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# Massless tensor fields with the mixed symmetry $(k, 1)$: Lagrangian description and BRST symmetry

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**Abstract**

The full Lagrangian description of a free, massless tensor field that transforms in an irreducible representation of $GL(D, \mathbb{R})$, corresponding to a two-column Young diagram with $(k+1)$ cells and $k \geq 2$ rows is studied in detail following the general principle of gauge invariance. Finally, the antifield-BRST symmetry for this model is constructed.

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## 1 Introduction

An interesting class of field theories is represented by tensor fields in "exotic" representations of the Lorentz group, characterized by a mixed Young symmetry type [1, 2, 3, 4, 5], which are known to appear in superstring theories, supergravities or supersymmetric high spin theories. This type of models became of special interest lately due to the many desirable featured exhibited, like the dual formulation of field theories of spin two or higher [6, 7, 8, 9, 10, 11], the impossibility of consistent cross-interactions in the dual formulation of linearized gravity [12] or a Lagrangian first-order approach [13, 14] to some classes of massless or partially massive mixed symmetry-type tensor gauge fields, suggestively resembling to the tetrad formalism of General Relativity. A basic problem involving mixed symmetry-type tensor fields is the approach to their local BRST cohomology, since it is helpful at solving many Lagrangian and Hamiltonian aspects, like, for instance the determination of their consistent interactions [15] with higher-spin gauge theories [16, 17, 18, 19, 20, 21, 22, 23]. The present paper proposes the investigation of the Lagrangian formulation and BRST formulation for a massless tensor field $t_{\mu_1 \cdots \mu_k|\alpha}$ that transforms in an irreducible representation of $GL(D, \mathbb{R})$, corresponding to a two-column Young diagram with $(k+1)$ cells and $k \geq 2$ rows.

In view of this, we firstly give the Lagrangian formulation of such a mixed symmetry tensor field from the general principle of gauge invariance and then systematically analyze

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this formulation from the general principle of gauge invariance. Secondly, we compute the associated free antifield-BRST symmetry $s$, which is found to split as the sum between the Koszul-Tate differential and the exterior longitudinal derivative only, $s = \delta + \gamma$. The results contained in this paper can be used at the determination of the local BRST cohomology of the free, massless tensor field with the mixed symmetry $(k, 1)$ at positive ghost number numbers and at maximum form degree.

# 2 Lagrangian formulation from the principle of gauge invariance

We consider a tensor field $t_{\mu_1 \cdots \mu_k|\alpha}$ that transforms in an irreducible representation of $GL(D, \mathbb{R})$, corresponding to a two-column Young diagram with $(k+1)$ cells and $k > 2$ rows

$$t_{\mu_1 \cdots \mu_k|\alpha} = \begin{array}{cc} \mu_1 & \alpha \\ \vdots & \\ \mu_k & \end{array}, \tag{1}$$

or, in a shortened version, a tensor field with the mixed symmetry $(k, 1)$. This means that $t_{\mu_1 \cdots \mu_k|\alpha}$ is separately antisymmetric in the first $k$ indices and satisfies the (algebraic) Bianchi I identity

$$t_{[\mu_1 \cdots \mu_k|\alpha]} \equiv 0. \tag{2}$$

Here and in the sequel the symbol $[\mu \cdots \nu]$ signifies the operation of complete antisymmetrization with respect to the indices between brackets, defined such as to include only the distinct terms for a tensor with given antisymmetry properties. For instance, the left-hand side of (2) contains precisely $(k+1)$ terms

$$\begin{aligned} t_{[\mu_1 \cdots \mu_k|\alpha]} \equiv\ & t_{\mu_1 \cdots \mu_k|\alpha} + (-)^k t_{\mu_2 \cdots \mu_k \alpha|\mu_1} \\ & + t_{\mu_3 \cdots \mu_k \alpha \mu_1|\mu_2} + \cdots + (-)^k t_{\alpha \mu_1 \cdots \mu_{k-1}|\mu_k}. \end{aligned} \tag{3}$$

Assume that this tensor field is defined on a pseudo-Riemannian manifold $\mathcal{M}$ of dimension $D$, like, for instance, a Minkowski-flat spacetime of dimension $D$, endowed with a metric tensor of 'mostly plus' signature $\sigma_{\mu\nu} = \sigma^{\mu\nu} = (- + \cdots +)$. The trace of this tensor field, to be denoted by $t_{\mu_1 \cdots \mu_{k-1}}$, is defined by

$$t_{\mu_1 \cdots \mu_{k-1}} = t_{\mu_1 \cdots \mu_k|\alpha} \sigma^{\mu_k \alpha}. \tag{4}$$

Obviously, $t_{\mu_1 \cdots \mu_{k-1}}$ is a completely antisymmetric tensor field of degree $(k-1)$.

We are interested in the Lagrangian description of a single, free, massless tensor field with this type of mixed symmetry, which is known to describe exotic spin-two particles for $k \geq 2$. For $k = 2$, $D = 5$ and respectively $k = 3$, $D = 6$ we obtain nothing but the dual formulations of linearized gravity [12, 21]. *The construction of the Lagrangian action for such a tensor field relies on the general principle of gauge invariance, combined with the requirements of locality, Lorentz covariance, Poincaré invariance, zero mass and the natural assumptions that the field equations are linear in the field, second-order derivative and do not break the PT invariance.* In view of all these, a natural point to start with is to stipulate the (infinitesimal) gauge invariance of the action such that to recover the already known gauge symmetries for $k = 2$ and $k = 3$ [12, 21]. The simplest way to achieve this is to ask that the Lagrangian action $S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]$ is invariant under the (infinitesimal) gauge transformations

$$\begin{aligned}\delta_{\theta,\epsilon}t_{\mu_1\cdots\mu_k|\alpha} &= \partial_{[\mu_1}\overset{(1)}{\theta}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_k]\alpha} + (-)^{k+1}k\partial_\alpha\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k} \\ &= \partial_{[\mu_1}\overset{(1)}{\theta}_{\mu_2\ldots\mu_k]|\alpha} + \partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_k\alpha]} + (-)^{k+1}(k+1)\,\partial_\alpha\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}.\end{aligned} \tag{5}$$

The mixed symmetry properties of the gauge parameters $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}$ and $\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}$ follow from those of the left-hand side of (5) once we require that the mixed symmetry of $t_{\mu_1\cdots\mu_k|\alpha}$ is inherited by its gauge variation. As a consequence we find that $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}$ displays the mixed symmetry $(k-1,1)$

$$\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha} = \begin{array}{cc}\mu_1 & \alpha \\ \vdots & \\ \mu_{k-1} & \end{array}, \tag{6}$$

so it is separately antisymmetric in the first $(k-1)$ indices and satisfies the identity

$$\overset{(1)}{\theta}_{[\mu_1\ldots\mu_{k-1}|\alpha]} \equiv 0, \tag{7}$$

while $\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}$ is a completely antisymmetric tensor field of degree $k$. Formula (7) has the role to enforce that $\delta_{\theta,\epsilon}t_{\mu_1\cdots\mu_k|\alpha}$ satisfies a Bianchi I identity similar to that of the field itself, namely, (2). In order to check the correctness of (5) we take the limits $k=2$ and $k=3$ and re-obtain precisely the results from [12, 21]. For subsequent purposes it is useful to compute the gauge variation of the trace of $t_{\mu_1\cdots\mu_k|\alpha}$

$$\begin{aligned}\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}}t_{\mu_1\cdots\mu_{k-1}} &= (-)^{k-1}\partial^\alpha\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha} + \partial_{[\mu_1}\overset{(1)}{\theta}_{\mu_2\ldots\mu_{k-1}]} \\ &\quad + (k+1)\,\partial^\alpha\overset{(1)}{\epsilon}_{\alpha\mu_1\ldots\mu_{k-1}}.\end{aligned} \tag{8}$$

In the above the notation $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-2}}$ stands for the trace of $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}$, $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-2}} = \overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}\sigma^{\mu_{k-1}\alpha}$, which is obviously a fully antisymmetric tensor field.

At this stage we introduce the tensor

$$F_{\mu_1\cdots\mu_{k+1}|\alpha} = \partial_{[\mu_1}t_{\mu_2\cdots\mu_{k+1}]|\alpha} \tag{9}$$

with the mixed symmetry $(k+1,1)$ and observe that it is invariant under the gauge transformations from (5) involving the parameters $\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}$, but not under the part containing $\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}$. Indeed, it is easy to see that

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}}F_{\mu_1\cdots\mu_{k+1}|\alpha} = (-)^{k+1}k\partial_\alpha\partial_{[\mu_1}\overset{(1)}{\epsilon}_{\mu_2\ldots\mu_{k+1}]}. \tag{10}$$

Its trace is a completely antisymmetric tensor, $F_{\mu_1\cdots\mu_k} = F_{\mu_1\cdots\mu_{k+1}|\alpha}\sigma^{\mu_{k+1}\alpha}$, with the gauge transformations

$$\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}}F_{\mu_1\cdots\mu_k} = -k\partial^\alpha\partial_{[\alpha}\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k]}. \tag{11}$$

Generalizing the results from [12, 21], we try to construct the Lagrangian action for $t_{\mu_1\cdots\mu_k|\alpha}$ in terms of $F_{\mu_1\cdots\mu_{k+1}|\alpha}$ and its trace. The above mentioned hypotheses on $S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]$ lead to the most general expression of the form

$$S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]=\int d^Dx\left(c_1F_{\mu_1\cdots\mu_{k+1}|\alpha}F^{\mu_1\cdots\mu_{k+1}|\alpha}+c_2F_{\mu_1\cdots\mu_k}F^{\mu_1\cdots\mu_k}\right). \tag{12}$$

Computing the variation of (12) under (5), it follows that

$$\begin{aligned}\delta_{\overset{(1)}{\theta},\overset{(1)}{\epsilon}}S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right] &= 2k\left(-\right)^{k+1}\left[\left(k+1\right)c_1+c_2\right]\times\\ &\times\int d^Dx\left(\partial_\beta t^{\mu_1\cdots\mu_k|\beta}\right)\partial^\alpha\partial_{[\alpha}\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k]}.\end{aligned} \tag{13}$$

As a consequence, we find that the gauge invariance of $S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]$ under (5) is equivalent to the condition

$$c_2=-\left(k+1\right)c_1, \tag{14}$$

which replaced back in (12) leads to

$$S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]=c_1\int d^Dx\left(F_{\mu_1\cdots\mu_{k+1}|\alpha}F^{\mu_1\cdots\mu_{k+1}|\alpha}-\left(k+1\right)F_{\mu_1\cdots\mu_k}F^{\mu_1\cdots\mu_k}\right). \tag{15}$$

The constant $c_1$ must be fixed to the value

$$c_1=-\frac{1}{2\cdot\left(k+1\right)!} \tag{16}$$

in order to render the actions from [12, 21] in the limits $k=2$ and $k=3$ respectively. Expressing (15) in terms of $t_{\mu_1\cdots\mu_k|\alpha}$ via definition (9), we finally get

$$\begin{aligned}S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right] &= -\frac{1}{2\cdot k!}\int d^Dx\left[\left(\partial_\mu t_{\mu_1\cdots\mu_k|\alpha}\right)\partial^\mu t^{\mu_1\cdots\mu_k|\alpha}\right.\\ &-\left(\partial^\alpha t_{\mu_1\cdots\mu_k|\alpha}\right)\partial_\beta t^{\mu_1\cdots\mu_k|\beta}-k\left(\partial^\lambda t_{\lambda\mu_1\cdots\mu_{k-1}|\alpha}\right)\partial_\rho t^{\rho\mu_1\cdots\mu_{k-1}|\alpha}\\ &-k\left(\partial_\mu t_{\mu_1\cdots\mu_{k-1}}\right)\partial^\mu t^{\mu_1\cdots\mu_{k-1}}+2\left(-\right)^{k+1}k\left(\partial^\alpha t_{\mu_1\cdots\mu_k|\alpha}\right)\partial^{\mu_1}t^{\mu_2\cdots\mu_k}\\ &\left.+k\left(k-1\right)\left(\partial^\lambda t_{\lambda\mu_1\cdots\mu_{k-2}}\right)\partial_\rho t^{\rho\mu_1\cdots\mu_{k-2}}\right].\end{aligned} \tag{17}$$

It can be checked that (5) is a generating set of gauge transformations for action (17). This generating set is Abelian and off-shell $(k-1)$-order reducible. Indeed, if we make the transformations

$$\begin{aligned}\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha} &= \partial_{[\mu_1}\overset{(2)}{\theta}_{\mu_2\ldots\mu_{k-1}]|\alpha}+\partial_{[\mu_1}\overset{(2)}{\epsilon}_{\mu_2\ldots\mu_{k-1}]\alpha}\\ &+\left(-\right)^k\left(k-1\right)\partial_\alpha\overset{(2)}{\epsilon}_{\mu_1\ldots\mu_{k-1}}\\ &= \partial_{[\mu_1}\overset{(2)}{\theta}_{\mu_2\ldots\mu_{k-1}]|\alpha}+\partial_{[\mu_1}\overset{(2)}{\epsilon}_{\mu_2\ldots\mu_{k-1}\alpha]}+\left(-\right)^k k\partial_\alpha\overset{(2)}{\epsilon}_{\mu_1\ldots\mu_{k-1}},\end{aligned} \tag{18}$$

$$\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}=\frac{k-1}{k+1}\partial_{[\mu_1}\overset{(2)}{\epsilon}_{\mu_2\ldots\mu_k]}, \tag{19}$$

with $\overset{(2)}{\theta}_{\mu_1\ldots\mu_{k-2}|\alpha}$ and $\overset{(2)}{\epsilon}_{\mu_1\ldots\mu_{k-1}}$ some arbitrary tensor fields on $\mathcal{M}$ displaying the mixed symmetry $(k-2,1)$ and respectively $(k-1,0)$ (fully antisymmetric), then the gauge transformations of the tensor field vanish identically

$$\delta_{\overset{(1)}{\theta}\left(\overset{(2)}{\theta},\overset{(2)}{\epsilon}\right),\overset{(1)}{\epsilon}\left(\overset{(2)}{\epsilon}\right)}t_{\mu_1\cdots\mu_k|\alpha}=0. \tag{20}$$

Next, if we perform the transformations

$$
\begin{aligned}
\overset{(2)}{\theta}_{\mu_1\ldots\mu_{k-2}|\alpha} &= \partial_{[\mu_1}\overset{(3)}{\theta}_{\mu_2\ldots\mu_{k-2}]|\alpha} + \partial_{[\mu_1}\overset{(3)}{\epsilon}_{\mu_2\ldots\mu_{k-2}]\alpha} \\
&+ (-)^{k-1}(k-2)\,\partial_\alpha\overset{(3)}{\epsilon}_{\mu_1\ldots\mu_{k-2}} \\
&= \partial_{[\mu_1}\overset{(3)}{\theta}_{\mu_2\ldots\mu_{k-2}]|\alpha} + \partial_{[\mu_1}\overset{(3)}{\epsilon}_{\mu_2\ldots\mu_{k-2}\alpha]} + (-)^{k-1}(k-1)\,\partial_\alpha\overset{(3)}{\epsilon}_{\mu_1\ldots\mu_{k-2}},
\end{aligned} \tag{21}
$$

$$
\overset{(2)}{\epsilon}_{\mu_1\ldots\mu_{k-1}} = \frac{k-2}{k}\partial_{[\mu_1}\overset{(3)}{\epsilon}_{\mu_2\ldots\mu_{k-1}]}, \tag{22}
$$

with $\overset{(3)}{\theta}_{\mu_1\ldots\mu_{k-3}|\alpha}$ and $\overset{(3)}{\epsilon}_{\mu_1\ldots\mu_{k-2}}$ some arbitrary tensor fields on $\mathcal{M}$ displaying the mixed symmetry $(k-3,1)$ and respectively $(k-2,0)$, then we find that the gauge transformed parameters (18) and (19) strongly vanish

$$
\overset{(1)}{\theta}_{\mu_1\ldots\mu_{k-1}|\alpha}\left(\overset{(2)}{\theta}\left(\overset{(3)}{\theta},\overset{(3)}{\epsilon}\right),\overset{(2)}{\epsilon}\left(\overset{(3)}{\epsilon}\right)\right) = 0, \tag{23}
$$

$$
\overset{(1)}{\epsilon}_{\mu_1\ldots\mu_k}\left(\overset{(2)}{\epsilon}\left(\overset{(3)}{\epsilon}\right)\right) = 0. \tag{24}
$$

Along a similar line it can be shown that if we perform the changes

$$
\begin{aligned}
\overset{(m)}{\theta}_{\mu_1\ldots\mu_{k-m}|\alpha} &= \partial_{[\mu_1}\overset{(m+1)}{\theta}_{\mu_2\ldots\mu_{k-m}]|\alpha} + \partial_{[\mu_1}\overset{(m+1)}{\epsilon}_{\mu_2\ldots\mu_{k-m}]\alpha} \\
&+ (-)^{k-m+1}(k-m)\,\partial_\alpha\overset{(m+1)}{\epsilon}_{\mu_1\ldots\mu_{k-m}} \\
&= \partial_{[\mu_1}\overset{(m+1)}{\theta}_{\mu_2\ldots\mu_{k-m}]|\alpha} + \partial_{[\mu_1}\overset{(m+1)}{\epsilon}_{\mu_2\ldots\mu_{k-m}\alpha]} \\
&+ (-)^{k-m+1}(k-m+1)\,\partial_\alpha\overset{(m+1)}{\epsilon}_{\mu_1\ldots\mu_{k-m}},
\end{aligned} \tag{25}
$$

$$
\overset{(m)}{\epsilon}_{\mu_1\ldots\mu_{k-m+1}} = \frac{k-m}{k-m+2}\partial_{[\mu_1}\overset{(m+1)}{\epsilon}_{\mu_2\ldots\mu_{k-m+1}]}, \tag{26}
$$

for $3 \leq m \leq k-2$, with $\overset{(m+1)}{\theta}_{\mu_1\ldots\mu_{k-m-1}|\alpha}$ and $\overset{(m+1)}{\epsilon}_{\mu_1\ldots\mu_{k-m}}$ some arbitrary tensor fields on $\mathcal{M}$, with the mixed symmetries $(k-m-1,1)$ and $(k-m,0)$ respectively, then

$$
\overset{(m-1)}{\theta}_{\mu_1\ldots\mu_{k-m+1}|\alpha}\left(\overset{(m)}{\theta}\left(\overset{(m+1)}{\theta},\overset{(m+1)}{\epsilon}\right),\overset{(m)}{\epsilon}\left(\overset{(m+1)}{\epsilon}\right)\right) = 0, \tag{27}
$$

$$
\overset{(m-1)}{\epsilon}_{\mu_1\ldots\mu_{k-m+2}}\left(\overset{(m)}{\epsilon}\left(\overset{(m+1)}{\epsilon}\right)\right) = 0. \tag{28}
$$

In particular, for $m = k-2$ the parameters $\overset{(k-1)}{\theta}_{\mu_1|\alpha}$ are symmetric (mixed symmetry $(1,1)$), $\overset{(k-1)}{\theta}_{\mu_1|\alpha} = \overset{(k-1)}{\theta}_{\alpha|\mu_1}$. Finally, if we perform the transformations

$$
\overset{(k-1)}{\theta}_{\mu_1|\alpha} = \partial_{\mu_1}\overset{(k)}{\epsilon}_\alpha + \partial_\alpha\overset{(k)}{\epsilon}_{\mu_1} \equiv \partial_{(\mu_1}\overset{(k)}{\epsilon}_{\alpha)}, \tag{29}
$$

$$
\overset{(k-1)}{\epsilon}_{\mu_1\mu_2} = \frac{1}{3}\partial_{[\mu_1}\overset{(k)}{\epsilon}_{\mu_2]}, \tag{30}
$$

with $\overset{(k)}{\epsilon}_{\mu}$ an arbitrary vector field on $\mathcal{M}$, then

$$\overset{(k-2)}{\theta}_{\mu_1\mu_2|\alpha}\left(\overset{(k-1)}{\theta}\left(\overset{(k)}{\epsilon}\right),\overset{(k-1)}{\epsilon}\left(\overset{(k)}{\epsilon}\right)\right) = 0, \tag{31}$$

$$\overset{(k-2)}{\epsilon}_{\mu_1\mu_2\mu_3}\left(\overset{(k-1)}{\epsilon}\left(\overset{(k)}{\epsilon}\right)\right) = 0. \tag{32}$$

The reducibility order stops at $(k-1)$ since

$$\overset{(k-1)}{\theta}_{\mu_1|\alpha}\left(\overset{(k)}{\epsilon}\right) = 0, \quad \overset{(k-1)}{\epsilon}_{\mu_1\mu_2}\left(\overset{(k)}{\epsilon}\right) = 0 \tag{33}$$

take place *simultaneously* if and only if

$$\overset{(k)}{\epsilon}_{\mu} = 0. \tag{34}$$

The tensor fields $\left(\overset{(m+1)}{\theta}_{\mu_1\cdots\mu_{k-m-1}|\alpha},\overset{(m+1)}{\epsilon}_{\mu_1\cdots\mu_{k-m}}\right)_{1\leq m\leq k-2}$ will be called reducibility parameters of order $m$ and $\overset{(k)}{\epsilon}_{\mu}$ reducibility parameters of order $(k-1)$. Along the same line, relations (20), (23)–(24), (27)–(28), and (31)–(32) will be called reducibility relations of orders 1, 2, $m$, and $(k-1)$ respectively.

And now, a few words on the restrictions on the dimension of $\mathcal{M}$. According to the properties of the tensor field (1) it follows that at each point $x\in\mathcal{M}$ there are precisely $k\cdot(D+1)!/(k+1)!\cdot(D-k)!$ independent components of $t_{\mu_1\cdots\mu_k|\alpha}$. In view of the reducibility structure exhibited by the generating set (5) of gauge transformations, it follows that the number of physical degrees of freedom of $t_{\mu_1\cdots\mu_k|\alpha}$ at each $x\in\mathcal{M}$ is equal to

$$N_{\text{phys}} = \frac{(D-2)!D(D-k-2)k}{(D-k-1)!\cdot(k+1)!}. \tag{35}$$

It is now clear that (35) is invariant under the transformation $k\leftrightarrow D-k-2$ which corresponds to a Hodge duality transformation. On the one hand, the requirement on the $(k,1)$ tensor field to display a non-negative number of physical degrees of freedom imposes that

$$D\geq k+2. \tag{36}$$

On the other hand, since in the limit $k=1$ one recovers nothing but the linearized limit of Einstein–Hilbert action (Pauli–Fierz model) [25], it follows that a $(k,1)$ tensor field is dual to linearized gravity in exactly $D-k-2=1$, hence in $D=k+3$ spacetime dimensions.

The field equations

$$\frac{\delta S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]}{\delta t_{\nu_1\cdots\nu_k|\alpha}} \equiv \frac{1}{k!}T^{\nu_1\cdots\nu_k|\alpha}\approx 0, \tag{37}$$

involve the tensor $T^{\nu_1\cdots\nu_k|\alpha}$, which is linear in the tensor field $t_{\mu_1\cdots\mu_k|\alpha}$, second-order in its derivatives and displays the mixed symmetry $(k,1)$. Its concrete expression reads as

$$\begin{aligned} T^{\nu_1\cdots\nu_k|\alpha} = {} & \Box t^{\nu_1\cdots\nu_k|\alpha} + \partial_\mu\left((-)^k\,\partial^{[\nu_1}t^{\nu_2\cdots\nu_k]\mu|\alpha} - \partial^\alpha t^{\nu_1\cdots\nu_k|\mu}\right) \\ & + (-)^{k+1}\,\partial^\alpha\partial^{[\nu_1}t^{\nu_2\cdots\nu_k]} + \sigma^{\alpha[\nu_1}\left[(-)^k\,\Box t^{\nu_2\cdots\nu_k]}\right. \end{aligned}$$

$$\partial_\mu \left( (-)^{k+1} \partial_\beta t^{\nu_2 \cdots \nu_k]\mu|\beta} - \partial^{\nu_2} t^{\nu_3 \cdots \nu_k]\mu} \right) \Big] . \tag{38}$$

It is also interesting to compute $T^{\nu_1 \cdots \nu_k|\alpha}$ in terms of $F_{\mu_1 \cdots \mu_{k+1}|\alpha}$ from (15)

$$T^{\nu_1 \cdots \nu_k|\alpha} = \partial_\mu F^{\mu\nu_1 \cdots \nu_k|\alpha} + (-)^{k+1} \partial^\alpha F^{\nu_1 \cdots \nu_k} - \sigma^{\alpha[\nu_1} \partial_\mu F^{\nu_2 \cdots \nu_k]\mu}. \tag{39}$$

In a somehow abusive language we will name the components of this tensor the Euler-Lagrange (E.L.) derivatives of the action (17). The trace of $T^{\nu_1 \cdots \nu_k|\alpha}$ will be denoted by $T^{\nu_1 \cdots \nu_{k-1}}$, being understood that it is defined in a manner similar to (4). Its expression in terms of $t$ and respectively $F$ reads as

$$\begin{aligned} T^{\nu_1 \cdots \nu_{k-1}} &= (k+1-D)\, \partial_\mu F^{\mu\nu_1 \cdots \nu_{k-1}} \qquad (40) \\ &= (k+1-D)\, [\Box t^{\nu_1 \cdots \nu_{k-1}} \\ &\quad + \partial_\mu \left( (-)^{k-1} \partial^{[\nu_1} t^{\nu_2 \cdots \nu_{k-1}]\mu} - \partial_\alpha t^{\nu_1 \cdots \nu_{k-1}\mu|\alpha} \right) \Big] . \qquad (41) \end{aligned}$$

The gauge invariance of the Lagrangian action (17) under the transformations (5) is equivalent to the fact that the functions defining the field equations are not all independent, but rather obey the Noether identities

$$\partial_{\nu_1} T^{\nu_1 \cdots \nu_k|\alpha} \equiv 0, \;\; \partial_\alpha T^{\nu_1 \cdots \nu_k|\alpha} \equiv 0, \tag{42}$$

while the presence of the reducibility shows that not all of the above Noether identities are independent. As a consequence, the trace of $T^{\nu_1 \cdots \nu_k|\alpha}$ also verifies the "Noether identity"

$$\partial_{\nu_1} T^{\nu_1 \cdots \nu_{k-1}} \equiv 0. \tag{43}$$

It can be checked that the functions (38) defining the field equations, the gauge generators, as well as all the reducibility functions, satisfy the general regularity assumptions from [26], such that the model under discussion is described by a normal gauge theory of Cauchy order equal to $(k+1)$.

# 3 Curvature tensor. Relationship with the Lagrangian formulation

It can be shown by direct computation that the most general gauge-invariant quantities constructed out of the tensor field $t_{\mu_1 \cdots \mu_k|\alpha}$ and its spacetime derivatives is the curvature tensor

$$K_{\mu_1 \cdots \mu_{k+1}|\alpha\beta} = \partial_\alpha F_{\mu_1 \cdots \mu_{k+1}|\beta} - \partial_\beta F_{\mu_1 \cdots \mu_{k+1}|\alpha} = \partial_{[\mu_1} t_{\mu_2 \cdots \mu_{k+1}]|[\beta,\alpha]}, \tag{44}$$

together with its spacetime derivatives of all orders. In the above we used the standard notation $f_{,\mu} \equiv \partial_\mu f$ and the definition (9). The curvature tensor is linear in the field $t_{\mu_1 \cdots \mu_k|\alpha}$, second-order in its spacetime derivatives and transforms in an irreducible representation of $GL(D, \mathbb{R})$ corresponding to a Young diagram with $(k+3)$ cells, two columns and $(k+1)$ rows

$$K_{\mu_1 \cdots \mu_{k+1}|\alpha\beta} = \begin{array}{cc} \mu_1 & \alpha \\ \mu_2 & \beta \\ \vdots & \\ \mu_{k+1} & \end{array} . \tag{45}$$

This means that it is separately antisymmetric in its first $(k+1)$ indices and respectively last two indices and obeys the (algebraic) Bianchi I identity

$$K_{\left[\mu_1\cdots\mu_{k+1}|\alpha\right]\beta}\equiv 0. \tag{46}$$

The curvature tensor satisfies the (differential) Bianchi II identities

$$\partial_{[\mu_1}K_{\mu_2\cdots\mu_{k+2}]|\alpha\beta}\equiv 0,\quad K_{\mu_1\cdots\mu_{k+1}|[\alpha\beta,\gamma]}\equiv 0. \tag{47}$$

In what follows we will need the trace and double trace of the curvature tensor. By direct computation it follows that (we express both kinds of traces both in terms of $F$ and of $t$)

$$\begin{aligned}
K_{\mu_1\cdots\mu_k|\alpha} &\equiv K_{\mu_1\cdots\mu_{k+1}|\alpha\beta}\sigma^{\mu_{k+1}\beta}=\partial_\alpha F_{\mu_1\cdots\mu_k}+(-)^{k+1}\partial^\beta F_{\beta\mu_1\cdots\mu_k|\alpha} && (48)\\
&= (-)^{k+1}\Box t_{\mu_1\cdots\mu_k|\alpha}-\partial^\beta\partial_{[\mu_1}t_{\mu_2\cdots\mu_k]\beta|\alpha}\\
&\quad +\partial_\alpha\left(\partial_{[\mu_1}t_{\mu_2\cdots\mu_k]}+(-)^k\partial^\beta t_{\mu_1\cdots\mu_k|\beta}\right), && (49)
\end{aligned}$$

$$\begin{aligned}
K_{\mu_1\cdots\mu_{k-1}} &\equiv K_{\mu_1\cdots\mu_k|\alpha}\sigma^{\mu_k\alpha}=2(-)^{k-1}\partial^\alpha F_{\alpha\mu_1\cdots\mu_{k-1}}= && (50)\\
&\quad 2(-)^{k-1}\left(\Box t_{\mu_1\cdots\mu_{k-1}}+(-)^{k-1}\partial^\alpha\partial_{[\mu_1}t_{\mu_2\cdots\mu_{k-1}]\alpha}\right.\\
&\quad \left.-\partial^\alpha\partial^\beta t_{\mu_1\cdots\mu_{k-1}\alpha|\beta}\right). && (51)
\end{aligned}$$

The Bianchi II identities (47) for the curvature tensor induce some corresponding identities at the level of its traces. Indeed, taking repeatedly the possible contractions of both formulas from (47) we find the following identities

$$\begin{aligned}
\partial_{[\mu_1}K_{\mu_2\cdots\mu_{k+1}]|\alpha}-(-)^k\partial^\beta K_{\mu_1\cdots\mu_{k+1}|\alpha\beta} &\equiv 0, && (52)\\
K_{\mu_1\cdots\mu_k|[\beta,\alpha]}-(-)^{k-1}\partial^\gamma K_{\gamma\mu_1\cdots\mu_k|\alpha\beta} &\equiv 0, && (53)\\
\partial_{[\mu_1}K_{\mu_2\cdots\mu_k]}-2(-)^{k+1}\partial^\alpha K_{\mu_1\cdots\mu_k|\alpha} &\equiv 0, && (54)\\
\partial_\alpha K_{\mu_1\cdots\mu_{k-1}}-2(-)^{k+1}\partial^\gamma K_{\gamma\mu_1\cdots\mu_{k-1}|\alpha} &\equiv 0, && (55)\\
\partial^\gamma K_{\gamma\mu_1\cdots\mu_{k-2}} &\equiv 0. && (56)
\end{aligned}$$

We complete our discussion by displaying the connection of the functional derivatives of the Lagrangian action $S^{\mathrm{L}}\left[t_{\mu_1\cdots\mu_k|\alpha}\right]$, namely $T^{\nu_1\cdots\nu_k|\alpha}$, and its trace (see formulas (38)–(41)) to the curvature tensor and its traces and also the relationship between the Noether identities (42)–(43) and the Bianchi II identities (47) and (52)–(56). In view of this, we start from formula (39) and employ relations (48) and (50), which then yield

$$\begin{aligned}
T^{\nu_1\cdots\nu_k|\alpha} &= (-)^{k+1}K^{\nu_1\cdots\nu_k|\alpha}-\frac{1}{2}\sigma^{\alpha[\nu_1}K^{\nu_2\cdots\nu_k]}, && (57)\\
T^{\nu_1\cdots\nu_{k-1}} &= \frac{(-)^k}{2}(D-k-1)K^{\nu_1\cdots\nu_{k-1}}. && (58)
\end{aligned}$$

First, from (57) we notice that the field equations (37) are completely equivalent with the vanishing of the simple trace of the curvature tensor

$$T^{\nu_1\cdots\nu_k|\alpha}\approx 0\Longleftrightarrow K^{\nu_1\cdots\nu_k|\alpha}\approx 0. \tag{59}$$

Further, from (57) and (58) we compute the left-hand sides of the Noether identities (42) and (43) and find that

$$\partial_\lambda T^{\lambda\nu_1\cdots\nu_{k-1}|\alpha} = -\frac{1}{2}\left(\partial^\alpha K^{\nu_1\cdots\nu_{k-1}} + 2\,(-)^k\,\partial_\lambda K^{\lambda\nu_1\cdots\nu_{k-1}|\alpha}\right)$$

$$+\frac{1}{2}\,(-)^k\,\sigma^{\alpha[\nu_1}\partial_\lambda K^{\nu_1\cdots\nu_{k-1}]\lambda} \equiv 0, \tag{60}$$

$$\partial_\alpha T^{\nu_1\cdots\nu_k|\alpha} = \frac{1}{2}\left(2\,(-)^{k+1}\,\partial_\alpha K^{\nu_1\cdots\nu_k|\alpha} - \partial^{[\nu_1}K^{\nu_2\cdots\nu_k]}\right) \equiv 0, \tag{61}$$

$$\partial_\lambda T^{\lambda\nu_1\cdots\nu_{k-2}} = \frac{(-)^{k+1}}{2}\,(k+1-D)\,\partial_\lambda K^{\lambda\nu_1\cdots\nu_{k-2}} \equiv 0. \tag{62}$$

The identical vanishing of all the above quantities is guaranteed by the identities (52)–(56) satisfied by the traces of the curvature tensor. In conclusion, we can state that the Bianchi II identities for the curvature tensor enforces the Noether identities (42) for action (17).

The Lagrangian formulation of the free, massless tensor field based on action (17) may be interpreted in terms of an operator $\bar{d}$ nilpotent of order three, $\bar{d}^3 = 0$, which acts in the vector space $\Omega_2(\mathcal{M})$ of tensor fields with mixed symmetries corresponding to some two-column Young diagrams, defined on a pseudo-Riemannian manifold $\mathcal{M}$ of dimension $D$. Both the curvature tensor and the original tensor field $t_{\mu_1\cdots\mu_k|\alpha}$ belong to this vector space and are correlated via the operator $\bar{d}$. This correlation is expressed via the proportionality relation $K_{\mu_1\cdots\mu_{k+1}|\alpha\beta} \sim \left(\bar{d}^2 t\right)_{\mu_1\cdots\mu_{k+1}\alpha\beta}$. The Bianchi II identities (47) follow from the third-order nilpotency of $\bar{d}$, $\bar{d}K \sim \bar{d}^3 t \equiv 0$. We remark that formula (57) connects the field equations (37) to the curvature tensor through a generalized Hodge duality. In the case where $\mathcal{M}$ is endowed with the trivial topology of $\mathbb{R}^D$, then the generalized cohomology of $\bar{d}$ in $\Omega_2(\mathcal{M})$ is related to some interesting aspects of the free model under study. For instance, if $\bar{T}^{\mu_1\cdots\mu_k|\alpha}$ is a covariant tensor field with the mixed symmetry of a Young diagram with $(k+1)$ cells, two columns and $k$ rows, which in addition satisfies the equations

$$\partial_{\mu_1}\bar{T}^{\mu_1\cdots\mu_k|\alpha} = 0, \qquad \partial_\alpha\bar{T}^{\mu_1\cdots\mu_k|\alpha} = 0, \tag{63}$$

then there exists a tensor field $\bar{\Phi}^{\mu_1\cdots\mu_{k+1}|\alpha\beta}$ with the mixed symmetry $(k+1,2)$ of the curvature tensor such that

$$\bar{T}^{\mu_1\cdots\mu_k|\alpha} = \partial_\xi\partial_\beta\bar{\Phi}^{\xi\mu_1\cdots\mu_k|\alpha\beta}. \tag{64}$$

This result in particularly useful when $\bar{T}^{\mu_1\cdots\mu_k|\alpha} = T^{\mu_1\cdots\mu_k|\alpha}$ from the field equations (37).

# 4 BRST symmetry

In agreement with the general setting of the antibracket-antifield formalism, the construction of the BRST symmetry for the free theory under consideration starts with the identification of the BRST algebra on which the BRST differential $s$ acts. The generators of the BRST algebra are of two kinds: fields/ghosts and antifields. The ghost spectrum for the model under study comprises the tensor fields

$$\left(\overset{(m+1)}{C}{}_{\mu_1\cdots\mu_{k-m-1}|\alpha}, \overset{(m+1)}{\eta}{}_{\mu_1\cdots\mu_{k-m}}\right)_{m=\overline{0,k-2}}, \overset{(k)}{\eta}{}_\mu \tag{65}$$

with the Grassmann parities

$$\varepsilon\left(\overset{(m+1)}{C}_{\mu_1\cdots\mu_{k-m-1}|\alpha}\right) = (m+1)\ \mathrm{mod}\ 2,\ m=\overline{0,k-2}, \tag{66}$$

$$\varepsilon\left(\overset{(m+1)}{\eta}_{\mu_1\cdots\mu_{k-m}}\right) = (m+1)\ \mathrm{mod}\ 2,\ m=\overline{0,k-2}, \tag{67}$$

$$\varepsilon\left(\overset{(k)}{\eta}_{\mu}\right) = k\ \mathrm{mod}\ 2. \tag{68}$$

The fermionic ghosts $\overset{(1)}{C}_{\mu_1\cdots\mu_{k-1}|\alpha}$ and $\overset{(1)}{\eta}_{\mu_1\cdots\mu_k}$ are associated with the gauge parameters $\overset{(1)}{\theta}_{\mu_1\cdots\mu_{k-1}|\alpha}$ and $\overset{(1)}{\epsilon}_{\mu_1\cdots\mu_k}$ from the transformations (5), while the rest of the ghosts are due to the reducibility parameters of various orders that appear in the relations (18)–(19), (21)–(22), (25)–(26), and (29)–(30). In order to make compatible the behavior of the ghosts with that of the gauge and reducibility parameters, we ask that $\overset{(m+1)}{C}_{\mu_1\cdots\mu_{k-m-1}|\alpha}$ for $m=\overline{0,k-2}$ display the mixed symmetry $(k-m-1,1)$, so they are separately antisymmetric in the first $(k-m-1)$ indices and satisfy the identities

$$\overset{(m+1)}{C}_{[\mu_1\cdots\mu_{k-m-1}|\alpha]} \equiv 0,\ m=\overline{0,k-2}, \tag{69}$$

while $\overset{(m+1)}{\eta}_{\mu_1\cdots\mu_{k-m}}$ for $m=\overline{0,k-2}$ are completely antisymmetric. The antifield spectrum is organized into the antifields

$$t^{*\mu_1\cdots\mu_k|\alpha},\ \left(\overset{(m+1)}{C}{}^{*\mu_1\cdots\mu_{k-m-1}|\alpha},\ \overset{(m+1)}{\eta}{}^{*\mu_1\cdots\mu_{k-m}}\right)_{m=\overline{0,k-2}},\ \overset{(k)}{\eta}{}^{*\mu} \tag{70}$$

corresponding to the original tensor field and to the ghosts, of statistics opposite to that of the associated fields/ghosts

$$\varepsilon\left(t^{*\mu_1\cdots\mu_k|\alpha}\right) = 1, \tag{71}$$

$$\varepsilon\left(\overset{(m+1)}{\eta}{}^{*\mu_1\cdots\mu_{k-m}}\right) = m\ \mathrm{mod}\ 2 = \varepsilon\left(\overset{(m+1)}{C}{}^{*\mu_1\cdots\mu_{k-m-1}|\alpha}\right), \tag{72}$$

$$\varepsilon\left(\overset{(k)}{\eta}{}^{*\mu}\right) = (k+1)\ \mathrm{mod}\ 2. \tag{73}$$

Obviously, the antifields exhibit the same mixed symmetry/antisymmetry properties like the associated field/ghosts. In particular, this means that they satisfy the identities

$$t^{*[\mu_1\cdots\mu_k|\alpha]} \equiv 0,\ \overset{(m+1)}{C}{}^{*[\mu_1\cdots\mu_{k-m-1}|\alpha]} \equiv 0,\ m=\overline{0,k-2}. \tag{74}$$

As both the gauge generators and reducibility functions for this model are field-independent, it follows that the associated BRST differential ($s^2=0$) splits into

$$s = \delta + \gamma, \tag{75}$$

where $\delta$ represents the Koszul-Tate differential ($\delta^2=0$), graded by the antighost number agh ($\mathrm{agh}\,(\delta)=-1$), while $\gamma$ stands for the exterior derivative along the gauge orbits and turns out to be a true differential ($\gamma^2 = 0$) that anticommutes with $\delta$ ($\delta\gamma + \gamma\delta = 0$), whose degree is named pure ghost number pgh ($\mathrm{pgh}(\gamma) = 1$). These two degrees do not interfere ($\mathrm{agh}(\gamma) = 0$, $\mathrm{pgh}(\delta) = 0$). The overall degree that grades the BRST differential is known as the ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that $\mathrm{gh}(s) = \mathrm{gh}(\delta) = \mathrm{gh}(\gamma) = 1$. According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued like

$$\mathrm{pgh}\left(t_{\mu_1\cdots\mu_k|\alpha}\right) = 0,\ \mathrm{pgh}\left(\overset{(k)}{\eta}_{\mu}\right) = k, \tag{76}$$

$$\mathrm{pgh}\left(\overset{(m+1)}{C}_{\mu_1\cdots\mu_{k-m-1}|\alpha}\right) = (m+1),\ m = \overline{0,k-2}, \tag{77}$$

$$\mathrm{pgh}\left(\overset{(m+1)}{\eta}_{\mu_1\cdots\mu_{k-m}}\right) = (m+1),\ m = \overline{0,k-2}, \tag{78}$$

$$\mathrm{pgh}\left(t^{*\mu_1\cdots\mu_k|\alpha}\right) = 0 = \mathrm{pgh}\left(\overset{(k)}{\eta}{}^{*\mu}\right), \tag{79}$$

$$\mathrm{pgh}\left(\overset{(m+1)}{C}{}^{*\mu_1\cdots\mu_{k-m-1}|\alpha}\right) = 0,\ m = \overline{0,k-2}, \tag{80}$$

$$\mathrm{pgh}\left(\overset{(m+1)}{\eta}{}^{*\mu_1\cdots\mu_{k-m}}\right) = 0,\ m = \overline{0,k-2}, \tag{81}$$

$$\mathrm{agh}\left(t_{\mu_1\cdots\mu_k|\alpha}\right) = 0 = \mathrm{agh}\left(\overset{(k)}{\eta}_{\mu}\right), \tag{82}$$

$$\mathrm{agh}\left(\overset{(m+1)}{C}_{\mu_1\cdots\mu_{k-m-1}|\alpha}\right) = 0,\ m = \overline{0,k-2}, \tag{83}$$

$$\mathrm{agh}\left(\overset{(m+1)}{\eta}_{\mu_1\cdots\mu_{k-m}}\right) = 0,\ m = \overline{0,k-2}, \tag{84}$$

$$\mathrm{agh}\left(t^{*\mu_1\cdots\mu_k|\alpha}\right) = 1,\ \mathrm{agh}\left(\overset{(k)}{\eta}{}^{*\mu}\right) = k+1, \tag{85}$$

$$\mathrm{agh}\left(\overset{(m+1)}{C}{}^{*\mu_1\cdots\mu_{k-m-1}|\alpha}\right) = m+2,\ m = \overline{0,k-2}, \tag{86}$$

$$\mathrm{agh}\left(\overset{(m+1)}{\eta}{}^{*\mu_1\cdots\mu_{k-m}}\right) = m+2,\ m = \overline{0,k-2}, \tag{87}$$

while the actions of $\delta$ and $\gamma$ on them are given by

$$\gamma t_{\mu_1\cdots\mu_k|\alpha} = \partial_{[\mu_1}\overset{(1)}{C}_{\mu_2\cdots\mu_k]|\alpha} + \partial_{[\mu_1}\overset{(1)}{\eta}_{\mu_2\cdots\mu_k\alpha]} + (-)^{k+1}(k+1)\,\partial_\alpha \overset{(1)}{\eta}_{\mu_1\cdots\mu_k}, \tag{88}$$

$$\begin{aligned}\gamma \overset{(m)}{C}_{\mu_1\cdots\mu_{k-m}|\alpha} &= \partial_{[\mu_1}\overset{(m+1)}{C}_{\mu_2\cdots\mu_{k-m}]|\alpha} + \partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\cdots\mu_{k-m}\alpha]} \\ &+ (-)^{k-m+1}(k-m+1)\,\partial_\alpha \overset{(m+1)}{\eta}_{\mu_1\cdots\mu_{k-m}},\ m = \overline{1,k-2},\end{aligned} \tag{89}$$

$$\gamma \overset{(k-1)}{C}_{\mu_1|\alpha} = \partial_{(\mu_1}\overset{(k)}{\eta}_{\alpha)}, \tag{90}$$

$$\gamma \overset{(m)}{\eta}_{\mu_1\cdots\mu_{k-m+1}} = \frac{k-m}{k-m+2}\partial_{[\mu_1}\overset{(m+1)}{\eta}_{\mu_2\cdots\mu_{k-m+1}]},\ m = \overline{1,k-1}, \tag{91}$$

$$\gamma \overset{(k)}{\eta}_{\mu} = 0, \tag{92}$$

$$\gamma t^{*\mu_1 \cdots \mu_k|\alpha} = 0 = \gamma \overset{(m+1)}{C}{}^{*\mu_1 \cdots \mu_{k-m-1}|\alpha}, \; m = \overline{0, k-2}, \tag{93}$$

$$\gamma \left( \overset{(m+1)}{\eta}{}^{*\mu_1 \cdots \mu_{k-m}} \right) = 0, \; m = \overline{0, k-2}, \; \gamma \overset{(k)}{\eta}{}^{*\mu} = 0, \tag{94}$$

$$\delta t_{\mu_1 \cdots \mu_k|\alpha} = 0 = \delta \overset{(m)}{C}_{\mu_1 \cdots \mu_{k-m}|\alpha}, \; m = \overline{0, k-2}, \tag{95}$$

$$\delta \overset{(m)}{\eta}_{\mu_1 \cdots \mu_{k-m+1}} = 0, \; m = \overline{1, k-1}, \; \delta \overset{(k)}{\eta}_{\mu} = 0, \tag{96}$$

$$\delta t^{*\mu_1 \cdots \mu_k|\alpha} = -\frac{1}{k!} T^{\nu_1 \cdots \nu_k|\alpha}, \tag{97}$$

$$\delta \overset{(1)}{C}{}^{*\mu_1 \cdots \mu_{k-1}|\alpha} = -\partial_\lambda \left( k t^{*\lambda\mu_1 \cdots \mu_{k-1}|\alpha} + (-)^k t^{*\mu_1 \cdots \mu_{k-1}\alpha|\lambda} \right), \tag{98}$$

$$\begin{aligned} \delta \overset{(m)}{C}{}^{*\mu_1 \cdots \mu_{k-m}|\alpha} &= (-)^m \partial_\lambda \left( (k-m+1) \overset{(m-1)}{C}{}^{*\lambda\mu_1 \cdots \mu_{k-m}|\alpha} \right. \\ &\left. + (-)^{k-m+1} \overset{(m-1)}{C}{}^{*\mu_1 \cdots \mu_{k-m}\alpha|\lambda} \right), \; m = \overline{2, k-2}, \end{aligned} \tag{99}$$

$$\delta \overset{(k-1)}{C}{}^{*\mu_1|\alpha} = (-)^{k-1} \partial_\lambda \overset{(k-2)}{C}{}^{*\lambda(\mu_1|\alpha)}, \tag{100}$$

$$\delta \overset{(1)}{\eta}{}^{*\mu_1 \cdots \mu_k} = (-)^k (k+1) \partial_\alpha t^{*\mu_1 \cdots \mu_k|\alpha}, \tag{101}$$

$$\begin{aligned} \delta \overset{(m)}{\eta}{}^{*\mu_1 \cdots \mu_{k-m+1}} &= (-)^k (k-m+2) \partial_\alpha \overset{(m-1)}{C}{}^{*\mu_1 \cdots \mu_{k-m+1}|\alpha} \\ + (-)^m \frac{(k-m+2)(k-m+1)}{k-m+3} &\partial_\lambda \overset{(m-1)}{\eta}{}^{*\lambda\mu_1 \cdots \mu_{k-m+1}}, \; m = \overline{2, k}, \end{aligned} \tag{102}$$

with $T^{\nu_1 \cdots \nu_k|\alpha}$ like in (38) and both $\delta$ and $\gamma$ taken to act like right derivations.

The antifield-BRST differential is known to admit a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, $s\cdot = (\cdot, S)$, where $(,)$ signifies the antibracket and $S$ denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero involving both the field/ghost and antifield spectra, which obeys the classical master equation

$$(S, S) = 0. \tag{103}$$

The classical master equation is equivalent with the second-order nilpotency of $s$, $s^2 = 0$, while its solution encodes the entire gauge structure of the associated theory. Taking into account formulas (88)–(102), as well as the actions of $\delta$ and $\gamma$ in canonical form, we find that the complete solution to the master equation for the model under study reads as

$$\begin{aligned} S &= S^{\mathrm{L}} \left[ t_{\mu_1 \cdots \mu_k|\alpha} \right] + \int d^D x \left[ t^{*\mu_1 \cdots \mu_k|\alpha} \left( \partial_{[\mu_1} \overset{(1)}{C}_{\mu_2 \cdots \mu_k]|\alpha} \right. \right. \\ &\left. + \partial_{[\mu_1} \overset{(1)}{\eta}_{\mu_2 \cdots \mu_k \alpha]} + (-)^{k+1} (k+1) \partial_\alpha \overset{(1)}{\eta}_{\mu_1 \cdots \mu_k} \right) \end{aligned}$$

$$
+\overset{(k-1)}{C}{}^{*\mu_1|\alpha}\partial_{(\mu_1}\overset{(k)}{\eta}{}_{\alpha)}+\sum_{m=1}^{k-2}\overset{(m)}{C}{}^{*\mu_1\cdots\mu_{k-m}|\alpha}\left(\partial_{[\mu_1}\overset{(m+1)}{C}{}_{\mu_2\cdots\mu_{k-m}]|\alpha}+\right.
$$

$$
\left.+\partial_{[\mu_1}\overset{(m+1)}{\eta}{}_{\mu_2\cdots\mu_{k-m}\alpha]}+(-)^{k-m+1}(k-m+1)\,\partial_\alpha\overset{(m+1)}{\eta}{}_{\mu_1\cdots\mu_{k-m}}\right)
$$

$$
\left.+\sum_{m=1}^{k-1}\frac{k-m}{k-m+2}\overset{(m)}{\eta}{}^{*\mu_1\cdots\mu_{k-m+1}}\partial_{[\mu_1}\overset{(m+1)}{\eta}{}_{\mu_2\cdots\mu_{k-m+1}]}\right]. \tag{104}
$$

The main ingredients of the antifield-BRST symmetry derived here will be useful at the analysis of the BRST cohomology for the free, massless tensor field with the mixed symmetry $(k,1)$.

# 5 Conclusion

To conclude with, in this paper we have derived the basic properties of the Lagrangian BRST formulation for a free, massless tensor field $t_{\mu_1\cdots\mu_k|\alpha}$ that transforms in an irreducible representation of $GL(D,\mathbb{R})$, corresponding to a two-column Young diagram with $(k+1)$ cells and $k\geq 2$ rows and constructed the associated antifield-BRST symmetry. The results derived here will be employed in a future paper at the computation of the local BRST cohomology of this model and hence at the construction of consistent interacting field theories containing such a tensor field in its spectrum.

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