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Generalized winding number in area preserving maps

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Abstract

For area preserving maps we introduce an operator whose spectrum give the classical rotation numbers or generalised rotation numbers in chaotic sea. We prove that some classical properties of rotation number survive as properties of this new operator.

1 Introduction

Magnetically confined controlled fusion devices, cyclic particle accelerators, storage rings, Saturn ring system are only few examples of man made or natural many particle physical systems with increased predictability of the statistical evolution. The increased predictability is a result either of sophisticated technological effort or by "natural selection". As a results, the mathematical models of these systems are very close to the integrable Hamiltonian systems and have a very complicated phase space structure and complex time behavior. Even if we restrict ourselves to smallest invariant components, there are models [1] which are only weakly mixing. It is well known that the absence of mixing prevents the relaxation to equilibrium distribution [2], [3], [4].

The general study of generic, non-integrable Hamiltonian systems is too complicated, even from numerical point of view. Consequently even in the classical mechanics we are lead to the idea to give up the ambition to study of individual trajectories and concentrate only on the long time behavior statistical properties of the families of trajectories whose initial conditions are not too sharply localized. This old statistical (metric) approach has some general similarities with differentiable dynamics [5].

Translated in the terminology of statistical physics, we restrict our study to the infinite time behavior of mean values, arising from initial conditions given by non-singular probability density functions (PDF). Both in statistical mechanics [2], [6] and ergodic theory [3],[4], exist a suitable Hilbert space approach. In this approach a central role plays the projector on the invariant PDF. The existence of this projector is given by von Neumann mean ergodic theorem [3],[4].

In both these approaches the PDF is assumed square integrable relative to the invariant (in many interesting cases Liouville) measure. Remark that for systems of finite measure square integrability implies integrability. The methods of kinetic theory, in particular the possibility of Markovian approximation, combined with the ergodic theory methods was used in the study of very chaotic regime of the standard map in [7] (references therein). The case of nearly integrable case, the so called weekly stochastic regime of the standard map, was also investigated [8].

Despite of huge complexity of the phase space portrait there are lot of regularities which must be explained and correlated. A central role play the set of invariants, the rotation numbers.

Winding or rotation numbers and related safety factor, rotational transform, play an important role and its applications in magnetic field line dynamics in controlled fusion [9], [10], [11], [12], [13] and in fundamental problems of classical Hamiltonian dynamics [14], [3]. The rotation numbers, are well defined for periodic orbits, island chains, cantories, invariant circles. Nevertheless the rotation numbers was not defined globally. The main obstacle is because in the chaotic sea, generically there exist a denumerable dense set of periodic points chains, with different rotation numbers. Other category of counter example is presented in [14].

By our statistical or Hilbert space approach, this gap is filled. We will associate to our physical system, modelled, for sake of simplicity by area preserving map, a whole family of invariant mathematical objects: operators and functions, which generalizes the classical rotation numbers. We will explore how the classical properties of the rotation number can be generalized and how to compute.

Our approach is close to [15], where the mean angular velocity of an infinitesimal vector is defined in statistical sense. Our definition will be a natural extension of definitions used in Hamiltonian dynamics and tokamak physics.

2 Preliminaries

Suppose that the topology of the phase space of the systems under investigation is similar to the structure of integrable systems.

For simplicity, we consider area-preserving maps

$$\mathbf{T} : \mathbf{x}_n = (p_n, q_n) \longrightarrow \mathbf{x_{n+1}} = (p_{n+1}, q_{n+1}) = \mathbf{T}(p_n, q_n)$$

obtained from integration of time periodic Hamiltonian system.

More explicitly $q_{n+1} = q_n + a(p_n, q_n)$ and suppose that $a(p, q) = a(\mathbf{x})$ (called *angular function* [20]) is periodic in q and continuous. Suppose also that the Hamilton function is periodic in q and by integration we obtained the true value, not only the fractional part, of the angular function. Under these conditions we can consider the map $\mathbf{T}$ acting on the cylinder $C = S^1 \times \mathbb{R}$. Suppose for technical reasons that there is sub-domain D of the cylinder C which is invariant under $\mathbf{T}$, limited by two invariant circles. The cyclic, angle variables q, are supposed to be normalized to 1.

All of our results can be extended easily to many degrees of freedom Hamiltonian systems, discrete or continuous time, even subjected to random perturbations.

If we measure the angle by another variable Q, related to the old variable by $Q = q + s(p, q)$ where $s(p, q)$ is continuous and periodic in q then $Q_{n+1} = Q_n + A(p_n, q_n)$ and the new, re-parametrized angular function is given by

$$A(\mathbf{x}) = a(\mathbf{x}) + s(\mathbf{T}(\mathbf{x})) - s(\mathbf{x}) \tag{1}$$

The case of general Hamiltonian maps can be treated in similar manner. Consider the standard Hilbert space $H = L^2(D, dpdq)$, of the square integrable functions in D, with the scalar product:

$$\langle \varphi, \psi \rangle = \int_D \varphi^*(\mathbf{x}).\psi(\mathbf{x}) dpdq \tag{2}$$

The unitary Koopman operator $\mathbf{U}$ formalizes the unit time evolution of observable, like coordinates and their functions. The action over a vector, is defined by

$$\mathbf{U}\psi(\mathbf{x}) = \psi(\mathbf{T}(\mathbf{x})) \tag{3}$$

The adjoint $\mathbf{U}^{+}$ is the unit time evolution operator for probability or particle density functions. To any bounded and continuous function $b(\mathbf{x})$ in we associate a bounded multiplication operator $\mathbf{b}$ defined by

$$(\mathbf{b}\psi)(\mathbf{x}) = b(\mathbf{x}).\psi(\mathbf{x}) \tag{4}$$

If $\mathbf{b}'$ is the operator defined by

$$(\mathbf{b}'\psi)(\mathbf{x}) = b(\mathbf{T}(\mathbf{x})).\psi(\mathbf{x})$$

then

$$\mathbf{b}' = \mathbf{U}\mathbf{b}\mathbf{U}^{-1} \tag{5}$$

Denote by $\mathbf{a}$, $\mathbf{A}$ and $\mathbf{s}$ the multiplication operators associated to $a(\mathbf{x})$ and $A(\mathbf{x})$ and $s(\mathbf{x})$. Then 1 can be rewritten as

$$\mathbf{A} = \mathbf{a} + \mathbf{U}\mathbf{s}\mathbf{U}^{-1} - \mathbf{s} \tag{6}$$

Denote by H_{inv}, the subspace of H spanned by the invariant vectors, under the action of $\mathbf{U}$ and by $H_{\perp}$ its ortogonal complement. By definition, $\psi \in H_{inv}$ if and only if $\mathbf{U}\psi = \psi$, or $\psi(\mathbf{T}(\mathbf{x})) = \psi(\mathbf{x})$. Denote by $\mathbf{P}$, the projector over H_{inv} . By von Neumann mean ergodic theorem $\mathbf{P}$ is given by [3], [4]

$$\mathbf{P}\psi = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{U}^k \psi = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{U}^{-k} \psi \tag{7}$$

and both limits exists for all $\psi \in H$, understood as Hilbert space limits. In similar manner we introduce the projector $\mathbf{P}^{(n)}$ which projects on the subspace of functions invariants under $\mathbf{U}^n$.

Clearly

$$\mathbf{P}^{(n)}\mathbf{U}^{\pm n} = \mathbf{U}^{\pm n}\mathbf{P}^{(n)} = \mathbf{P}^{(n)} = \mathbf{P}^{(-n)} \tag{8}$$

$$\mathbf{P} \triangleq \mathbf{P}^{(1)} = \frac{1}{|n|} \sum_{k=0}^{|n|-1} \mathbf{U}^k \mathbf{P}^{(n)}$$

and $\mathbf{P}^{(n)}\ \mathbf{U} = \mathbf{U}\mathbf{P}^{(n)}$.

From the definition of $\mathbf{P}^{(n)}$ results the following simple relation

$$\mathbf{P}^{(n)}\mathbf{P}^{(m)} = \mathbf{P}^{(d)}$$

where d is the greatest common divisor of positive integers m and n.

The projector $\mathbf{P}$ itself is very interesting for statistical studies of Hamiltonian systems. Suppose that $\rho(\mathbf{x})$ is a (square integrable) particle density function, then $\mathbf{U}^{-t}\rho$ gives it time evolution $\langle f, \mathbf{U}^{-t}\rho\rangle$ is the phase-space average at time t of the function $f(\mathbf{x})$. So $\langle f, \mathbf{P}\rho\rangle$ is the phase space and time average of the function f $(\mathbf{x})$, which is well defined, irrespective on the existence of equilibrium state. For instance if $f(\mathbf{x})$ is equal to the characteristic function of a sub domain, then $\langle f, \mathbf{P}\rho\rangle$ is the mean visiting frequency. For continuous time systems we must only replace the summation by integral.

The matrix elements $\langle f, \mathbf{P}\rho\rangle$ of the projector $\mathbf{P}$ can be computed if we can compute the diagonal elements $\langle \psi, \mathbf{P}\psi\rangle$. We need the following

Lemma 1 *[4]: Any $\chi \in H_\perp$ can be approximated to any precision ε by vectors of the form $\chi_\varepsilon = (\mathbf{U} - \mathbf{1})g_\varepsilon$, where g_ε is a vector from H.*

Then we can prove the following variational principle.

Proposition 2 *The matrix elements $\langle \psi, \mathbf{P}\psi\rangle$ are given by*

$$\langle \psi, \mathbf{P}\psi\rangle = \inf_{g\in H} \|\psi - (\mathbf{U} - \mathbf{1})g\|^2 \tag{9}$$

Proof. $\langle \psi, \mathbf{P}\psi\rangle = \|\mathbf{P}\psi\|^2 = \inf_{\chi\in H\perp} \|\psi - \chi\|^2$. Then by previous lemma we obtain 9.

Remark 1 *From the previous proposition we can obtain the following result for general autonomous Hamiltonian systems, considered as limiting case of discrete time systems:*

$$\langle \psi, \mathbf{P}\psi\rangle = \inf_{g\in H_d} \|\psi - \{\mathcal{H}, g\}\|^2$$

Proposition 3 *where $\mathcal{H}$ is the Hamilton function and H_d is the subspace of continuously differentiable functions, $\{\mathcal{H}, g\}$ is the Poisson bracket.*

3 Generalized rotation numbers, algebraic properties

We define our basic operator $\mathbf{\Omega}$ by

$$\mathbf{\Omega} = \mathbf{PaP} \tag{10}$$

It is invariant under re-parametrization. Indeed, from 1 we obtain

$$\mathbf{A} = \mathbf{a} + \mathbf{UsU}^{-1} - \mathbf{s}$$

and if we define the re-parametrized operator $\mathbf{\Omega}'= \mathbf{PAP}$ from 8 results $\mathbf{\Omega}'= \mathbf{\Omega}$.

Its action on invariant function $\psi(\mathbf{x}) \in H_{inv}$ is gives another invariant function $\psi' = \mathbf{\Omega}\psi \in H_{inv}$ given by

$$\psi'(\mathbf{x}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} a(\mathbf{T}^{(k)}(\mathbf{x}))\psi(\mathbf{x}) \tag{11}$$

Here $\mathbf{T}^{(k)}(\mathbf{x})$ is the $k-th$ iterate of the map $\mathbf{T}(\mathbf{x})$. Thus the restriction of $\mathbf{\Omega}_1$ to H_{inv} defines an invariant, bounded almost everywhere (a.e.) function $\omega(p,q) = \omega_1(\mathbf{x})$, given by

$$\omega(\mathbf{x}) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} a(\mathbf{T}^{(k)}(\mathbf{x})) \tag{12}$$

The limits in 7, 11, 12 exists a.e.. The invariant function $\omega_1(\mathbf{x})$ is constant a.e. in the ergodic components of the chaotic domain. If $\mathbf{x}$ belongs to invariant circles, then the limit always exists and equals to the classical rotation number. Observe that even if the map $\mathbf{T}$ satisfies the twist condition, the function $\omega_1(\mathbf{x})$ is not necessarily monotone with respect to variation of p. Clearly $\omega_1(\mathbf{x})$ is invariant under angular re-parametrization.

I a similar manner if we replace $\mathbf{T}$ by $\mathbf{T}^{(n)}$ the corresponding angular function $a^{(n)}(\mathbf{x}) = \sum_{k=0}^{n-1} a(\mathbf{T}^{(k)}(\mathbf{x}))$ for $n > 0$, $a_{-1}(\mathbf{x}) = -a_1(\mathbf{T}^{(-1)}(\mathbf{x}))$ and we define

$$\mathbf{\Omega}^{(n)} = \mathbf{P}^{(n)}\mathbf{a}^{(n)}\mathbf{P}^{(n)}$$

We have $\mathbf{\Omega}^{(n)}\mathbf{U}^{\pm n} = \mathbf{U}^{\pm n}\mathbf{\Omega}^{(n)} = \mathbf{\Omega}^{(n)}$ and

$$\mathbf{\Omega}^{(n)}\mathbf{U} = \mathbf{U}\mathbf{\Omega}^{(n)} \tag{13}$$

The corresponding $\omega^{(n)}(\mathbf{x})$ function, invariant under $\mathbf{T}^{(n)}$, due to 13 is invariant also under $\mathbf{T}$: $\omega^{(n)}(\mathbf{x}) = \omega^{(n)}(\mathbf{T}(\mathbf{x}))$ and from 12 we obtain

$$\omega^{(n)}(\mathbf{x}) = n\omega^{(1)}(\mathbf{x})$$

for all n ($\mathbf{T}^{(0)}(\mathbf{x}) = \mathbf{x}$). This is a remnant of the corresponding property of the classical rotation number. Moreover, by simple algebra we obtain:

$$n\mathbf{\Omega} = \mathbf{\Omega}^{(n)} \frac{1}{|n|} \sum_{k=0}^{|n|-1} \mathbf{U}^k$$

In the case of systems with many degree of freedom we can define a whole algebra of rotation number operators and functions with respect to periodic points of different periodicity.

4 Continuity

4.1 Reduction to the continuity of projectors

In the classical case, small modification of the map produces small variations of the rotation number [16]. In our general case the situation is more complicated. Suppose there is a family of maps $\mathbf{T}_\xi$, depending on the parameter ξ and approximating the map $\mathbf{T} = \mathbf{T}_0$. Denote by $\mathbf{U}_\xi$, $\mathbf{P}_\xi$ and $\mathbf{\Omega}_\xi$ the associated Koopman operators, projectors and rotation number operators. We say that $\mathbf{T}_\xi \to \mathbf{T} = \mathbf{T}_0$ if for every $\psi \in H$ we have $\|(\mathbf{U}_\xi - \mathbf{U})\psi\| \to 0$ when $\xi \to 0$ (strong convergence of $\mathbf{U}_\xi$) [17][18]). We will suppose moreover that associated angular functions a_ξ converges uniformly to a, consequently

$$\lim_{\xi\to 0} \|\mathbf{a}_\xi - \mathbf{a}\| = 0 \tag{14}$$

In these conditions we have the following

Lemma 4 *If for some $\psi \in H$ we have $\lim_{\xi\to 0}\|(\mathbf{P}_\xi - \mathbf{P})\psi\| = 0$ (strong convergence of $\mathbf{P}_\xi$) then for every $\psi, \varphi \in H$, $\mathbf{\Omega}_\xi$ converges weakly to $\mathbf{\Omega}$, i.e. $\lim_{\xi\to 0}\langle\varphi, (\mathbf{\Omega}_\xi - \mathbf{\Omega})\psi\rangle = 0$.*

By linearity it is sufficient to proof the four diagonal matrix elements. Suppose $\|\psi\| = 1$ for simplicity. We obtain, by 10 we obtain

$$\langle\psi, \mathbf{\Omega}_\xi\psi\rangle = \langle\psi, \mathbf{P}_\xi\mathbf{a}_\xi\mathbf{P}_\xi\psi\rangle = \langle\psi, \mathbf{P}_\xi(\mathbf{a}_\xi - \mathbf{a})\mathbf{P}_\xi\psi\rangle + \langle\psi, \mathbf{P}_\xi\mathbf{a}\mathbf{P}_\xi\psi\rangle \tag{15}$$

the first term vanishes in the limit $\xi \to 0$ because 14

$$\left|\langle\psi, \mathbf{P}_\xi(\mathbf{a}_\xi - \mathbf{a})\mathbf{P}_\xi\psi\rangle\right| \leq \|\mathbf{a}_\xi - \mathbf{a}\|\,\|\mathbf{P}_\xi\psi\|^2 \leq \|\mathbf{a}_\xi - \mathbf{a}\| \tag{16}$$

The second term from 15 can be rewritten as

$$\langle\psi, \mathbf{P}_\xi\mathbf{a}\mathbf{P}_\xi\psi\rangle = \langle\psi, \mathbf{P}_\xi(\mathbf{a}+m\mathbf{I})\mathbf{P}_\xi\psi\rangle - m\langle\psi, \mathbf{P}_\xi\mathbf{P}_\xi\psi\rangle \tag{17}$$

where the constant m was chosen such that $(\mathbf{a}+m\mathbf{I}) > \mathbf{0}$. Denoting $(\mathbf{a}+m\mathbf{I}) = \mathbf{r}^2$ from 15 and 17 we obtain

$$\lim_{\xi\to 0}\langle\psi, (\mathbf{\Omega}_\xi - \mathbf{\Omega})\psi\rangle = \lim_{\xi\to 0}\left[\langle\psi, \mathbf{P}_\xi\mathbf{r}^2\mathbf{P}_\xi\psi\rangle - \langle\psi, \mathbf{P}\mathbf{r}^2\mathbf{P}\psi\rangle\right] - m\lim_{\xi\to 0}\langle\psi, (\mathbf{P}_\xi - \mathbf{P})\psi\rangle \tag{18}$$

The second limit is zero because $|\langle\psi, (\mathbf{P}_\xi - \mathbf{P})\psi\rangle| \leq \|(\mathbf{P}_\xi - \mathbf{P})\psi\|$. For the first term we obtain

$$\left|\langle\psi, \mathbf{P}_\xi\mathbf{r}^2\mathbf{P}_\xi\psi\rangle - \langle\psi, \mathbf{P}\mathbf{r}^2\mathbf{P}\psi\rangle\right| = \left|\|\mathbf{r}\mathbf{P}_\xi\psi\|^2 - \|\mathbf{r}\mathbf{P}\psi\|^2\right| \leq K\left\|\mathbf{r}(\mathbf{P}_\xi - \mathbf{P})\psi\right\| \tag{19}$$

where K is a constant with $\|\mathbf{r}\mathbf{P}_\xi\psi\| + \|\mathbf{r}\mathbf{P}\psi\| \leq K$. Consequently from 19 we obtain

$$\lim_{\xi\to 0}|\langle\psi, (\mathbf{\Omega}_\xi - \mathbf{\Omega})\psi\rangle| \leq K\lim_{\xi\to 0}\left\|\mathbf{r}(\mathbf{P}_\xi - \mathbf{P})\psi\right\| \leq K\,\|\mathbf{r}\|\lim_{\xi\to 0}\left\|(\mathbf{P}_\xi - \mathbf{P})\psi\right\| = 0$$

which completes the proof.

4.2 Some general continuity results

Remains to investigate the limit $\lim_{\xi\to 0}\|(\mathbf{P}_\xi - \mathbf{P})\psi\|$. The following result is valid for general, abstract dynamically systems. We consider a general abstract, discrete dynamic system $(M, \mathcal{A}, \mu, \mathbf{T})$ where M is the phase space, with its family of measurable parts $\mathcal{A}$, $\mu(A)$ is the probability measure (in some particular cases the conserved Liouville measure, or the area) of the subset $A \in \mathcal{A}$ of M. $\mathbf{T}$ is a measure - preserving automorphism (i.e. one to one almost everywhere) [4]. Suppose that $\mu(M) = 1$ and $\mathbf{T}$ preserves the measure μ:

$$\mu(\mathbf{T}(A)) = \mu(A) \tag{20}$$

for all $A \in \mathcal{A}$. The scalar product is given in analogy to 2, according to [3] and [4]

$$\langle\psi, \varphi\rangle = \int_M \psi^*(x)\varphi(x)d\mu(x) \tag{21}$$

for $\psi(x), \varphi(x)$ from $H = L^2(M, \mu)$. The meaning of the operators $\mathbf{P}$, $\mathbf{U}$ and the subspace H_{inv} is the same as before.

Remark 2 *By triangle inequality we can study separately the cases when $\psi \in H_{inv}$ and $\psi \in H_{\perp}$, where $H_{\perp}$ is the subspace annihilated by $\mathbf{P}$.*

Remark 3 *Because $\mathbf{P}_\xi$ and $\mathbf{P}$ are projectors, then the strong convergence holds, iff $\mathbf{P}_\xi$ converges weakly to $\mathbf{P}$. More explicitly, the condition: for all $\psi \in H$, $\lim_{\xi\to 0} \|(\mathbf{P}_\xi - \mathbf{P})\psi\| = 0$ is equivalent to: for all $\psi \in H$ $\lim_{\xi\to 0}\langle\psi, (\mathbf{P}_\xi - \mathbf{P})\psi\rangle$.*

Lemma 5 *If $\psi \in H_\perp$,where $H_\perp$ is the subs then $\lim_{\xi\to 0} \|(\mathbf{P}_\xi - \mathbf{P})\psi\| = 0$.*

Proof. By lemma 1 there exist for each ε a vector $g_\varepsilon \in H$ such that $\|\psi - (\mathbf{U} - \mathbf{I})g_\varepsilon\| \leq \frac{\varepsilon}{2}$. Because $\mathbf{P}\psi = \mathbf{P}g_\varepsilon = 0$, we obtain

$$\|(\mathbf{P}_\xi - \mathbf{P})\psi\| = \|\mathbf{P}_\xi\psi\| \leq \|\mathbf{P}_\xi(\psi - (\mathbf{U} - \mathbf{I})g_\varepsilon)\| + \|\mathbf{P}_\xi(\mathbf{U} - \mathbf{I})g_\varepsilon\| \leqslant$$

$$\leq \frac{\varepsilon}{2} + \|\mathbf{P}_\xi(\mathbf{U}_\xi - \mathbf{I})g_\varepsilon\| + \|\mathbf{P}_\xi(\mathbf{U} - \mathbf{U}_\xi)g_\varepsilon\| \tag{22}$$

the second term in 22 is zero and for the third term we use the fact that a projector's norm is one. We get

$$\|(\mathbf{P}_\xi - \mathbf{P})\psi\| \leq \frac{\varepsilon}{2} + \|(\mathbf{U} - \mathbf{U}_\xi)g_\varepsilon\| \tag{23}$$

But by strong convergence of $\mathbf{U}_\xi$ there exist δ such that for all $|\xi| \leq \delta$ the inequality $\|(\mathbf{U} - \mathbf{U}_\xi)g_\varepsilon\| \leq \frac{\varepsilon}{2}$ is satisfied. This fact, together with 23 completes the proof.

4.3 Phase space portrait continuity

Now we study the limit

$$\lim_{\xi\to 0} \|(\mathbf{P}_\xi - \mathbf{P})\psi\| \tag{24}$$

when $\psi \in H_{inv}$. Observe that if ψ is an invariant functions then the sets $\psi = const$ give the phase space portrait. Observe that when the our dynamic system is ergodic, then H_{inv} contains only constant functions. In this case from $\psi \in H_{inv}$ it follows $\mathbf{P}_\xi\psi = \mathbf{P}\psi = \psi$ consequently from previous lemma we get

Corollary 6 *If a dynamic system is ergodic then for all $\psi \in H$ we have $\lim_{\xi\to 0} \|(\mathbf{P}_\xi - \mathbf{P})\psi\| = 0$, and for every $\psi, \varphi \in H$, $\mathbf{\Omega}_\xi$ converges weakly to $\mathbf{\Omega}$, i.e. $\lim_{\xi\to 0}\langle\varphi, (\mathbf{\Omega}_\xi - \mathbf{\Omega})\psi\rangle = 0$.*

In the continuation we will restrict ourselves to the area preserving maps and study the another extreme case. Suppose that the conditions of KAM theorem [3] are fulfilled. Consider some real, positive and bounded function $\psi \in H_{inv}$. Then $\mathbf{P}_\xi\psi$ decomposes in two components, one of them, ψ_ξ^s, with the support in stochastic part and another, ψ_ξ^r, with the support in the regular part. Both subspaces, spanned by all ψ_ξ^s , respectively ψ_ξ^r , are invariants under the action of $\mathbf{U}_\xi$

$$\mathbf{P}_\xi\psi = \psi_\xi^s + \psi_\xi^r \tag{25}$$

But due to the KAM theorem the measure of the stochastic domain is small and the value of $\mathbf{P}_\xi\psi$ is bounded by the same bound as ψ, it follows that $\lim_{\xi\to 0} \left\|\psi_\xi^s\right\| = 0$.

5 Conclusions

We proved that in the framework of the ergodic theory it is possible to extend the concept of the winding number and also ins inverse, the safety factor, outside of the ideal standard model of the nested magnetic surfaces. The winding number is defined almost everywhere and it is of L^{∞} class. This aspect is in contrast to the usual winding number, or safety factor profile function, that are supposed to be defined as a continuos and perhaps also differentiable function. In the weakly perturbed ideal magnetic field line structures the winding number on the undestroied invariant magnetic surfaces is identical to the classical winding number. On the ergodic domains it is defined almost everywhere (with respect to the invariant Liouville measure) and it is equivalent to constant function. Due to the persistence of the elliptic fixed points embedded in stable islands, originating from the destruction of rational magnetic surfaces, all points of these islands has the same winding number as the undestroied rational magnetic surfaces. By increasing the perturbation, in the ergodic domain will be embedded many islands corresponding to the different winding number. This phenomenon give rise to the apparent non monotone safety profile.

References

[1] G. M. Zaslavsky, M. Edelman, Chaos, **11**, 295-305 (2001).

[2] R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, Wiley, N.Y., 1975.

[3] V. I. Arnold, A. Avez, Ergodic Problems of Classical Mechanics, Benjamin, New York

[4] I. P. Kornfeld, Ya. G. Sinai, S. V. Fomin, Ergodic Theory, Science, Moscow 1980, (in Russian).

[5] Z. Nitecki, Differentiable Dynamics, M.I.T. Press, Cambridge, 1971.

[6] R. Balescu, Statistical Dynamics: Matter out of Equilibrium, Imperial College Press, 1997.

[7] R. Balescu, Journal of Statistical Physics, **98**, 1169 (2000).

[8] R. Balescu, Journal of Plasma Physics, **64** ,379, (2000).

[9] R. Balescu, Transport Processes in Plasma, (North Holland, Amsterdam, 1988)

[10] B. V. Chirikov, Plasma Physics **1**, 253 (1960).

[11] R. Balescu, M. Vlad, F. Spineanu, Phys. Rev. E **58**, 951, (1998).

[12] R. Balescu, Phys. Rev. **E 58 ,** 378 (1998).

[13] J. H. Misguich, Phys. of Plasmas **8**, 2132 (2001).

[14] R. S. MacKay, Renormalization in Area-Preserving Maps, World Scientific, 1993.

[15] Ludwig Arnold, Random Dynamical Systems, Springer Verlag, Berlin Heidelberg 1998.

[16] Chapitres Supplémentaires de la Théorie des Équations Différentielles Ordinaires, Mir, Moscow, 1980.

[17] P. R. Halmos, Approximation Theories for measure preserving transformations, Trans. Amer. Math. Soc. **55**, (1944), 1-18.

[18] S. Kakutani, Selected Works, vol 2, pp. 292-404, Birkhauser,1986.

[19] A. Ionescu-Tulcea, On the category of certain classes of transformations in ergodic theory, Trans. Amer. Math. Soc. **114**, (1965) 261-279.

[20] V. I. Arnold, V. V.Kozlov, A. I. Neichshtadt, Mathematical Aspects of Classical and Celestial Mechanics, Wiley, N.Y. 1995.

[21] K. Davidson, C*-Algebras by Example, Field Institute Monograph, Amer. Math. Soc., Providence, 1996.

[22] S. Kakutani, Selected Works, vol 2, pp. 150-164, Birkhauser,1986., 1968.

[23] R. Balescu, G. Steinbrecher, Perturbative kinetic theory study of the area preserving maps, Work presented at TEC Conference, Bruxelles, E.R.M.-K.M.S. ,14 November 2001.