces. The MaxEnt principle formulated for the Rényi entropy can be reformulated as a problem of geometry on convex sets in Banach or complete metric linear spaces. The result of computation of probability

density functions (PDF) by MaxEnt principle with or without restrictions give the same PDF both for Rényi and Tsallis. Nevertheless the entropy introduced by A. Rényi, has many elegant physical and mathematical properties. First of all, the Rényi entropy is additive, so the study of the thermodynamic limit of non interacting or weakly interacting subsystems can be studied by suitable perturbation methods. We can introduce also the intensive quantity, the Rényi entropy per particle, or per volume. Despite this property (the possibility to express the entropy of composed system by entropy of subsystems) can be transferred to the Tsallis entropy, the formula is quite complicated. When the number of non-interacting subsystems is of the order of magnitude of the Avogadro number, the numerical value of the Tsallis entropy contains a very large exponent. In the case of the continuous distributions, the Rényi entropy has several new advantages over the Tsallis entropy. First, at the rescaling of the variables, by the change of units measurements, the Rényi entropy changes by an additive constant, exactly in the same way as the classical Shannon Boltzmann Gibbs entropy. Moreover, when using Rényi entropy, we can use instead of the probability density function the particle density function, or in the case of identical particles, the mass density function. In all of these cases the variation of Rényi entropy remains unchanged, like in the case of the Shannon-Boltzmann-Gibbs entropy.

The adaptation of the Tsallis entropy to these changes (scaling of variables, replacing PDF by particle density function, study of systems composed of subsystems whose number is of the order of magnitude of Avogadro number) is still possible. In fact all of the reasonings performed in the framework of the Rényi entropy can be translated easily in the language of the Tsallis entropy. In particular in the case of composing a large system from independent subsystems, the addition of the Rényi entropy is translated in a new, apparently non linear operation, that allows to compute the Tsallis entropy of the composed system. So it remains at the taste of the researchers the choice.

In conclusion, the physical arguments for the use of Tsallis entropy ([3]) are valid also for the use of the Rényi entropy, as well as the Markov chain argument ([1]) for the use of the Rényi entropy applies to Tsallis entropy.

Finally, we can relate the Rényi entropy to well studied concepts in functional analysis: Banach space norms or pseudo-norms in complete metric vector spaces. The interpretation of the Rényi entropy in the language of convex analysis and geometry in  $L_p$  spaces, allows to use mathematical results, that could give a simple proof to useful properties of the Tsallis entropy.

The structure of this work is the following. In Section 2, for the sake of self consistency, we expose the definition and main properties of the Rényi entropy. In Section 3 we discuss the properties of the Rényi entropy from the viewpoint of geometry of Banach spaces respectively from the perspective of the geometry of exotic, non local convex Lebesgue spaces with exponent  $0 < p < 1$ . Here, in the subsection 3.1 we establish a new result, the duality property of the Rényi entropy, that we consider that will be useful in numerical methods for performing optimized approximation of the solutions of the Fokker-Planck equations. In the Section 4 we compute the Rényi entropy for the Levy distribution in a domain when the numerical methods fails.

## 2 The equivalence of entropy of C. Tsallis and A. Rényi

### 2.1 Discrete distribution

In the simplest possible case of discrete probability field with  $N$  states and with associated probabilities  $\{p_1, \dots, p_N\}$ , the Tsallis respectively Rényi entropies, with parameter  $q > 0$  and  $q \neq 1$ , are defined as follows

$$S_{T,q}(p_i) = \frac{1}{1-q} \left( 1 - \sum_{i=1}^N p_i^q \right) \quad (1)$$

$$S_{R,q}(p_i) = \frac{1}{1-q} \log \left( \sum_{i=1}^N p_i^q \right) \quad (2)$$

It is easy to see that in the limit case  $q \rightarrow 1$  both  $S_{T,q}(p_i)$  and  $S_{R,q}(p_i)$  approach the Shannon entropy.

$$S_{Shannon}(p_i) = - \sum_{i=1}^N p_i \log p_i$$

Their equivalence, when discussing the problem of finding the probability distributions that obey maximal entropy principle with some restrictions, results in a straightforward manner from Eqs(1, 2)

$$S_{T,q}(p_i) = \frac{1}{q-1} [\exp((q-1) S_{R,q}(p_i)) - 1] \quad (3)$$

**Remark 1** *Observe that Eq.(3) will be valid also in the case of continuous, multi variate distributions*

In the subsequent part, we will discuss the Rényi entropies also for the case  $0 < q < 1$ . It is clear from Eq.(3) that in the case of the small system both entropies can be computed easily and give the same amount in information. In particular the dependence of  $S_{T,q}(p_i)$  over  $S_{R,q}(p_i)$  is monotonous, for all  $0 < q$ , so the maximal entropy principles, eventually with restrictions, in both case give rise to the same probabilities. Moreover, in the Rényi formulation, we can consider the limit  $q \rightarrow \infty$  too. From Eq.(2) in the limit  $q \rightarrow \infty$  we still obtain a meaningful result:

$$S_{R,q}(p_i) = - \frac{q}{q-1} \log \left( \sum_{i=1}^N p_i^q \right)^{1/q} \xrightarrow{q \rightarrow \infty} - \log \left[ \max_i (p_1, \dots, p_n) \right] > 0$$

Observe that in this limit the maximal entropy principle give rise to the min-max problem with restrictions In the case of the system composed of large number of identical and approximately independent subsystems, the use of  $S_{R,q}$  is more suitable, because it is extensive and we can define the intensive "specific Rényi entropy". Indeed consider a complex system with  $M \times N$  states specified by the probabilities  $\{p_{i,j} | 1 \leq i \leq N; 1 \leq j \leq M\}$ . In this case the Rényi entropy is

$$S_{R,q}(p_{i,j}) = \frac{1}{1-q} \log \left( \sum_{i=1}^N \sum_{j=1}^M p_{i,j}^q \right) \quad (4)$$

In the case when the large system is composed of independent subsystems with  $N$  respectively  $M$  states with probabilities  $\{p'_1, \dots, p'_N\}$  respectively  $\{p''_1, \dots, p''_M\}$  we have

$$p_{i,j} = p'_i p''_j$$

and from Eq.(4) results

$$S_{R,q}(p_{i,j}) = S_{R,q}(p'_i) + S_{R,q}(p''_j) \quad (5)$$

In the continuation, due to Eqs.(3 , 5 ) we discuss the Rényi entropy. The Eq.(5) can be easily generalized, so in the case of a large system composed of a large number of  $\mathcal{N}$  identical and independent (non-interacting) subsystems , each of them with  $N$  states and with associated probabilities  $\{p_1, \dots, p_N\}$  , the entropy  $S_{\mathcal{N},q}$  of the resulting large system is simply

$$S_{\mathcal{N},q} = \mathcal{N} S_{R,q}(p_i) \quad (6)$$

which is advantageous when  $\mathcal{N}$  is of the order of magnitude of Avogadro number. For physical systems with short range or weak interactions, it is meaningful to attempt to compute  $S_{\mathcal{N},q}/\mathcal{N}$  in the thermodynamic limit.

Like Shannon entropy, the Rényi entropy is determined uniquely by a system of physically meaningful axioms ([1]).

### 3 General distributions, Rényi entropy and Lebesgue space norm

Suppose we have a measure space  $(\Omega, m)$  with positive not necessary finite, measure  $m$  over the space  $\Omega$  . For sake of simplicity the  $\sigma$  algebra will be omitted. Suppose we have the probability density function  $\rho(\mathbf{x})$  on  $\Omega$  , where  $\mathbf{x} \in \Omega$ , such that

$$\int_{\Omega} \rho(\mathbf{x}) dm(\mathbf{x}) = 1 \quad (7)$$

**Remark 2** Without loss of generality we can consider that  $\rho(\mathbf{x}) > 0$  excepting a null set  $A$  that is statistically negligible, i.e.  $m(A) = 0$

In this case the Tsallis and Rényi entropies, for  $q > 0$  and  $q \neq 1$  are defined as follows, if the integrals exists

$$S_{T,q}(\rho) = \frac{1}{1-q} \left( 1 - \int_{\Omega} [\rho(\mathbf{x})]^q dm(\mathbf{x}) \right) \quad (8)$$

$$S_{R,q}(\rho) = \frac{1}{1-q} \log \left[ \int_{\Omega} [\rho(\mathbf{x})]^q dm(\mathbf{x}) \right] \quad (9)$$

Observe that the relation Eq.( 3 ) between the entropies remains valid . From the Eq.(9) results:

$$S_{R,q}(\rho) = \frac{q}{1-q} \log \left[ \|\rho\|_{L_q(\Omega, dm)} \right] ; q > 1 \quad (10)$$

$$S_{R,q}(\rho) = \frac{1}{1-q} \log N_{L_q(\Omega, dm)}(\rho); 0 < q < 1 \quad (11)$$

where the notations from ([5], [6]) are used.

The unusual  $L_q(\Omega, dm)$  spaces with  $0 < q < 1$  are important for the study of the distributions with heavy tail ([7], [8]).

It is convenient to define the probability measure

$$dP(x) = \rho(\mathbf{x}) dm(\mathbf{x}) \quad (12)$$

that has the property

$$\int_{\Omega} dP(\mathbf{x}) = 1 \quad (13)$$

The Rényi entropy functional, according to Eq.(9), associated to this distribution is (see Remark(2))

$$S_q\{\rho\} = -\log \left[ \int_{\Omega} [\rho(x)]^{q-1} dP(x) \right]^{\frac{1}{q-1}} \quad (14)$$

The following proposition results from Eq.(14) and are well known.

**Proposition 3 Remark 4** *In the case where  $\int_{\Omega} dP(x) = 1$ , i.e.  $P$  is a probability measure, then for  $0 < a_1 < a_2$  from the Hölder inequality ([5], [6]) results*

$$\left[ \int_{\Omega} |f(\mathbf{x})|^{a_1} dP(\mathbf{x}) \right]^{1/a_1} \leq \left[ \int_{\Omega} |f(\mathbf{x})|^{a_2} dP(\mathbf{x}) \right]^{1/a_2} \quad (15)$$

**Proposition 5** *The Rényi and Tsallis entropies for  $0 < q$  decreases with respect to  $q$ .*

**Proof.** In the case  $1 < q_1 < q_2$  we use Eq.(15) with  $a_1 = q_1 - 1$ ,  $a_2 = q_2 - 1$  and  $f(\mathbf{x}) = \rho(\mathbf{x})$  combined with Eq.(14). In the case  $0 < q_1 < q_2 < 1$  we have (See Remark (2))

$$S_q\{\rho\} = \log \left[ \int_{\Omega} \left[ \frac{1}{\rho(x)} \right]^{1-q} dP(x) \right]^{\frac{1}{1-q}} \quad (16)$$

By using Eq.(15) with  $a_1 = 1 - q_2$ ,  $a_2 = 1 - q_1$  and  $f(\mathbf{x}) = 1/\rho(\mathbf{x})$  combined with Eq.(16) completes the proof. The corresponding property for the Tsallis entropy results from Eq.(3) and Remark(1). ■

Now it is easy to prove the following

**Proposition 6** *In the limit  $q \searrow 1$  the Rényi entropy approaches the Shannon entropy.*

**Proof.** From Proposition (4) results that the limit  $\lim_{q \searrow 1} \|\rho\|_{L_{q-1}(\Omega, dP)}$  exists. In the the case when  $\rho(\mathbf{x}) > 0$  the function  $f(q) = \|\rho\|_{L_{q-1}(\Omega, dP)}$  is analytic near  $q = 1$ . By series expansion in  $q - 1$  in Eq.( 9) results that

$$\lim_{q \searrow 1} S_q\{\rho\} = - \int_{\Omega} \rho(\mathbf{x}) \log[\rho(\mathbf{x})] dm(\mathbf{x}) = S_{Shannon}\{\rho\}$$

The the extension to the case when on some domains  $\rho(\mathbf{x}) = 0$  is obtained by approximating the PDF  $\rho(\mathbf{x})$  with non zero PDF. By extending the function  $f(x) = x \log x$  by continuity such that  $f(0) = 0$  completes the proof. ■

### 3.1 Duality properties

Let  $p \geq 1$  and  $q \geq 1$  such that  $1/p + 1/q = 1$ . We define the scalar product of the functions  $f(x) \in L_p(\Omega, dm)$  and  $g(x) \in L_q(\Omega, dm)$  as follows

$$\langle f, g \rangle = \int_{\Omega} f(\mathbf{x})g(\mathbf{x})dm(\mathbf{x}) \quad (17)$$

From duality between  $L_p(\Omega, dm)$  and  $L_q(\Omega, dm)$  we have([5], [6])

$$\|f\|_{L_p(\Omega, dm)} = \sup_{g \in B_q} |\langle f, g \rangle| \quad (18)$$

where  $B_q$  is the unit sphere in the space  $L_q(\Omega, dm)$  :

$$B_q = \left\{ g | g \in L_q(\Omega, dm); \|g\|_{L_q(\Omega, dm)} = 1 \right\} \quad (19)$$

From Eqs. (18, 19) and from the relation between norm and entropy Eq.(10), we obtain the following duality between entropies

$$S_p\{\rho\} = \frac{p}{1-p} \sup_{\rho' \in B_q} \langle \rho, \rho' \rangle \quad (20)$$

$$\frac{1}{p} + \frac{1}{q} = 1 \quad (21)$$

This relation can be reformulated in the term of entropies: We have  $\{\rho' | \rho' \in B_q\} = \{\rho' | S_q(\rho') = 0\}$  so we obtain from Eq.(20)

$$S_p\{\rho\} = \frac{p}{1-p S_q(\rho')=0} \sup \langle \rho, \rho' \rangle; \frac{1}{p} + \frac{1}{q} = 1 \quad (22)$$

## 4 The Rényi entropy of the symmetric Lévy distributions

The Lévy distribution plays an important role in the study of processes with heavy tail PDF, that are encountered in the statistical physics of anomalous transport processes. In this section we compute the Rényi entropy of the PDF of Levy distribution in the range of large values of the Rényi  $q$  parameter, the domain when the numerical method does not works.

The generating function  $g_{\alpha}(t)$  of the symmetric  $\alpha$  stable distribution (Lévy), with convenient normalization, having PDF  $\rho_{\alpha}(x)$ , is given by

$$g_{\alpha}(t) \equiv \int_{-\infty}^{\infty} \rho_{\alpha}(x) \exp(itx) dx = \exp(-k|t|^{\alpha}) \quad (23)$$

Because the stability of the distribution specified by the generating function  $g_{\alpha}(t)$  is related to the variable  $x$ , in the sense that the sum of independent variables distributed according to Lévy distribution is again from the same family, it is natural to use the Lebesgue measure for the definition of the Rényi entropy

$$S_{R,q}(\rho_{\alpha}) = \int_{-\infty}^{\infty} [\rho_{\alpha}(x)]^q dx \quad (24)$$

The inversion of the Fourier transform in Eq.(23) cannot be obtained via elementary functions, nevertheless we can obtain easily the first terms in the Taylor expansion near  $x = 0$  as follows. From Eq.(23) results (with the normalization  $k = 1$ )

$$\rho_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx - |t|^\alpha) dx \quad (25)$$

From Eq.(25) results that for  $\alpha > 1$  the PDF  $\rho_\alpha(x)$  is in fact an entire analytic function in the variable  $x$ , for  $\alpha = 1$  is meromorphic with poles at  $x = \pm i$  and for  $0 < \alpha < 1$  is from the  $C^\infty$  class. Consequently from Eq.(25), after taking into account the symmetry, we obtain

$$\rho_\alpha(0) = \frac{1}{\pi} \int_0^{\infty} \exp(-|t|^\alpha) dx = \frac{1}{\pi\alpha} \Gamma\left(\frac{1}{\alpha}\right) \quad (26)$$

$$\left[ \frac{d\rho_\alpha(x)}{dx} \right]_{x=0} = \left[ \frac{d^3\rho_\alpha(x)}{dx^3} \right]_{x=0} = 0 \quad (27)$$

$$\left[ \frac{d^2\rho_\alpha(x)}{dx^2} \right]_{x=0} = -\frac{1}{\pi\alpha} \Gamma\left(\frac{3}{\alpha}\right) \quad (28)$$

$$\left[ \frac{d^4\rho_\alpha(x)}{dx^4} \right]_{x=0} = -\frac{1}{\pi\alpha} \Gamma\left(\frac{5}{\alpha}\right) \quad (29)$$

The integral from Eq.(24) can be rewritten as follows

$$S_{R,q}(\rho_\alpha) = \int_{-\infty}^{\infty} \exp[f_\alpha(x)] dx \quad (30)$$

$$f_\alpha(x) = \log \rho_\alpha(x) \quad (31)$$

The asymptotic expansion of  $S_{R,q}(\rho_\alpha)$  for the large values of the Rényi parameter  $q$  can be obtained according to Laplace method, taking into account that according to Eqs.(31, 27)

$$\left[ \frac{df_\alpha(x)}{dx} \right]_{x=0} = \left[ \frac{d^3f_\alpha(x)}{dx^3} \right]_{x=0} = 0$$

Consequently we obtain

$$S_{R,q}(\rho_\alpha) \underset{q \rightarrow +\infty}{\asymp} \frac{\exp[qf(0)] \sqrt{2\pi}}{\sqrt{q|f^{(2)}(0)|}} \left[ 1 + \frac{f^{(4)}(0)}{72q|f^{(2)}(0)|} \right] \quad (32)$$

By using Eqs.(31, 26, 28, 29) we obtain

$$f^{(2)}(0) = -\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} \quad (33)$$

$$f^{(4)}(0) = \frac{\Gamma\left(\frac{5}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} - 3 \left[ \frac{\Gamma\left(\frac{3}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)} \right]^2 \quad (34)$$

$$f(0) = \log \left[ \frac{1}{\pi\alpha} \Gamma\left(\frac{1}{\alpha}\right) \right] \quad (35)$$

From Eqs.(32-35) we obtain

$$S_{R,q}(\rho_\alpha) \underset{q \rightarrow +\infty}{\asymp} \left[ \frac{1}{\pi\alpha} \Gamma\left(\frac{1}{\alpha}\right) \right]^q \left[ \frac{2\pi\Gamma(1/\alpha)}{q\Gamma(3/\alpha)} \right]^{1/2} \times \left\{ 1 + \frac{1}{72q} \left[ \frac{\Gamma(5/\alpha)}{\Gamma(3/\alpha)} - 3 \frac{\Gamma(3/\alpha)}{\Gamma(1/\alpha)} \right] \right\}$$

## 5 Conclusions

In the perspective of the geometric interpretation of the Rényi entropy we obtained a duality relation. In general the duality principles are largely used in the theory and applications of the convex optimization methods, so it can be used in the interpretation and applications of the MaxEnt principles for the Rényi entropy, for the optimal approximation of the solution of the Fokker-Planck equation.

In the domain of the large values of the Rényi parameter, when the numerical methods does not work, we obtained an asymptotic expansion of the Rényi entropy for the symmetric stable Lévy distribution.

## References

- [1] Rényi, Alfréd (1961). "On measures of information and entropy". Proceedings of the fourth Berkeley Symposium on Mathematics, Statistics and Probability 1960. pp. 547–561.
- [2] [http://en.wikipedia.org/wiki/R%C3%A9nyi\\_entropy](http://en.wikipedia.org/wiki/R%C3%A9nyi_entropy).
- [3] Tsallis, C. (1988). "Possible generalization of Boltzmann-Gibbs statistics". Journal of Statistical Physics 52: 479–487.
- [4] [http://en.wikipedia.org/wiki/Tsallis\\_entropy](http://en.wikipedia.org/wiki/Tsallis_entropy)
- [5] Reed M., Simon B. Functional Analysis, Vol. 1 (Methods of Modern Mathematical Physics), (Vol. 1), Academic Press **1981**
- [6] Rudin, W. *Real and Complex Analysis*; McGraw Hill Inc. 3rd Ed. 1987.
- [7] Luschgy H.; Pagès G. Moment estimates for Lévy processes. Electronic Communication in Probability **2008**, *13*, 422-434.
- [8] Steinbrecher G., Garbet X., Weyssow B. "Large time behavior in random multiplicative processes", **2010**, arXiv:1007.0952v1.
- [9] Besov O. V., Il'in V.P., Nikol'skii S. M. *"Integral representations of functions and embedding theorems"*, Ed."Nauka", Moscow, 1975. (in Russian).
- [10] G. Sonnino, G. Steinbrecher, A. Cardinali, A. Sonnino, M. Tlidi, Phys. Rev. **E** 87, 014104-1, 014104-5 (2013).
- [11] Boyd, Stephen; Vandenberghe, Lieven. Convex Optimization. Cambridge: University Press. P. 244 (2004).