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Kinetic theory of Thomson scattering of an electromagnetic field by non-equilibrium plasma

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Abstract

The kinetic theory of incoherent (Thomson) scattering of an electromagnetic field by a non-equilibrium plasma is derived. We show, that in the non-equilibrium collisional regime the Callen-Welton formula should be revised. Applying the Langevin approach and the momentum method we show that not only the imaginary part, but also the derivatives of the real part of the dielectric susceptibility determine the amplitude and the width of the spectral lines of electrostatic field fluctuations, as well as the form factor. As a result of inhomogeneity, these properties become asymmetric with respect to inversion of the sign of the frequency. In the kinetic regime the form factor is more sensitive to space gradients than to the spectral function of the electrostatic field fluctuations. This asymmetry of lines can be used as a new diagnostic tool to measure local gradients in the plasma.

When transverse electromagnetic waves propagate in a plasma, wave scattering, due to interaction with fluctuational oscillations, occurs, that can be accompanied by a change of the frequency and wave vector. The intensity of scattered waves depends on both the intensity of the incident wave and the level of plasma fluctuations. Since the spectrum of fluctuations exhibits sharp maxima at proper plasma frequencies, the spectrum of scattered waves will also exhibit sharp maxima at frequencies differing from the frequency of the given wave by the proper plasma frequencies. The interaction of waves, propagating in the plasma, with flucutation oscillations may also lead to transformation of the waves, for instance, to transformation of a transverse wave into a longitudinal wave, and vice versa. The probability of these processes, like the probability of scattering processes, depends on the spectrum of electron density fluctuations. The shift, width and shape of spectral lines carry information on such parameters of the plasma as its density, temperature, mean velocity, ion composition etc. A method of remote probing of a medium, termed Thomson, or incoherent, scattering was developed in the sixties of the past century [1], and it is still successfully applied for remote diagnosis both of laboratory plasma, for example, in tokomaks, and of ionospheric plasma.

The field of an incident wave interacting with the fluctuation field gives rise to scattered waves, therefore, when a wave propagates in a plasma, the total electric field can be represented as follows:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^0(\mathbf{r},t) + \boldsymbol{\delta}\mathbf{E}(\mathbf{r},t) + \mathbf{E}'(\mathbf{r},t) \qquad (1)$$

where $\mathbf{E}^0(\mathbf{r},t)-$ is the field of the incident wave, $\boldsymbol{\delta}\mathbf{E}(\mathbf{r},t)-$ is the fluctuation field and $\mathbf{E}'(\mathbf{r},t)-$ is the field of the scattered wave.

The field of the scattered wave is determined by Maxwell's equation:

$$\nabla\times(\nabla\times\mathbf{E}')+\frac{1}{c^2}\frac{\partial^2\mathbf{E}'}{\partial t^2}=-\frac{4\pi}{c^2}\frac{\partial\mathbf{J}'}{\partial t} \tag{2}$$

where $\mathbf{J}'-$ is the density of the current, giving rise to the scattered field $\mathbf{E}'$, and is related to the deviation of the distribution function $f_a^{'}$ $(a=e,i)$, which is caused by the scattered wave, by the following relationship:

$$\mathbf{J}'=\sum_a e_a\int\mathbf{v}f_a^{'}d\mathbf{v} \tag{3}$$

The deviation of the distribution function is determined by the following linearized kinetic equation:

$$\frac{\partial}{\partial t}f_a^{'}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}f_a^{'}+\frac{e_a}{m_a}(\mathbf{E}'+\frac{1}{c}[\mathbf{v}\times\mathbf{B}^{'}])\cdot\frac{\partial}{\partial\mathbf{v}}f_a^{'}\mathbf{E}+$$

$$+\frac{e_a}{m_a}(\mathbf{E}^0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}^0])\cdot\frac{\partial}{\partial\mathbf{v}}\delta f_a+\frac{e_a}{m_a}(\delta\mathbf{E}+\frac{1}{c}[\mathbf{v}\times\boldsymbol{\delta}\mathbf{B}])\cdot\frac{\partial}{\partial\mathbf{v}}f_a^0=0 \tag{4}$$

where f_a^0- is the deviation of the distribution function due to the incident wave, δf_a- is the fluctuation of the distribution function, $\mathbf{B}^0$ and $\mathbf{B}^{'}-$ represent the magntic fields of the incident and scattered waves, respectively, $\boldsymbol{\delta}\mathbf{B}-$ is the fluctuation magnetic field.

The functions f_a^0 and δf_a- are determined by the following equations:

$$\frac{\partial}{\partial t}f_a^0+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}f_a^0+\frac{e_a}{m_a}(\mathbf{E}^0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}^0])\cdot\frac{\partial}{\partial\mathbf{v}}f_a^0=0 \tag{5}$$

$$[\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}+\frac{e_a}{m_a}(\mathbf{E}^0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}^0])\cdot\frac{\partial}{\partial\mathbf{v}}]\delta f_a+\frac{e_a}{m_a}(\delta\mathbf{E}+\frac{1}{c}[\mathbf{v}\times\delta\mathbf{B}])\cdot\frac{\partial}{\partial\mathbf{v}}f_a^0=\delta I_a\delta f_a+y_a \tag{6}$$

where $\delta I_a\delta f_a-$ is the linearized collision integral, and y_a - represents a random Langevin force.

Under certain assumptions the set of equations (3-5) can be resolved, and the expression for the current density $\mathbf{J}'$ can be represented as follows:

$$\mathbf{J}'_{\mathbf{k}'\omega'}=-i\omega\frac{\varepsilon_l(\mathbf{k}'\omega')-1}{4\pi}\mathbf{E}'_{\mathbf{k}'\omega'}-$$

$$-i\sum_a\frac{e_a^2}{m_a}\int d\mathbf{v}\frac{\mathbf{v}}{\omega-\mathbf{k}'\mathbf{v}}\{(\mathbf{E}^0+\frac{1}{c}[\mathbf{v}\times\mathbf{B}^0])\cdot\frac{\partial}{\partial\mathbf{v}}](\delta f_a)_{\mathbf{q}\boldsymbol{\Delta\omega}}+(\delta\mathbf{E}_{\mathbf{q}\boldsymbol{\Delta\omega}}+\frac{1}{c}[\mathbf{v}\times\delta\mathbf{B}_{\mathbf{q}\boldsymbol{\Delta\omega}}])\cdot\frac{\partial}{\partial\mathbf{v}}f_a^0 \tag{7}$$

where $\mathbf{q}=\mathbf{k}-\mathbf{k}'$, $\boldsymbol{\Delta\omega}=\boldsymbol{\omega}'-\boldsymbol{\omega}$.

Applying Maxwell's equations we obtain the differential scattering cross section of electromagnetic waves $d\Xi$, within an elementary solid angle $d\theta'$ and for a frequency interval $d\omega'$:

$$d\Xi=\frac{1}{4\pi}(\frac{e^2}{m_ec^2})^2\frac{\omega'^2}{\omega^2}\sqrt{\frac{\varepsilon(\omega')}{\varepsilon(\omega)}}(1+\cos^2\theta)(\delta n_e\delta n_e)_{\mathbf{q}\boldsymbol{\Delta\omega}}d\theta'd\omega'. \tag{8}$$

Thus, the problem reduces to finding the spectral characteristics of electron density fluctuations (electron form factor). Due to the Poisson equation, the electron form factor in the spatially homogeneous system is directly linked to the electrostatic field fluctuations. In thermodynamic equilibrium, the electrostatic field fluctuations satisfy the famous Callen-Welton fluctuation-dissipation theorem [2], linking their intensity to the *imaginary* part of the dielectric function and to the temperature.

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \frac{8\pi\ T\,\mathrm{Im}\,\varepsilon(\omega,\mathbf{k})}{\omega\,|\varepsilon(\omega,\mathbf{k})|^2} \tag{9}$$

For local equilibrium case one used

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \sum_a \frac{8\pi\ T_a\,\mathrm{Im}\,\chi_a(\omega,\mathbf{k})}{(\omega-\mathbf{k}\mathbf{V}_a)\,|\varepsilon(\omega,\mathbf{k})|^2} \tag{10}$$

where $\chi_a(\omega,\mathbf{k})-$ is the dielectric susceptibility of the a component.

The matter becomes more delicate in the local-equilibrium case. We have indeed shown, that in the *collisional regime* the Callen-Welton formula should be revised [3]. There then appear new terms explicitly displaying dissipative nonequilibrium contributions and containing the interparticle collision frequency, the differences in the temperatures and the velocities, and also functions of the *real* parts of the dielectric susceptibilities:

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \frac{8\pi\ T_e\,\mathrm{Im}\,\chi_e(\omega,\mathbf{k})}{(\omega-\mathbf{k}\mathbf{V}_e)\,|\varepsilon(\omega,\mathbf{k})|^2} + \frac{8\pi\ T_i\,\mathrm{Im}\,\chi_i(\omega,\mathbf{k})}{(\omega-\mathbf{k}\mathbf{V}_i)\,|\varepsilon(\omega,\mathbf{k})|^2} +$$
$$+\nu_{ei}(T_e-T_i)\Phi_1(\mathrm{Re}\,\varepsilon(\omega,\mathbf{k})) + \nu_{ei}(\mathbf{k}\mathbf{V}_e-\mathbf{k}\mathbf{V}_i)\Phi_2(\mathrm{Re}\,\varepsilon(\omega,\mathbf{k})) \tag{11}$$

The non-equilibrium correction in (11) can be amounted to 10% for the intensity of the ion-acoustic line.

However, it is not evident that the plasma parameters can be kept *constant* in both space and time. Inhomogeneities in space and time of these quantities will certainly also contribute to the fluctuations. In this paper, using the Langevin approach and the time-space multiscale technique, we show that not only the *imaginary* part but also the derivatives of the *real* part of the dielectric susceptibility determine the amplitude and the width of the spectral lines of the electrostatic field fluctuations, as well as the form factor. As a result of the inhomogeneity, these properties become asymmetric with respect to the inversion of the sign of the frequency. In the kinetic regime the form factor is more sensitive to space gradients than the spectral function of the electrostatic field fluctuations. Note that for simple fluids and gases a general theory of hydrodynamic fluctuations for nonequilibrium stationary inhomogeneous states has been developed in a series of publications [4]. In particular, it has been found that there exist an asymmetry of the spectrum for Brillouin scattering from a fluid in a shear flow or in a temperature gradient. The situation for the plasma problem we are considering is, however, quite different.

To treat the problem, a kinetic approach is required, especially when the wavelength of the fluctuations is larger than the Debye wavelength. To derive nonlocal expressions for the spectral function of the electrostatic field fluctuation and for the electron form factor we use the Langevin approach to describe kinetic fluctuations [5]. The starting point of our procedure is the same as in [6]. A kinetic equation for the fluctuation δf_a of the one-particle distribution function (DF) with respect to the reference state f_a is considered. In the general case the reference state is a none-equilibrium DF which varies in space and time both on the kinetic scale (mean free path l_{ei} and interparticle collision time ${\nu_{ei}}^{-1}$) and on the larger hydrodynamic scales. These scales are much larger than the characteristic fluctuation time ω^{-1}. In the non-equilibrium case we can therefore introduce a small parameter $\mu = \nu_{ei}/\omega$, which allows us to describe fluctuations on the basis of a multiple space and time scale analysis. Obviously, the fluctuations vary on both the "fast" $(\mathbf{r}, t)$ and the "slow" $(\mu\mathbf{r}, \mu t)$ time and space scales: $\delta f_a(\mathbf{x},t) = \delta f_a(\mathbf{x}, t, \mu t, \mu\mathbf{r})$ and $f_a(\mathbf{x},t) = f_a(\mathbf{p}, \mu t, \mu\mathbf{r})$. Here $\mathbf{x}$ stands for the phase-space coordinates $(\mathbf{r}, \mathbf{p})$. The Langevin kinetic equation for δf_a has the form [6]

$$\widehat{L}_{a\mathbf{x}t}(\delta f_a(\mathbf{x},t) - \delta f_a^S(\mathbf{x},t)) = -e_a \delta\mathbf{E}(\mathbf{r}, t)\cdot\frac{\partial f_a(\mathbf{x},t)}{\partial\mathbf{p}} \tag{12}$$

where e_a is the charge of the particle of specie a, $\delta\mathbf{E}$ - electrostatic field fluctuation, and the operator $\widehat{L}_{a\mathbf{x}t}$ is defined by

$$\widehat{L}_{a\mathbf{x}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}} + \widehat{\Gamma}_a(\mathbf{x},t) \tag{13}$$

$$\widehat{\Gamma}_a(\mathbf{x},t) = e_a\mathbf{E}\cdot\frac{\partial}{\partial\mathbf{p}} - \delta\widehat{I}_a \tag{14}$$

and $\delta\widehat{I}_a$ is the linearized collision operator. The Langevin source δf_a^S in Eq. (12) is determined by the following equation [6]:

$$\widehat{L}_{a\mathbf{x}t}\overline{\delta f_a(\mathbf{x},t)\delta f_b(\mathbf{x}',t')}^S = \delta_{ab}\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')f_b(\mathbf{x}',t') \tag{15}$$

The solution of Eq. (12) has the form

$$\delta f_a(\mathbf{x},t) = \delta f_a^S(\mathbf{x},t) -$$

$$-\sum_b \int d\mathbf{x}' \int_{-\infty}^{t} dt' G_{ab}(\mathbf{x},t,\mathbf{x}',t') e_b \delta\mathbf{E}(\mathbf{r}',t')\cdot\frac{\partial f_b(\mathbf{x}',t')}{\partial\mathbf{p}'} \tag{16}$$

where the Green function $G_{ab}(\mathbf{x},t,\mathbf{x}',t')$ of the operator $\widehat{L}_{a\mathbf{x}t}$ is determined by

$$\widehat{L}_{a\mathbf{x}t}G_{ab}(\mathbf{x},t,\mathbf{x}',t') = \delta_{ab}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{17}$$

with the causality condition:

$$G_{ab}(\mathbf{x},t,\mathbf{x}',t') = 0 \tag{18}$$

when $t < t'$.

Thus, $\overline{\delta f_a(\mathbf{x},t)\delta f_b(\mathbf{x}',t')}^S$ and $G_{ab}(\mathbf{x},t,\mathbf{x}',t')$ are connected by the relation:

$$\overline{\delta f_a(\mathbf{x},t)\delta f_b(\mathbf{x}',t')}^S = G_{ab}(\mathbf{x},t,\mathbf{x}',t')f_b(\mathbf{x}',t'), \quad t > t' \tag{19}$$

For stationary and spatially uniform systems, the DF f_a and the operator $\widehat{\Gamma}_a$ do not depend on time and space. In this case, the dependence on time and space of the Green function $G_{ab}(\mathbf{x},t,\mathbf{x}',t')$ appears only through the differences $t - t'$ and $\mathbf{r} - \mathbf{r}'$. However, when the DF $f_a(\mathbf{p},\mu\mathbf{r},\mu t)$ and $\widehat{\Gamma}_a(\mathbf{p},\mu\mathbf{r},\mu t)$ are slowly varying quantities in time and space, and when nonlocal effects are considered, the time and space dependence of $G_{ab}(\mathbf{x},t,\mathbf{x}',t')$ is more subtle:

$$G_{ab}(\mathbf{x},t,\mathbf{x}',t') = G_{ab}(\mathbf{p},\mathbf{p}',\mathbf{r}-\mathbf{r}',t-t',\mu\mathbf{r}',\mu t') \tag{20}$$

For the homogeneous case this non-trivial result was obtained for the first time in our previous work [7]. This result was extended to inhomogeneous systems [8]. Here we want to stress that the nonlocal effects appear due to the slow time and space dependences $\mu\mathbf{r}'$ and $\mu t'$.

Relationship (20) is directly linked with the constitutive relation between the electric displacement and the electric field

$$D_i(\mathbf{r},t) = \int d\mathbf{r}' \int\limits_{-\infty}^{t} dt' \varepsilon_{ij}(\mathbf{r},\mathbf{r}',t,t') E_j(\mathbf{r}',t') \tag{21}$$

Previously two kinds of constitutive relations were proposed phenomenologically for a weakly inhomogeneous and slowly time-varying medium. Kadomtsev [9] formulated the so-called *symmetrized* constitutive relation

$$D_i(\mathbf{r},t) = \int d\mathbf{r}' \int\limits_{-\infty}^{t} dt' \varepsilon_{ij}(\mathbf{r}-\mathbf{r}',t-t',\mu\frac{\mathbf{r}+\mathbf{r}'}{2},\mu\frac{t+t'}{2}) E_j(\mathbf{r}',t') \tag{22}$$

Rukhadze and Silin [10] proposed a *nonsymmetrized* constitutive relation

$$D_i(\mathbf{r},t) = \int d\mathbf{r}' \int\limits_{-\infty}^{t} dt' \varepsilon_{ij}(\mathbf{r}-\mathbf{r}',t-t',\mu\mathbf{r},\mu t) E_j(\mathbf{r}',t') \tag{23}$$

Both phenomenological formulations are unsatisfactory. The correct expression should be

$$D_i(\mathbf{r},t) = \int d\mathbf{r}' \int\limits_{-\infty}^{t} dt' \varepsilon_{ij}(\mathbf{r}-\mathbf{r}',t-t',\mu\mathbf{r}',\mu t') E_j(\mathbf{r}',t') \tag{24}$$

At first order, the expansion with respect to μ, Eq. (16) leads to

$$\delta f_a(\mathbf{x},t) = \delta f_a^S(\mathbf{x},t) - \sum_b e_b \int d\mathbf{p}' d\boldsymbol{\rho} \int\limits_0^{\infty} d\tau (1 - \mu\tau\frac{\partial}{\partial \mu t} - \mu\boldsymbol{\rho}\cdot\frac{\partial}{\partial \mu\mathbf{r}}) \times$$
$$\times G_{ab}(\boldsymbol{\rho},\tau,\mathbf{p},\mathbf{p}',\mu t,\mu\mathbf{r}) \delta\mathbf{E}(\mathbf{r}-\boldsymbol{\rho},t-\tau) \cdot \frac{\partial f_b(\mathbf{p}',\mu t,\mu\mathbf{r})}{\partial \mathbf{p}'} \tag{25}$$

with $\boldsymbol{\rho} = \mathbf{r}-\mathbf{r}'$ and $\tau = t - t'$.

From the Poisson equation

$$\delta\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial \mathbf{r}} \sum_b e_b \int \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta f_b(\mathbf{x}',t) d\mathbf{x}' \tag{26}$$

and performing the Fourier-Laplace transformation for the fast variables

$$\delta\mathbf{E}(\mathbf{k},\omega) = \int\limits_0^{\infty} dt \int d\mathbf{r} \delta\mathbf{E}(\mathbf{r},t) \exp(-\Delta t + i\omega t - i\mathbf{k}\cdot\mathbf{r}) \tag{27}$$

we have

$$\delta \mathbf{E}(\mathbf{k},\omega,t,\mathbf{r}) = \delta \mathbf{E}^s(\mathbf{k},\omega)+$$
$$+\sum_a 4\pi i e_a^2 \int d\mathbf{p}(1+i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{\partial}{\partial \mathbf{k}}\cdot\frac{\partial}{\partial \mathbf{r}})\frac{\mathbf{k}}{k^2}\widehat{L}^{-1}_{a\omega\mathbf{k}}\delta\mathbf{E}(\mathbf{k},\omega,\mathbf{r},t)\cdot\frac{\partial f_a(\mathbf{p},\mathbf{r},t)}{\partial \mathbf{p}} \tag{28}$$

Here and in the following for simplicity we omit μ, keeping in mind that derivatives over coordinates and time are taken with respect to the slowly varying variables. The resolvent $\widehat{L}^{-1}_{a\omega\mathbf{k}}$ in Eq. (28) is determined by the following relation:

$$\widehat{L}^{-1}_{a\omega\mathbf{k}}\delta_{ab}\delta(\mathbf{p}-\mathbf{p}') = \int d\boldsymbol{\rho}\int_0^\infty d\tau \exp(-\Delta\tau + i\omega\tau - i\mathbf{k}\cdot\boldsymbol{\rho})G_{ab}(\boldsymbol{\rho},\tau,\mathbf{p},\mathbf{p}',\mu t,\mu\mathbf{r}) \tag{29}$$

One should bear in mind that the derivatives $\frac{\partial}{\partial\omega}$ and $\frac{\partial}{\partial\mathbf{k}}$ in Eq. (28) act only on the operator $\frac{\mathbf{k}}{k^2}\widehat{L}^{-1}_{a\omega\mathbf{k}}$. The approximation in which Eq. (28) was derived corresponds to the geometric optics approximation [13]. At first-order and after some manipulations, one obtains from Eq. (28) the transport equation in the geometric optics approximation, which is not considered in the present paper, and the equation for the spectral function of the electrostatic field fluctuations:

$$\operatorname{Re}\varepsilon(\omega,\mathbf{k})[(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} - \frac{1}{|\widetilde{\varepsilon}(\omega,\mathbf{k})|^2}(\delta\mathbf{E}\delta\mathbf{E})^S_{\omega,\mathbf{k}}] = 0 \tag{30}$$

where we introduced

$$\widetilde{\varepsilon}(\omega,\mathbf{k}) = 1+\sum_a \widetilde{\chi}_a(\omega,\mathbf{k});\quad \varepsilon(\omega,\mathbf{k}) = 1+\sum_a \chi_a(\omega,\mathbf{k}) \tag{31}$$

$$\widetilde{\chi}_a(\omega,\mathbf{k}) = (1+i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{\partial}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{k}})\chi_a(\omega,\mathbf{k},t,\mathbf{r}) \tag{32}$$

and where

$$\chi_a(\omega,\mathbf{k},t,\mathbf{r}) = -\frac{4\pi i e_a^2}{k^2}\int d\mathbf{p}\widehat{L}^{-1}_{a\omega\mathbf{k}}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}f_a(\mathbf{p},t,\mathbf{r}) \tag{33}$$

is the susceptibility for a collisional plasma.

In the same approximation the spectral function of the Langevin source $(\delta\mathbf{E}\delta\mathbf{E})^S_{\omega,\mathbf{k}}$ takes the form

$$(\delta\mathbf{E}\delta\mathbf{E})^S_{\omega,\mathbf{k}} = 32\pi^2\sum_a e_a^2 \operatorname{Re}\int d\mathbf{p}(1+i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}})\frac{1}{k^2}\widehat{L}^{-1}_{a\omega\mathbf{k}}f_a(\mathbf{p},\mathbf{r},t) \tag{34}$$

If $\operatorname{Re}\boldsymbol{\varepsilon}(\omega,\mathbf{k}) \neq 0$, it follows from Eqs. (30) and (34) that the spectral function of the nonequilibrium electrostatic field fluctuations is determined by the expression:

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \frac{32\pi^2\sum_a e_a^2 \operatorname{Re}\int d\mathbf{p}(1+i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}})\frac{1}{k^2}\widehat{L}^{-1}_{a\omega\mathbf{k}}f_a(\mathbf{p},\mathbf{r},t)}{|\widetilde{\varepsilon}(\omega,\mathbf{k})|^2} \tag{35}$$

The effective dielectric function $\widetilde{\varepsilon}(\omega,\mathbf{k})$ in the denominator of Eq. (35) determines the spectral properties of the electrostatic field fluctuations and its imaginary part

$$\mathrm{Im}\,\widetilde{\varepsilon}(\omega,\mathbf{k}) = \mathrm{Im}\,\varepsilon(\omega,\mathbf{k}) + \frac{\partial}{\partial\omega}\frac{\partial}{\partial t}\mathrm{Re}\,\varepsilon(\omega,\mathbf{k},t,\mathbf{r}) - \frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}}\mathrm{Re}\,\varepsilon(\omega,\mathbf{k},t,\mathbf{r}) \tag{36}$$

determines the width of the spectral lines near the resonance. Note that when expanding the Green function in Eq. (25) in terms of the small parameter μ, there appear additional terms at first order. It is important to note that the *imaginary* part of the dielectric susceptibility is now replaced by the *real* part, which is greater than *imaginary* part by the factor μ^{-1}. Therefore, the second and third terms in Eq. (36) in the kinetic regime have an effect comparable to that of the first term. At second order in the expansion in μ the corrections appear only in the *imaginary* part of the susceptibility, and they can reasonably be neglected. It is therefore sufficient to retain the first order corrections to solve the problem.

For the local equilibrium case where the reference state f_a is Maxwellian, we have the identity:

$$\int d\mathbf{p}(1 + i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}})\frac{1}{k^2}\widehat{L}^{-1}_{a\omega\mathbf{k}}f_a(\mathbf{p},t,\mathbf{r}) =$$
$$= \frac{i}{\omega_a}\int d\mathbf{p} f_a(\mathbf{p},t,\mathbf{r}) - \frac{iT_a}{\omega_a 4\pi e_a^2}\widetilde{\chi}_a(\omega,\mathbf{k}) \tag{37}$$

where ($\omega_a = \omega - \mathbf{k}\cdot\mathbf{V}_a$, and T_a is the temperature in energy units) and Eq.(35) takes the form

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \sum_a \frac{8\pi\ T_a}{\omega_a\,|\widetilde{\varepsilon}(\omega,\mathbf{k})|^2}\,\mathrm{Im}\,\widetilde{\chi}_a(\omega,\mathbf{k}) \tag{38}$$

In this case the small parameter μ is determined on the slower hydrodynamic scale. For the case of equal temperatures and $\mathbf{V}_a = 0$ one obtains a generalized expression for the Callen-Welton formula:

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \frac{8\pi\ T\,\mathrm{Im}\,\widetilde{\varepsilon}(\omega,\mathbf{k})}{\omega\,|\widetilde{\varepsilon}(\omega,\mathbf{k})|^2} \tag{39}$$

To calculate explicitly $(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}}$ we will restrict our analysis to the vicinity of the resonance, *i.e.* $\omega = \pm\omega_0$, where $\mathrm{Re}\,\varepsilon(\omega_0,\mathbf{k}) = 0$. We can develop

$$\widetilde{\varepsilon}(\omega,\mathbf{k}) = (\omega - \omega_0 sgn\omega)\frac{\partial\,\mathrm{Re}\,\varepsilon}{\partial\omega}\lfloor_{\omega=\omega_0 sgn\omega} + i[\mathrm{Im}\,\varepsilon + (\frac{\partial^2}{\partial\omega\partial t} - \frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}})\,\mathrm{Re}\,\varepsilon]\lfloor_{\omega=\omega_0 sgn\omega} \tag{40}$$

Thus

$$(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}} = \frac{\widetilde{\gamma}}{(\omega-\omega_0 sgn\omega)^2 + \widetilde{\gamma}^2}\frac{8\pi T}{\omega\partial Re\varepsilon/\partial\omega}\lfloor_{\omega=\omega_0}, \tag{41}$$

where

$$\widetilde{\gamma} = [\mathrm{Im}\,\varepsilon + \frac{\partial^2}{\partial\omega\partial t}\mathrm{Re}\,\varepsilon - \frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}}\mathrm{Re}\,\varepsilon]/\frac{\partial\,\mathrm{Re}\,\varepsilon}{\partial\omega}\lfloor_{\omega=\omega_0 sgn\omega} \tag{42}$$

is the effective damping decrement. For the case where the system parameters are homogeneous in space but vary in time, the correction is still symmetric with respect to the change of sign of ω, but the intensities and broadening are different, and the intensity integrated over the frequencies remains the same as in the stationary case. However, when the plasma parameters are space dependent this symmetry is lost. In the same manner as for simple fluids and gases [4] the spectral asymmetry is related to the appearance of space anisotropy in inhomogeneous systems. The *real* part of the susceptibility $\mathrm{Re}\,\varepsilon$ is an even function of ω. This property implies that the contribution of the third term to the expression of the damping decrement (42) is an odd function of ω. Moreover this term gives rise to an anisotropy in k space. Let us estimate this correction for the plasma mode $(\omega_0 = \omega_L)$

$$\mathrm{Re}\,\varepsilon = 1 - \frac{\omega_L^2}{\omega^2}(1 + 3\frac{k^2 T}{m\omega^2}) \tag{43}$$

$$\mathrm{Im}\,\varepsilon = \frac{\omega_L^2}{\omega^2}\frac{\nu_{ei}}{\omega}, \omega_L^2 = \frac{4\pi n e^2}{m} = \frac{T k_D^2}{m} \tag{44}$$

and

$$\widetilde{\gamma} = [\nu_{ei} + \frac{2}{n}\frac{\partial n}{\partial t} + 6\frac{\omega_L}{n k_D^2}\mathbf{k}\cdot\frac{\partial n}{\partial \mathbf{r}} sgn\omega]/2 \tag{45}$$

On the hydrodynamic scale

$$\left|\frac{2}{n}\frac{\partial n}{\partial t}\right|, \left|\frac{6\omega_L}{n k_D^2}\mathbf{k}\cdot\frac{\partial n}{\partial \mathbf{r}}\right| < \nu_{ei} \tag{46}$$

and $\widetilde{\gamma} > 0$.

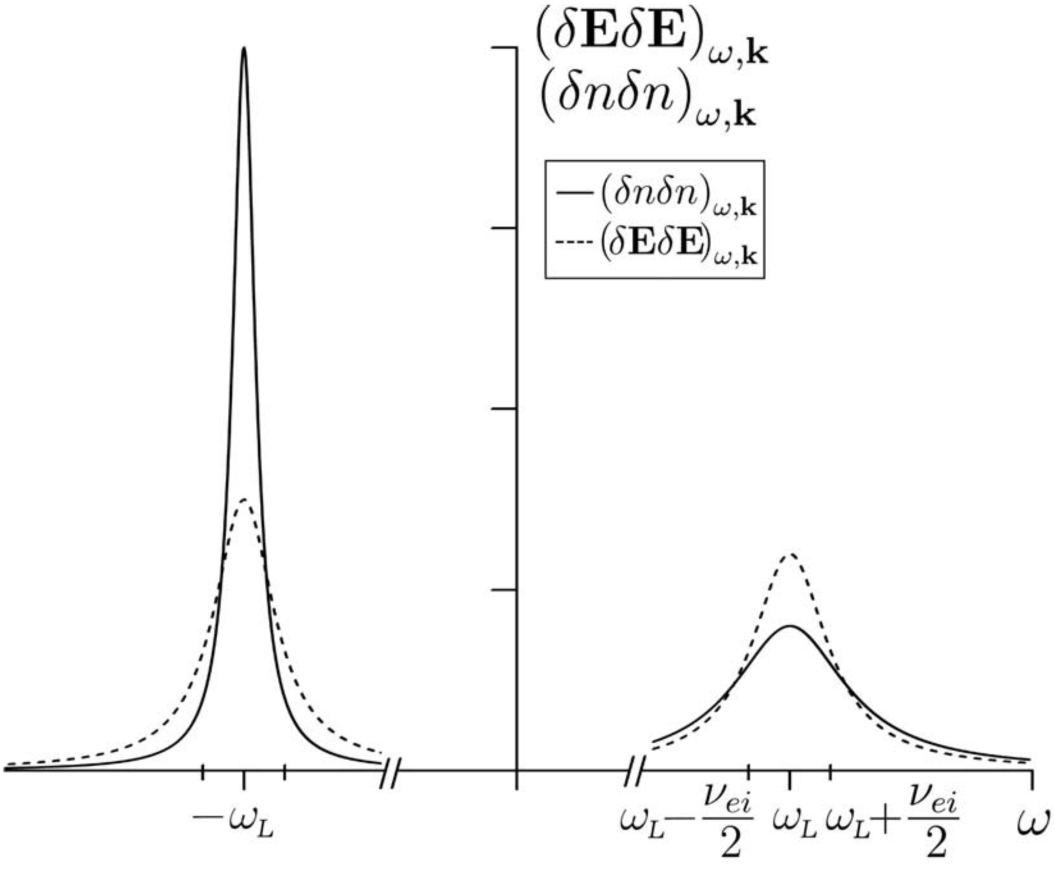

Figure 1: The electron form factor $(\delta n_e \delta n_e)_{\omega,\mathbf{k}}$ (solid line) and the spectral function of electrostatic field fluctuations $(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}}$ (dashed line) as a function of frequency. $\mathbf{k}\cdot\frac{\partial n}{\partial \mathbf{r}} = \frac{\nu_{ei} n k_D^2}{54\omega_L}$; $\frac{k_D}{k} = 6$

For the spatially homogeneous case there is no difference between the spectral properties of the longitudinal electric field and of the electron density. They are connected by the Poisson equation. This statement is no longer valid when considering an inhomogeneous plasma. Indeed the longitudinal electric field is linked to the particle density by the nonlocal Poisson relation (26). In the latter case, an analysis similar to that made above can also be performed for the particle density. From Eq. (16) there follows

$$\delta n_a(\mathbf{k},\omega,\mathbf{r},t) = \delta n_a^S(\mathbf{k},\omega,\mathbf{r},t) + \sum_b \frac{4\pi i\mathbf{k}e_b e_a}{k^2}\int d\mathbf{p}(1+$$

$$+i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{r}})\widehat{L}^{-1}_{a\omega\mathbf{k}}\delta n_b(\mathbf{k},\omega,\mathbf{r},t)\cdot\frac{\partial f_a(\mathbf{p},\mathbf{r},t)}{\partial\mathbf{p}} \tag{47}$$

One should remember that now the derivatives $\frac{\partial}{\partial\omega}$ and $\frac{\partial}{\partial\mathbf{k}}$ in Eq. (47) act only on the operator $\widehat{L}^{-1}_{a\omega\mathbf{k}}$. At the first order approximation and after some manipulations, one obtains the following expression for the electron form factor for a two-component ($a = e, i$) plasma:

$$(\delta n_e\delta n_e)_{\omega,\mathbf{k}} = \frac{2n_e k^2}{\omega_e k_D^2}\left|\frac{1+\underset{i}{\widehat{\chi}}(\omega,\mathbf{k})}{\widehat{\varepsilon}(\omega,\mathbf{k})}\right|^2 \operatorname{Im}\underset{e}{\widehat{\chi}}(\omega,\mathbf{k})$$

$$+\left|\frac{\underset{e}{\widehat{\chi}}(\omega,\mathbf{k})}{\widehat{\varepsilon}(\omega,\mathbf{k})}\right|^2 \frac{T_i}{T_e}\frac{2n_e k^2}{\omega_i k_D^2}\operatorname{Im}\underset{i}{\widehat{\chi}}(\omega,\mathbf{k}) \tag{48}$$

where we used for local equilibrium the following expression for the "source"

$$(\delta n_a\delta n_b)^S_{\omega,\mathbf{k}} = \delta_{ab}\frac{T_a}{\omega_a}\frac{k^2}{2\pi e_a^2}\operatorname{Im}\underset{a}{\widehat{\chi}}(\omega,\mathbf{k}) \tag{49}$$

and

$$\widehat{\varepsilon}(\omega,\mathbf{k}) = 1 + \sum_a \underset{a}{\widehat{\chi}}(\omega,\mathbf{k}) \tag{50}$$

$$\underset{a}{\widehat{\chi}}(\omega,\mathbf{k}) = (1 + i\frac{\partial}{\partial\omega}\frac{\partial}{\partial t} - i\frac{1}{k^2}\frac{\partial}{\partial r_i}k_j\frac{\partial}{\partial k_i}k_j)\chi_a(\omega,\mathbf{k},t,\mathbf{r}) \tag{51}$$

As above we can expand $\widehat{\varepsilon}(\omega,\mathbf{k})$ near the plasma resonance $\omega = \omega_L$. Thus, for the electron line,

$$(\delta n_e\delta n_e)_{\omega,\mathbf{k}} = \frac{\widehat{\gamma}}{(\omega - sign\omega_L)^2 + (\widehat{\gamma})^2}\frac{2n_e k^2}{\omega k_D^2\partial Re\varepsilon/\partial\omega}\lfloor_{\omega=\omega_L} \tag{52}$$

where

$$\widehat{\gamma} = [\operatorname{Im}\varepsilon + \frac{\partial^2\operatorname{Re}\varepsilon}{\partial t\partial\omega} - \frac{1}{k^2}\frac{\partial}{\partial r_i}k_j\frac{\partial}{\partial k_i}k_j\operatorname{Re}\varepsilon]/\frac{\partial\operatorname{Re}\varepsilon}{\partial\omega}\lfloor_{\omega=\omega_L sgn\omega} \tag{53}$$

is the effective damping decrement for the electron form factor. At this stage of calculation, let us note that the damping decrements for the electrostatic field fluctuations [Eq. (42)] and for the electron density fluctuations [Eq. (53)] are not the same. The origin of this difference is that the Green function for electrostatic field fluctuation and density particle fluctuations are not the same. This property holds only in the inhomogeneous situation. An estimation for the plasma mode is then:

$$\widetilde{\gamma} = [\nu_{ei} + \frac{2}{n}\frac{\partial n}{\partial t} + \frac{\omega_L}{nk^2}\mathbf{k}\cdot\frac{\partial n}{\partial \mathbf{r}}(1 + \frac{6k^2}{k_D^2})sgn\omega]/2. \qquad (54)$$

From this equation we see that the inhomogeneous correction in Eq.(54) is greater than the one in Eq. (45) by the factor $1 + k_D^2/6k^2$. For the same inhomogeneity; i.e., the same gradient of the density, we plot the form factor $(\delta n_e \delta n_e)_{\omega,\mathbf{k}}$ together with the $(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}}$ as functions of frequency [11] (Fig. 1). This figure shows that the asymmetry of the spectral lines is present both for $(\delta n_e \delta n_e)_{\omega,\mathbf{k}}$ and $(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}}$. However, this effect is more pronounced in $(\delta n_e \delta n_e)_{\omega,\mathbf{k}}$ than in $(\delta\mathbf{E}\delta\mathbf{E})_{\omega,\mathbf{k}}$. We have shown that the amplitude and the width of the spectral lines of the electrostatic field fluctuations and form factor are affected by new non-local dispersive terms. They are not related to Joule dissipation and appear because of an additional phase shift between the vectors of induction and electric field [12]. This phase shift results from the finite time needed to set the polarization in the plasma with dispersion [13]. Such a phase shift in the plasma with space dispersion appears due to the medium inhomogeneity. These results are important for the understanding and the classification of the various phenomena that may be observed in applications; in particular, the asymmetry of lines can be used as a diagnostic tool to measure local gradients in the plasma.

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