Physics AUC, vol. 19, 45-58 (2009)

PHYSICS AUC

Transport of solid particles in fluids

Léon Brenig*
Université Libre de Bruxelles, Faculty of Sciences,
1050 Brussels, Belgium

Abstract

A theoretical framework is presented for the description of active transport of solid particles in turbulent fluids. Active transport exists when large enough particle concentration is present in the fluid and when the mass of particles is large enough for inertia effects to be non negligible. In such situations, particles trajectories differ from fluid flow lines, the reaction forces of particles on the fluid are important and modify the Navier-Stokes equation, collisions between solid particles are non negligible and are affected by the turbulent fluctuations of the fluid velocity field. As a consequence the Navier-Stokes turbulent hierarchy is coupled to an inelastic Boltzmann-like kinetic equation for the phase-space one particle distribution of the solid particles. Effects of fluid velocity field fluctuations on binary collisions of particles are discussed and shown to modify the collision term of the kinetic equation. Moreover, we show that the Lagrangian equations of motion for the particles in the fluid are fractional differential equations. This is due to the non-Markovian character of the hydrodynamical forces acting on particles dispersed in unsteady flows.

Keywords: fluid, turbulence, solid particles, transport
PACS: 47.55.Kf

1 Introduction

In frequent geophysical, laboratory and industrial processes, transport of solid particles by fluids can be qualified as passive. This refers to cases where the particles are light enough so that inertial effects can be neglected, i.e. the trajectories of the particles are almost identical to the flow lines of the fluid. Moreover, if the particles concentration is low enough their reactions on the flow can be neglected.

However, when the solid particles have larger mass and are present in the fluid in large concentration, new phenomena occur [1]. Their trajectories are different from the flow lines. The force exerted by the fluid on the particles contains memory effects due to the unsteadiness of the flow of the fluid around the particles. As we show in the present article, this deeply modifies the structure of the governing equations of that kind of system. Furthermore, in cases where high enough concentrations are achieved, reaction of the particles on the fluid in response to the forces exerted by the latter on them can no longer be neglected. Moreover, in case of high concentrations, collisions between particles are frequent and induce a time evolution of the particles phase-space distribution.

*E-mail address: lbrenig@ulb.ac.be

Talk given at International Workshop "Anomalous Transport in Fusion Plasmas", October 6-8, 2008, Craiova, Romania

Hydrodynamic forces between a particle and the fluid generally depend on the relative velocity of the particle with respect to the local fluid velocity. Hence, back reaction to these forces by the particles on the fluid depend on the velocity distribution of the particles, i.e. on their phase-space distribution (PSD). As shown below, the direct consequence of this is that the Navier-Stokes equation for the fluid will now depend on the solid particles velocity distribution and, thus, this equation becomes coupled to the kinetic equation governing the latter. This structure reflects the difference of levels occuring in this description. The molecules constituting the fluid are described at the macroscopic resolution level while the mesoscopic solid particles are described at the kinetic level. This is of course related, first, to the fact that the average distance between the solid particles in suspension in the fluid is much larger than the average distance between the molecules of the fluid. Second, the relaxation time toward local equilibrium is much shorter for the molecules than for the system of particles due to the fact that there less collisions per time unit among the latters than among the formers. Hence, the hydrodynamic description is appropriated for the molecules of the fluid while the solid particles, viewed as a gas, are still in the kinetic regime.

Generally the fluids in which the solid particles are immersed are in turbulent flow regimes. This implies the existence of stochastic forces acting on the particles. These are due to the dependence of the hydrodynamic forces on the relative velocity of the particle with respect to the local flow velocity which, in turbulent regimes, become random. As a consequence the motion of the particles becomes a random process driven by two different sources of stochasticity: interactions with turbulent surrounding fluid and binary collisions between particles.

As compared to passive transport, the complexity of the description is largely increased due to the coupling of particle dynamics with the turbulent hydrodynamics of the fluids. Moreover, for high enough concentrations of particles, interactions between them , i.e. collisions, become important. In such cases, the Langevin description for the individual motion of particles is insufficient as it neglects collisions between particles. This description must be replaced by a kinetic Boltzmann-like equation for the particles and its coupling with the Navier-Stokes equation. The complexity of these coupled equations is such that simulations are quite difficult and theoretical insights are, therefore, necessary.

Our purpose in this article is to present a theoretical scheme for the description of active transport of particles in turbulent flows of neutral fluids. External fields such as gravitation or electrostatic forces acting on the solid particles are included in the description.

The plan of the article is the following. The second chapter presents the collisionless motion equation for the heavy particles in a moving fluid. The integro-differential character of this equation is discussed and shown to correspond to a fractional differential equation. Then, in chapter three, the Navier-Stokes equation for the fluid is established including the reaction terms of the particles on the fluid. The relevance of these terms is discussed as usually they are not taken into account explicitely. The reaction terms introduce the phase-space density of the particles in the Navier-Stokes equation and, thus, couples the latter with the phase-space kinetic equation for the particles.

In the fourth chapter the effect of turbulence on the description is discussed.

Generalization to situations where the neutral fluid is replaced by a plasma can be envisaged along the same lines of reasoning and is discussed in the conclusions.

2 Collisionless motion equation for the solid particles

We now describe the dynamics of small solid particles in an incompressible neutral but otherwise arbitrary fluid. In this chapter we do not analyze the effects of turbulence.

The classical non-relativistic dynamics of each particle of masse m_p and density ρ_p in a fluid of density ρ is governed by the Newton equation

$$
\begin{aligned}
m_p \frac{d\mathbf{v}}{\mathrm{dt}} &= \left(\rho_p - \rho\right) V_p \mathbf{g} + \ \mathbf{F}(\mathbf{v}, \mathbf{u}) + \mathbf{F}_{\mathrm{ext}} \\
\frac{d\mathbf{x}}{\mathrm{dt}} &= \mathbf{v}
\end{aligned} \tag{1}
$$

where $\mathbf{x}(t)$ and $\mathbf{v}(t)$ respectively, are the position and velocity of the particle and $\boldsymbol{u}$ is the Lagrangian velocity field of the fluid at point $\mathbf{x}(t)$. In a more accurate description, these variables would respectively represent the position and velocity of the particle's center of mass and the velocity field of the flow in the vicinity of the surface of the particle. The equation for the angular momentum of the particle does not appear. This come from the assumption that the size of the particles is small enough with respect to the characteristic lengths of the flow so that the particles can be thought as almost point-like. On the other hand, however, the forces exerted by the fluid on the particles such as the buoyancy or the drag force depend on their finite volumes. Equation (1) represents, thus, a hybrid approximation where in some aspects the particles are treated as point-like and in some others they are supposed to have a non-vanishing volume.

The right-hand side of (1) contains the gravitation and buoyancy force, the total force $\mathbf{F}(\mathbf{v}, \mathbf{u})$ exerted by the flow due to the relative motion of the particle with respect to the fluid. External forces other than gravity such as electrostatic and/or magnetic forces are globally denoted by $\mathbf{F}_{\mathrm{ext}}$. Interactions, i.e. collisions, between particles in the fluid are omitted in equation (1) and will be introduced latter in the chapter.

A very common approximation of the hydrodynamical force is the Stokes [2] form of the drag force

$$
\mathbf{F}(\mathbf{v}, \mathbf{u}) = \gamma(\mathbf{u} - \mathbf{v}) \tag{2}
$$

where the constant friction coefficient γ is positive and given by $6\pi\mu\mathcal{R}$ in which μ is the dynamical viscosity of the fluid and $\mathcal{R}$ is the average radius of the particle.

This approximation is, however, very drastic and in most realistic cases uncorrect. Its validity domain is limited to fluids at rest, and to very small Reynolds number of the flow around the particle.

The more accurate form of $\mathbf{F}(\mathbf{v}, \mathbf{u})$, however, is worth being considered for two reasons. First, it is necessary in the case of flows at higher values of the Reynolds number. Moreover, from a more mathematical point of view, it leads to one of the very rare occurence where a fractional differential equation is derived from fundamental principles as has been shown recently [7], [8]. Let us analyse into more details the form of that force.

The study of the motion of a non-ponctual particle embedded in an arbitrary neutral fluid started with the pioneering work of G.G.Stokes [2] in the 19th century. Formula (2) results from this work. Nearly thirty years later J.Boussinesq [3] followed three years later by A.B.Basset [4] showed the existence of three other forces characterizing the action of an unsteady flow on a finite radius solid particle moving in the fluid. The form of $\mathbf{F}(\mathbf{v}, \mathbf{u})$ they obtained is

$$
\mathbf{F}(\mathbf{v}, \mathbf{u}) = \gamma(\mathbf{u} - \mathbf{v}) + \rho V_p \frac{D\mathbf{u}}{\mathrm{Dt}} + \frac{\rho V_p}{2} \frac{d\left(\mathbf{u}(x(t), t) - \mathbf{v}(t)\right)}{dt} +
$$

$$\frac{\gamma\mathcal{R}\rho^{1/2}}{\pi^{1/2}\mu^{1/2}}\int_{0^-}^{t}\frac{d\left(\mathbf{u}(x(\tau),\tau)-\mathbf{v}(\tau)\right)}{d\tau}\frac{1}{(t-\tau)^{1/2}}d\tau \tag{3}$$

where
$$\frac{D\mathbf{u}}{Dt}=\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}.\boldsymbol{\nabla}\mathbf{u}$$
and
$$\frac{d\mathbf{u}}{dt}=\frac{\partial\mathbf{u}}{\partial t}+\mathbf{v}.\boldsymbol{\nabla}\mathbf{u}$$

The second term in the right hand side of equation (3) is due to the acceleration of the fluid displaced by the particle, the third is the so-called added mass term which corresponds to an effective increase of the inertia of the particle, and the last term corresponds to the memory or history force also called Basset force -though it has been first found by Boussinesq. This effect may be roughly described as the reaction on the particle of streamlines deformations previously produced by the particle on the neighbouring fluid.

The expression (3) is derived [5] by solving the Navier-Stokes equation for the velocity field $\mathbf{u}$ with no-slip boundary conditions at the surface of the moving spherical solid particle. In this approach the particle is considered as a spherical solid inclusion in the fluid. Whence $\mathbf{u}$ is known, the force exerted by the fluid on the particle is obtained by calculating the flux of the total constraint tensor of the fluid across the surface of the particle.

Let us define a Reynolds number Re associated to the flow of the fluid in the vicinity of the particle
$$\mathrm{Re}=\frac{2\mathcal{R}\rho\left|\mathbf{u}-\mathbf{v}\right|}{\mu} \tag{4}$$

It has been shown by direct numerical simulations and experiments that the validity of equation (3) extends to moderate but not very high Reynolds numbers. Another limitation of equation (3) is the fact that it has been derived with the assuption of weak spatial inhomogeneity of the flow. A more accurate form of the hydrodynamic force taking into account stronger inhomogneities in the flow has been obtained by Maxey and Riley [6]

$$\mathbf{F}(\mathbf{v},\mathbf{u})=\gamma(\mathbf{u}-\mathbf{v}+\rho V_p\frac{D\mathbf{u}}{\mathrm{Dt}}\mid_{x(t)})+\frac{\rho V_p}{2}\frac{d\left(\mathbf{u}-\mathbf{v}+\frac{\mathcal{R}^2}{10}\nabla^2 u\right)}{dt}\mid_{x(t)}+$$
$$\frac{\gamma\mathcal{R}\rho^{1/2}}{\pi^{1/2}\mu^{1/2}}\int_{0^-}^{t}\frac{d\left(\mathbf{u}(x(\tau),\tau)-\mathbf{v}(\tau)+\frac{\mathcal{R}^2}{6}\nabla^2 u\right)}{d\tau}\mid_{x(\tau)}\frac{1}{(t-\tau)^{1/2}}d\tau \tag{5}$$

Clearly, the terms of equation (3) are modified by the addition of a contribution representing the curvature of the flow $\nabla^2 u$.

The general form of the history force as it appears in equations (3) and (5) reveals a common structure of the integral term in the form of

$$I=\frac{1}{\pi^{1/2}}\int_{0^-}^{t}\frac{d\left(L\mathbf{u}-\mathbf{v}(\tau)\right)}{d\tau}\mid_{x(\tau)}\frac{1}{(t-\tau)^{1/2}}d\tau \tag{6}$$

where L is a linear differential operator that may include spatial derivatives terms. Here, the action of this operator on the fluid velocity field is evaluated at point $\boldsymbol{x}(t)$. The temporal integral kernel of the above memory force has been shown to correspond to the definition of the Riemann-Liouville fractional derivative of order one-half by Mainardi [7] and by Coimbra and Rangel [8]. This property has been recently rediscovered independently by the author of the present article. More precisely, the memory force is the derivative of order one-half of some functional of $\mathbf{u}$ as we now show. Let us recall the definition of the derivative of fractional order α (>0) of a function $f\ (t)$ in the sense of Riemann-Liouville

$$ {}_0D_t{}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t f(\tau) \frac{1}{(t-\tau)^{\alpha+1-n}} d\tau \tag{7} $$

where n $\in \mathbb{N}$ and n-1 $< \alpha <$ n.

Following the same lines of reasoning as Mainardi [7] we get

$$ I = {}_0D_t{}^{1/2} (L\mathbf{u} - \mathbf{v}(t)) \mid_{x(t)} \tag{8} $$

Thus, the total hydrodynamic force acting on the particle is

$$ \mathbf{F}\left(\mathbf{v},\mathbf{u}\right) = \gamma\left(\mathbf{u}-\mathbf{v}+\frac{\mathcal{R}^2}{6}\nabla^2\mathbf{u}\mid_{x(t)}\right) + \rho V_p \frac{D\mathbf{u}}{\mathrm{Dt}} \mid_{x(t)} + $$

$$ \frac{\rho V_p}{2} \frac{d\left(\mathbf{u}-\mathbf{v}+\frac{\mathcal{R}^2}{6}\nabla^2\mathbf{u}\mid_{x(t)}\right)}{dt} + \frac{\gamma\mathcal{R}\rho^{1/2}}{\pi^{1/2}\mu^{1/2}} {}_0D_t{}^{1/2}(L\mathbf{u}-\mathbf{v}(t))\mid_{x(t)} \tag{9} $$

With the above final expression of $\mathbf{F}(\mathbf{v},\mathbf{u})$, the Newton equation (1) for the particles becomes a fractional differential system of equations. When making explicit the random fluctuations of the velocity field for a turbulent flow in the equation of motion, a random noise appears and, hence, this equation becomes a fractional differential Langevin equation as shown below in section 4.

Let us mention that for high values of the Reynolds number corresponding to developped turbulence, the drag force $\gamma(\mathbf{u}-\mathbf{v})$ must be replaced by a quadratic function of the relative velocity of the particle with respect to the local fluid velocity. Moreover, for flows with large vorticities, the Saffman lift force should be introduced, and for rotating spherical particles the Magnus force should be taken into account in $\mathbf{F}(\mathbf{v},\mathbf{u})$ [9] .

3 Navier-Stokes equation with reaction terms

The equation of motion (1) is coupled to the Navier-Stokes equation for the velocity field of the flow, $\boldsymbol{u}$. We now study how this coupling affects the latter equation.

For the sake of simplicity we assume here that the fluid obeys the Navier-Stokes equation for an incompressible flow

$$ \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}.\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}^2\mathbf{u} + \rho\mathbf{g} + \frac{1}{\rho}\boldsymbol{\phi} \tag{10} $$

where ν is the kinematic viscosity of the fluid. Generalization to compressible flow is straightforward. We assume here that the motion of the fluid is affected by no other external fields than gravity. However, generalization to charged or polarized fluids in external electric or magnetic fields would not change the fundamental ideas of the discussion below.

In the last term of the right hand side of equation (10), ϕ denotes the force density due to the reaction of the particles on the fluid. This term is generally neglected or omitted in works about transport of small solid particles in fluids, though, it is required by the third Newton's principle. This approximation is, however, correct for weak concentrations of particles. More generally, another reason for not introducing this term is the fact that it can be taken into account via the boundary conditions as we discuss later.

This reaction force density is the vectorial sum of the reaction forces on the molecules of the fluid due to all the solid particles immersed in the fluid per unit volume. Due to the inertia of the solid particules, their mutual collisions and the action of the different external forces, their velocities generally differ from the local velocity of the fluid. More precisely, there exists a phase-space one-particle distribution, $f(r,\mathbf{v},t)$, for the particules from which the probability density for the velocity of a particle can be deduced.

Thus, using the third Newton's law, the reaction force density of the particles on the fluid is obtained as minus the average in velocity space over $f(r,\mathbf{v},t)$ of the buoyancy and the other hydrodynamic forces in the right-hand side of equation (1)

$$\boldsymbol{\phi}= -\int d^3\mathrm{v}[-\rho V_p\mathbf{g} + \ \mathbf{F}(\mathbf{v},\mathbf{u})]f(r,\ \mathbf{v},t) \tag{11}$$

Consequently, the modified Navier-Stokes equation is now

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}.\boldsymbol{\nabla}\mathbf{u}= -\frac{1}{\rho}\boldsymbol{\nabla}p+\nu\boldsymbol{\nabla}^2\mathbf{u}+\mathbf{g}+V_p\mathbf{g}\ c(r,t)-\frac{1}{\rho}\int d^3\mathrm{v}\ \mathbf{F}(\mathbf{v},\mathbf{u})f(r,\ \mathbf{v},t) \tag{12}$$

where $c(r,t)$ is the particle's concentration in the fluid

$$c(r,t)=\int d^3\mathrm{v}f(r,\mathbf{v},t) \tag{13}$$

At this level some remarks should be made. First, as already mentioned, in many works the reaction of the particles does not explicitely appear in the Navier-Stokes equation. The effect of each particle on the fluid is introduced via the boundary conditions of the fluid at the surface of the moving particle. This is a tractable approach for a small number of large particles dispersed in the fluid. However, in situations such as dust storms in the atmosphere where large concentrations of dust particles are achieved, that kind of description becomes cumbersome and unpracticable. More generally, when the transported particles are small with respect to the resolution of the fluid description and in large concentrations, one would rather adopt a phenomenological description that assumes almost point-like particles. Consequently, the action of the particles on the fluid can not be transmitted via the boundary conditions of the fluid at their surface since they are described as point-like. The only possibility is, thus, to invoke the third Newton's principle for point-like particles and introduce this action via reaction forces. This is what we are doing here leading to the modified Navier-Stokes (12).

Second, obviously, the Navier-Stokes equation (12) becomes a fractional partial differential equation due to the occurence of a fractional time derivative of order one-half in $\mathbf{F}(\mathbf{v},\mathbf{u})$ as obtained in equation (9). This modifies deeply the mathematical status of the Navier-Stokes equation. The study of the consequences of this change is out of the scope of the present article but it should be carried out more systematically in subsequent works.

Third, the modified Navier-Stokes depends on the phase-space distribution (PSD) $f(r,\mathbf{v},t)$ via two terms. It is, thus, coupled to the kinetic equation

$$\frac{\partial}{\partial t}f(r,\mathbf{v},t)+\mathbf{v}.\boldsymbol{\nabla}f(r,\mathbf{v},t)+\frac{1}{m_p}\frac{\partial}{\partial\mathbf{v}}.\left\{\left[\left(\rho_p-\rho\right)V_p\mathbf{g}+\mathbf{F}(\mathbf{v},\mathbf{u})+\mathbf{F}_{\mathrm{ext}}\right]f(r,\mathbf{v},t)\right\}=K\left\{f,f\right\} \tag{14}$$

where the second and third term in the left-hand side of equation (14) come from the one-particle Liouville equation associated to equation (1) for non-interacting particles while the collision term, $K\{f, f\}$, describes the interactions between particles. For dilute neutral particle concentrations it can be thought as the Boltzmann collision term for point-like particles. For charged particles it could be the Landau or the Balescu-Lenard kinetic equation. However, for larger concentrations of neutral particles, the effect of excluded volume become important and $K\{f, f\}$ should represent the Enskog collision term for non vanishing radius hard spheres . Moreover, the collisions between macroscopic particles being generally inelastic, the Boltzmann or Enskog collision terms must be modified to take into account inelastic collisions [10].

It is useless for our present purpose to give the explicit form of the collision term. The only property that matters here is that, quite generally, this term contains a gain and a loss contributions that are both quadratic in the PSD. Indeed, both of them depends on the product $f(r_1,\mathbf{v}_1,t)f(r_2,\mathbf{v}_2,t)$ where the respective positions and velocities of the particle 1 and particle 2 are their positions and velocities just before the collision.

We, thus, have built a system of coupled equations (12) and (14) that describes the interactions between the fluid and the particles and their temporal evolution. Before closing this chapter, the status of the PSD $f(r,\mathbf{v},t)$ should be discussed. This function represents a probability density function for a particle to have its position and velocity somewhere in a small phase-space volume around the point $(\mathbf{r},\ \mathbf{v})$. The random character of these two variables is, at this level, only due to two factors: the lack of knowledge of the initial position and velocity, and the random character of the binary collisions.

However, once the random nature of the turbulent fluid velocity field is introduced in the above equations, a further level of stochasticity is introduced and $f(r,\mathbf{v},t)$ becomes itself a random function. The consequences of this fact are studied in the next chapter.

4 Effects of turbulence

We now explicit the turbulent character of the flow in which the particles are immersed, assuming the Reynolds number is large enough. Let us split the random velocity field in its average $\mathbf{U}$ and turbulent fluctuation $\delta\boldsymbol{u}$

$$\mathbf{u}=\mathbf{U}+\delta u \tag{15}$$

This decomposition is inserted in equations (12) and (14) and the average over the probability density function (PDF) of $\delta\boldsymbol{u}$, also called turbulent average, is formally taken. However, in order to carry this operation out one must first calculate the fluctuation of $f(r,\mathbf{v},t)$.

In order to do so, let us insert the splitting (15) in the equations of motion of each particle in absence of collisions. We get the system

$$m_p\frac{d\mathbf{v}}{dt} = \left(\rho_p-\rho\right)V_p\mathbf{g} + \ \mathbf{F}(\mathbf{v},\mathbf{U}+\delta u) + \mathbf{F}_{\text{ext}} \tag{16}$$

and

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \tag{17}$$

Clearly, equation (16) contains a random noise, $\delta\boldsymbol{u}$. It, thus, becomes a Langevin equation. The precise probability density (PDF) of the random variable $\delta\boldsymbol{u}$ is not known. In the sequel we shall call averages with respect to that PDF, turbulent averages. We shall not need immediately the knowledge of that PDF. The main property that will be used now is that the noise explicitely affects the velocity $\mathbf{v}(t)$ through equation (16). Thus, the velocity of the particle becomes a random variable. This implies, in turn, that the position of that particle is also random via equation (17). The PSD $f(\mathbf{r},\mathbf{v},t)$ generally is a function of ($\mathbf{r}$-$\mathbf{r}(t)$) and of ($\mathbf{v}$-$\mathbf{v}(t)$). An example of this is the Dirac distribution $\delta(\mathbf{r}\text{-}\mathbf{r}(t))\delta(\mathbf{v}\text{-}\mathbf{v}(t))$. Hence, through its dependency in $\mathbf{v}(t)$, the PSD $f(\mathbf{r},\mathbf{v},t)$ is itself a random function. Consequently, the PSD can be decomposed into its average plus fluctuation

$$f(\mathbf{r},\mathbf{v},t) = P(\mathbf{r},\mathbf{v},t) + \delta f(\mathbf{r},\mathbf{v},t) \tag{18}$$

where $P(\mathbf{r},\mathbf{v},t)$ is the average of $f(\mathbf{r},\mathbf{v},t)$ over the turbulent fluctuations $\delta\boldsymbol{u}$. This means that P must be itself a PSD, i.e. non-negative and normalized. Hence, the integral over the whole phase-space of the fluctuation $\delta f(\mathbf{r},\mathbf{v},t)$ must vanish. Moreover, the turbulent average of $\delta f(\mathbf{r},\mathbf{v},t)$ must also vanish.

Let us now examine the dependency of $f(\mathbf{r},\mathbf{v},t)$ as a functional of $\mathbf{v}(t)$ which, in turn, is itself a functional of $\boldsymbol{u}(\mathbf{r}(t),\, t)$ via (16). First, we determine the functional dependance of $\mathbf{v}(t)$ in $\boldsymbol{u}(\mathbf{r}(t),\, t)$. This requires to express the solution of equation (16) in term of the fluctuating part of the force $\mathbf{F}(\mathbf{v},\mathbf{U}{+}\delta u)$. This implies isolating the contribution of δu in that force by expanding the latter in a Taylor series in power of δu. Then the terms of that series that depend on δu can be treated as source terms in equation (16) leading, in turn, to a formal solution of that equation. In practice, this formal solution is generally quite difficult to obtain as equation (16) for models (3) and (5) is a linear integro-differential equation for $\mathbf{v}$(t) with time dependent coefficients. For high Reynolds flows, it becomes even nonlinear as $\mathbf{F}(\mathbf{v},\mathbf{u})$becomes quadratic in $\mathbf{u}-\mathbf{v}$. In order to give a tractable example we now study the case of the particularly simple model (2).

Formal solution of equation (16) with model (2) leads to

$$\mathbf{v}(t) = e^{-t/\tau_p}\mathbf{v}(0) + \int_0^t e^{-(t-\tau)/\tau_p}[\frac{1}{\tau_p}\mathbf{U}(\mathbf{r}(\tau),\tau) + \frac{(\rho_p-\rho)}{m_p}V_p\mathbf{g} + \frac{\mathbf{F}_{\text{ext}}(\mathbf{r}(\tau),\tau)}{m_p}]d\tau + \frac{1}{\tau_p}\int_0^t e^{-(t-\tau)/\tau_p}\delta u(\mathbf{r}(\tau),\tau)d\tau \tag{19}$$

where

$$\tau_p \equiv \frac{m_p}{\gamma} \tag{20}$$

is the characteristic relaxation time of the particle's velocity towards the Stokes limit velocity.

Let us notice that in equation (16) the external forces $\mathbf{F}_{\text{ext}}$ have been assumed to be independent of the velocity of the particle. However, in the case of the Lorentz force, a similar formal solution could be obtained due to the linearity of that force in the velocity of the particle.

Relation (19) readily gives the explicit link between the particle velocity fluctuation $\delta\mathbf{v}$ and the random noise $\delta\boldsymbol{u}$ that we are looking for

$$\delta\mathbf{v}{=}\frac{1}{\tau_p}\int_0^t e^{-(t-\tau)/\tau_p}\delta u(\mathbf{r}(\tau),\tau)d\tau \tag{21}$$

We, thus, have obtained a linear functional relation between $\delta\mathbf{v}$ and $\delta\boldsymbol{u}$.

We now must find a relation between $\delta\mathbf{r}$ and $\delta\boldsymbol{u}$. This is a more difficult task as we have to solve the equation (17). Since this equation is expressed in terms of $\boldsymbol{U}(\mathbf{r}(t),$

t), $\mathbf{F}_{\rm ext}(\mathbf{r}(t),\ t)$ and $\delta\boldsymbol{u}(\mathbf{r}(t),\ t)$, it is in general highly nonlinear in $\mathbf{r}(t)$. In order to circumvent this difficulty we compute $\mathbf{r}(t)$ by iteration and stop at the first order in $\delta\boldsymbol{u}$. This gives us the fluctuation $\delta\mathbf{r}(t)$ since the other terms obtained when performing higher order steps in the iteration would not vanish in the turbulent average. The iteration is performed on the integral form of equation (17) for the model (2) and leads to

$$\mathbf{r}(t) = \mathbf{r}(0) + \tau_p(1 - e^{-t/\tau_p})[\mathbf{v}(0) + \frac{(\rho_p - \rho)}{m_p}V_p\mathbf{g}] +$$
$$\frac{1}{\tau_p}\int_0^t \mathrm{d}t' \int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}\delta u(\mathbf{r}(\tau),\tau) + \int_0^t \mathrm{d}t' \int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}[\frac{1}{\tau_p}\mathbf{U}(\mathbf{r}(\tau),\tau) + \frac{\mathbf{F}_{\rm ext}(\mathbf{r}(\tau),\tau)}{m_p}] \tag{22}$$

However, in the last term of the right hand side of the above equation the iteration must be still be carried out up to first order in $\delta\boldsymbol{u}$. The resulting fluctuation for $\mathbf{r}(t)$ is

$$\delta\mathbf{r}(t) = \frac{1}{\tau_p}\int_0^t \mathrm{d}t' \int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}\delta u(\mathbf{r}(\tau),\tau) +$$
$$\int_0^t \mathrm{d}t' \int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}\left\{\frac{1}{\tau_p}\boldsymbol{\nabla}\mathbf{U}(\mathbf{r}(\tau),\tau) + \frac{1}{m_p}\boldsymbol{\nabla}\mathbf{F}_{\rm exr}(\mathbf{r}(\tau),\tau)\right\}.$$
$$\frac{1}{\tau_p}\int_0^\tau \mathrm{d}t' \int_0^{t'} d\tau'^{-(t'-\tau')/\tau_p}\delta u(\mathbf{r}(\tau'),\tau') \tag{23}$$

The next step amounts to establish a relation between $\delta f(\mathbf{r},\ \mathbf{v},t)$ and $\delta\mathbf{v}$. Using the dependence of $f(\mathbf{r},\ \mathbf{v},t)$ in ($\mathbf{r}$-$\mathbf{r}(t)$) and in ($\mathbf{v}$-$\mathbf{v}(t)$), and the fact that the turbulent average of $\delta f(\mathbf{r},\ \mathbf{v},t)$ must vanish , that is only terms linear in $\delta\mathbf{r}(t)$ and in $\delta\mathbf{v}(t)$ can be taken into account, one gets

$$\delta f\,(\mathbf{r},\ \mathbf{v},t) = -\delta\mathbf{r}(t).\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial\mathbf{r}} - \delta\mathbf{v}(t).\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial\mathbf{v}} \tag{24}$$

The presence of higher order terms in equation (24) would involve contributions of $\delta\mathbf{v}(t)\delta\mathbf{v}(t)$, $\delta\mathbf{r}(t)\delta\mathbf{r}(t)$, $\delta\mathbf{r}(t)\delta\mathbf{v}(t)$ and higher powers of $\delta\mathbf{r}(t)$ and $\delta\mathbf{v}(t)$ whose turbulent averages would, clearly, not vanish. Consequently, $\delta f(\mathbf{r},\ \mathbf{v},t)$ would not satisfy the condition to be a fluctuation and would be contradictory with equation (18). The higher order terms are, in fact, taken into account in the average PDF $P(\mathbf{r},\ \mathbf{v},t)$. Remark also that as defined by equation (24), $\delta f(\mathbf{r},\ \mathbf{v},t)$ is normalized to zero as required.

Inserting the relations (21), (23) in equation (24) one gets

$$\delta f(\mathbf{r},\ \mathbf{v},t) = -\frac{1}{\tau_p}\int_0^t \mathrm{d}t' \int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}\delta u(\mathbf{r}(\tau),\tau).\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial\mathbf{r}} +$$
$$\int_0^t \mathrm{d}t' \int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}[\frac{1}{\tau_p}\boldsymbol{\nabla}\mathbf{U}(\mathbf{r}(\tau),\tau) + \frac{1}{m_p}\boldsymbol{\nabla}\mathbf{F}_{\rm ext}(\mathbf{r}(\tau),\tau)].$$
$$\frac{1}{\tau_p}\int_0^\tau \mathrm{d}t' \int_0^{t'} d\tau'^{-(t'-\tau')/\tau_p}\delta u(\mathbf{r}(\tau'),\tau').\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial\mathbf{r}} -$$
$$\frac{1}{\tau_p}\int_0^t d\tau e^{-(t-\tau)/\tau_p}\delta u(\mathbf{r}(\tau),\tau).\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial v} \tag{25}$$

Since expression (25) is a linear functional in δu , its average over the turbulent fluctuations vanishes. This is a necessary condition for the coherence of equation (18).

Whether relation (25) amounts to use a Gaussian approximation of the turbulent PDF is not clear. The Gaussian approximation, as given by the Furutsu-Novikov-Donsker functional relation [11], would lead to a similar expression at the level of the moments $<\delta\boldsymbol{u}(\boldsymbol{r},t)\delta f(\mathbf{r},\ \mathbf{v},t)>$. However, it is not clear where such an approximation has been made in our approach. The only condition leading to (25) is the fact that in equation (24) only terms linear in $\delta r(t)$ and in $\delta\mathbf{v}(t)$ can be retained. The latter condition seems to us to be imposed by the necessary coherence with the fact that in equation (18) $\delta f(\mathbf{r},\ \mathbf{v},t)$ must be a fluctuation. Hence, no Gaussian approximation seems to be hidden in equation (25).

Let us stress at this stage that (25) is expressed in term of the Lagrangian turbulent fluctuation $\delta u(\mathbf{r}(t),t)$. However, in order to be coherent with the fact that $\boldsymbol{r}$ and $\mathbf{v}$ are independent variables in the kinetic equation (14) and in the Navier-Stokes equation (12), one has to shift from the Lagrangian to the Eulerian description in equation (25). This not trivial question is discussed in details by L.I.Zaichik et al. [1] and we adopt here the approach described in that work to handle this problem.

We are now in position to average the equation (12) over the turbulent fluctuations $\delta\boldsymbol{u}$. This is achieved by, first, introducing in it the decompositions (15), (18). Next, the functional Taylor expansion of $\mathbf{F}(\mathbf{v},\mathbf{U}{+}\delta u)$in powers of $\delta\boldsymbol{u}(\boldsymbol{r},\ \mathrm{t})$ is performed and, then, the average of the equation (12) with respect to the PDF of $\delta\boldsymbol{u}(\mathbf{r},\ \mathrm{t})$ is taken. This leads

$$\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}.\boldsymbol{\nabla}\mathbf{U}+\boldsymbol{\nabla}.<\delta u(\mathbf{r},t)\delta u(\mathbf{r},t)> = -\frac{1}{\rho}\boldsymbol{\nabla}p+\nu\boldsymbol{\nabla}^2\mathbf{U}+V_p\mathbf{g}C-$$
$$\frac{1}{\rho}\int d^3\mathrm{v}\mathbf{F}(\mathbf{v},\mathbf{U})P(\mathbf{r},\mathbf{v},t)-\frac{1}{\rho}\int d^3\mathrm{v}\int d^3r'\int_0^\infty dt'\frac{D\mathbf{F}(\mathbf{v},\mathbf{U})}{D\mathbf{U}(\mathbf{r}',t')}.<\delta u(\mathbf{r},t')\delta f(\mathbf{r},\mathbf{v},t)>-$$
$$\frac{1}{2\rho}\int d^3\mathrm{v}\int d^3r'\int_0^\infty dt'\int d^3r"\int_0^\infty dt"\frac{D^2\mathbf{F}(\mathbf{v},\mathbf{U})}{D\mathbf{U}(\mathbf{r}',t')D\mathbf{U}(\mathbf{r}",t")}:<\delta u(\mathbf{r}',t')\delta u(\mathbf{r}",t")>P(\mathbf{r},\mathbf{v},t)-$$
$$\frac{1}{2\rho}\int d^3\mathrm{v}\int d^3r'\int_0^\infty dt'\int d^3r"\int_0^\infty dt"\frac{D^2\mathbf{F}(\mathbf{v},\mathbf{U})}{D\mathbf{U}(\mathbf{r}',t')D\mathbf{U}(\mathbf{r}",t")}:<\delta u(\mathbf{r}',t')\delta u(\mathbf{r}",t")\delta f(\mathbf{r},\mathbf{v},t)> \tag{26}$$

where $\frac{D}{D\mathbf{U}(\mathbf{r},t)}$ represents the functional derivative with respect to $U(\mathbf{r},\ \mathrm{t})$. The functional Taylor expansion of $\mathbf{F}(\mathbf{v},\mathbf{u})$ in the right hand side of equation (26) stops at the second order since $\mathbf{F}(\mathbf{v},\mathbf{u})$ for the models (2), (3) and (5) is a functional of at most degree two in $\mathbf{u}(\mathbf{r},t)$. This is due to the appearance of the material derivative $\frac{D\mathbf{u}}{Dt}$ in equations (3) and (5).

Notice that in equation (26) p and C denote respectively the pressure and the concentration of particles both averaged on the PDF of $\delta\boldsymbol{u}(\mathbf{r},\ \mathrm{t})$. More precisely, for the average concentration one has

$$C\left(r,t\right)=\int d^3\mathrm{v}P(\mathbf{r},\ \mathbf{v},t) \tag{27}$$

In order to illustrate the above theoretical scheme and to derive an explicit equation for $\mathbf{U}$, let us consider the case of the simple model (2) where the second order functional derivatives that appear in (26) vanish identically. We get

$$\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}.\boldsymbol{\nabla}\mathbf{U}+\boldsymbol{\nabla}.<\delta u(\mathbf{r},t)\delta u(\mathbf{r},t)>=-\frac{1}{\rho}\boldsymbol{\nabla}p+\nu\nabla^2\mathbf{U}+V_p\mathbf{g}\ C-$$

$$\frac{\gamma}{\rho}\int d^3v\ \ (\mathbf{U}-\mathbf{v})\,P(\mathbf{r},\ \mathbf{v},\ t)+\frac{\gamma}{\rho\tau_p}\int d^3v\int_0^t d\tau e^{-(t-\tau)/\tau_p} < \delta u(\mathbf{r},t)\delta u(\mathbf{r}(\tau),\tau)\mid \mathbf{r}> .\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial \mathbf{v}}+$$
$$\frac{\gamma}{\rho}\int d^3v\frac{1}{\tau_p}\int_0^t \mathrm{d}t'\int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p} < \delta u(\mathbf{r},t)\delta u(\mathbf{r}(\tau),\tau)\mid \mathbf{r}> .\frac{\partial P\,(\mathbf{r},\ \mathbf{v},t)}{\partial \mathbf{r}}+$$
$$\frac{\gamma}{\rho}\int d^3v\int_0^t \mathrm{d}t'\int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}[\frac{1}{\tau_p}\boldsymbol{\nabla}\mathbf{U}(\mathbf{r}(\tau),\tau)+\frac{1}{m_p}\boldsymbol{\nabla}\mathbf{F}_{\mathrm{exr}}(\mathbf{r}(\tau),\tau)].$$
$$\frac{1}{\tau_p}\int_0^{\tau} \mathrm{d}t'\int_0^{t'} d\tau'^{-(t'-\tau')/\tau_p} < \delta u(\mathbf{r},t)\delta u(\mathbf{r}(\tau'),\tau')\mid \mathbf{r}> .\frac{\partial P\,(\mathbf{r},\ \mathbf{v},t)}{\partial \mathbf{r}} \quad (28)$$

where $<\ \delta\mathbf{u}(\mathbf{r},t)\delta\mathbf{u}(\mathbf{r}(\tau),\tau)\ \mid\ \mathbf{r}\ >$ denotes the Eulerian version of the correlation $<\delta u(\mathbf{r},t)\delta u(\mathbf{r}(\tau),\tau)>$ with the condition that $\mathbf{r}(t)=\mathbf{r}$. Using the result established by L.I.Zaichik [1] one gets

$$< \delta u(\mathbf{r},t)\delta u(\mathbf{r}(\tau),\tau)\mid \mathbf{r}>=< \delta u(\mathbf{r},\frac{t+\tau}{2})\delta u(\mathbf{r},\frac{t+\tau}{2})>\Psi_L(t-\tau) \quad (29)$$

with

$$\Psi_L(t-\tau)=e^{-(t-\tau)/T_L}$$

where $T_L$is the Lagrangian fluid turbulent time scale and is given by

$$T_L=0.482\frac{k}{\varepsilon}$$

with

$$k=\frac{1}{2}<\delta u(\mathbf{r},t)\delta u(\mathbf{r},t)>$$

denoting the turbulent kinetic energy, and

$$\varepsilon=\frac{dk}{dt}$$

represents the turbulent dissipation rate of the flow.

As said earlier, for model (2), the contribution involving the second order functional derivative $\frac{D^2}{D\mathbf{U}(\mathbf{r},t)D\mathbf{U}(\mathbf{r}',t')}$ in the right hand side of equation (26) did not contribute to (28) since $\mathbf{F}(\mathbf{v},\mathbf{u})$ in model (2) is linear in $\boldsymbol{u}(\mathbf{r},t)$. For nonlinear models like (3) and (5), the second order functional derivative term does not vanish and the moments $<\delta u(\mathbf{r},t)\delta u(\mathbf{r}',t')\delta u(\mathbf{r}",t")>$ would, in these cases, contribute to the equation.

A last simplification in equation (28) comes from the fact that the fifth term in its right hand side vanishes. Indeed, the volume integral in velocity space $\int d^3\mathrm{v}$ contained in this term transforms into a surface integral on a surface at the infinity in the velocity space and $P(\mathbf{r},\ \mathbf{v},t)$ is supposed to vanish on that surface. Hence, the equation (28) finally reads in this particular case

$$\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}.\boldsymbol{\nabla}\mathbf{U}+\boldsymbol{\nabla}.<\delta u\delta u>=-\frac{1}{\rho}\boldsymbol{\nabla}p+\nu\nabla^2\mathbf{U}+V_p\mathbf{g}\ C-\frac{\gamma}{\rho}\int d^3\mathrm{v}\,(\mathbf{U}-\mathbf{v})\,P(\mathbf{r},\ \mathbf{v},t)+$$
$$\frac{\gamma}{\rho}\int d^3\mathrm{v}\frac{1}{\tau_p}\int_0^t \mathrm{d}t'\int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p} < \delta u(\mathbf{r},\frac{t+\tau}{2})\delta u(\mathbf{r},\frac{t+\tau}{2})>\Psi_L(t-\tau).\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial r}+$$
$$\frac{\gamma}{\rho}\int d^3\mathrm{v}\int_0^t \mathrm{d}t'\int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}[\frac{1}{\tau_p}\boldsymbol{\nabla}\mathbf{U}(\mathbf{r}(\tau),\tau)+\frac{1}{m_p}\boldsymbol{\nabla}\mathbf{F}_{\mathrm{ext}}(\mathbf{r}(\tau),\tau)].$$

$$\frac{1}{\tau_p}\int_0^{\tau} \mathrm{dt}' \int_0^{t'} d\tau'^{-(t'-\tau')/\tau_p} < \delta u(\mathbf{r}, \frac{t+\tau'}{2})\delta u(\mathbf{r}, \frac{t+\tau'}{2}) > \Psi_L(t-\tau').\frac{\partial P(\mathbf{r},\ \mathbf{v},t)}{\partial r} \quad (30)$$

This equation is not closed in $\mathbf{U}$ and , obviously, is the first equation of an infinite hierarchy of equations for the higher order turbulent moments of $\mathbf{u}$. An equation for $<\delta u \delta u>$ can be derived in the same way as above. It involves the third order moments $<\delta u \delta u \delta u>$and so on. A closure hypothesis must, thus, be applied here in order to close the hierarchy as is usually the case in developped turbulence.

Let us now turn to the turbulent average of the kinetic equation (14). For the sake of simplicity, we directly focus on the particular case of model (2). Inserting equations (24), (25) and (29) in equation (14) and taking the turbulent average, one gets

$$\begin{gathered}
\frac{\partial}{\partial t}P(\mathbf{r},\ \mathbf{v},t) + \mathbf{v}.\boldsymbol{\nabla}P(\mathbf{r},\ \mathbf{v},t) + \frac{1}{m_p}\frac{\partial}{\partial \mathbf{v}}.\left\{\left[\left(\rho_p-\rho\right)V_p\mathbf{g}+\gamma(\mathbf{U}-\mathbf{v})+\mathbf{F}_{\text{ext}}\right]P(\mathbf{r},\ \mathbf{v},t)\right\} - \\
\left(\frac{1}{\tau_p}\right)^2\int_0^t d\tau e^{-(t-\tau)/\tau_p} < \delta u(\mathbf{r}, \frac{t+\tau}{2})\delta u(\mathbf{r}, \frac{t+\tau}{2}) > \Psi_L(t-\tau) : \frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}}P(\mathbf{r},\ \mathbf{v},t) - \\
\left(\frac{1}{\tau_p}\right)^2\int_0^t \mathrm{dt}'\int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p} < \delta u(\mathbf{r}, \frac{t+\tau}{2})\delta u(\mathbf{r}, \frac{t+\tau}{2}) > \Psi_L(t-\tau) : \frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{r}}P(\mathbf{r},\ \mathbf{v},t) + \\
\left(\frac{1}{\tau_p}\right)^2\int_0^t \mathrm{dt}'\int_0^{t'} d\tau e^{-(t'-\tau)/\tau_p}[\frac{1}{\tau_p}\boldsymbol{\nabla}\mathbf{U}(\mathbf{r},\tau) + \frac{1}{m_p}\boldsymbol{\nabla}\mathbf{F}_{\text{ext}}(\mathbf{r},\tau)]. \\
\int_0^{\tau}\mathrm{dt}'\int_0^{t'} d\tau'^{-(t'-\tau')/\tau_p} < \delta u(\mathbf{r}, \frac{t+\tau'}{2})\delta u(\mathbf{r}, \frac{t+\tau'}{2}) > \Psi_L(t-\tau') : \frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{r}}P(\mathbf{r},\ \mathbf{v},t) = K\left\{P,P\right\} + \\
\frac{1}{2}\int d^3r\int d^3r'\int d^3\mathrm{v}\int d^3\mathrm{v}' < \delta f(\mathbf{r},\ \mathbf{v},t)\delta f(\mathbf{r}',\ \mathbf{v}',t) > \frac{D}{Df(\mathbf{r},\ \mathbf{v},t)}\frac{D}{Df(\mathbf{r}',\ \mathbf{v}',t)}K[f,f] + ...
\end{gathered} \quad (31)$$

where the operator $\frac{D}{Df(\mathbf{r},\ \mathbf{v},t)}$ denotes the functional derivative with respect to $f(\mathbf{r},\ \mathbf{v},t)$ and $K[P,\ P]$ is the collision term for the averaged velocity distribution $P(\mathbf{r},\ \mathbf{v},t)$. The explicit form of $< \delta f(\mathbf{r},\ \mathbf{v},t)\delta f(\mathbf{r}',\ \mathbf{v}',t) >$ is obtained by replacing δf in it by the expression (25) and calculating the turbulent averages. We do not write it down explicitely here as it is quite lengthy. The only feature that interests us at this level is that it is a linear functional of $<\delta u(\mathbf{r},t)\delta u(\mathbf{r}',t')>$ and, consequently, there appears a coupling between the collisions and the turbulent fluctuations.

In equation (31) the higher order terms after the quadratic term in the functional Taylor expansion vanish identically for collision terms that are quadratic functionals of $f(\mathbf{r},\ \mathbf{v},t)$. This hypothesis is generally fulfilled for binary interactions, in particular for the Boltzmann term for dilute particles or the Enskog collision term for higher concentrations.

5 Conclusions

The equations (30) and (31) are quite complex. However, they can be simplified in specific instances in which terms corresponding to certain mechanisms can be neglected. In frequent situations for atmospheric and oceanic flows, for example, the concentrations of particles are quite weak and most of the above effects can be neglected. So, the whole reaction term in the right hand side of equation (30) can be discarded in these situations. Furthermore, in these cases, the collisions between particles can be neglected. The kinetic equation (31), thus, becomes a Fokker-Planck equation with a diffusion tensor that is a linear functional of $<\delta u \delta u>$. This dependence couples this Fokker-Planck equation to the turbulent hierarchy of the moments of the flow velocity field.

In other situations like after atmospheric dust storms, the concentrations of particles can be significantly high while the Reynolds number is small. In those cases, the reaction terms in (30) and the collisions in (31) must be kept in the equation while the coupling to the turbulent moments $<\delta u(\mathbf{r}, t)\delta u(\mathbf{r}', t')>$ of the different effects is negligeable. The structure of the reaction term may even depend only on the particles' concentration if the velocities of the particles does not differ significantly from the local fluid velocity, that is when the inertia of the particles is weak and when the collisions between particles are unfrequent on the time-scales characterizing the variations of the flow. In the other cases, the Navier-Stokes becomes coupled to the phase-space distribution, that is to the kinetic equation. This already represents a considerable change in the mathematical structure of the governing equations.

For higher values of the Reynolds parameter, the coupling of the different terms to the turbulent moments should be analyzed term by term. In equation (30), the reaction effect is split into several terms corresponding to different couplings with the turbulent flow. These terms will be small in the cases where the particle relaxation time τ_p is much larger than the characteristic time of turbulence (Kolmogorov time, $t_K = (\nu/\varepsilon)^{1/2}$). Moreover, the importance of the terms involving the gradients $\boldsymbol{\nabla}\mathbf{U}(\mathbf{r}(\tau), \tau)$ and $\boldsymbol{\nabla}\mathbf{F}_{\mathrm{ext}}(\mathbf{r}(\tau), \tau)$ depends respectively on the length- or time-scale of the inhomogeneity of the average flow and of the external force field.

Also, in the kinetic equation, the turbulent corrections to the collision term depend on the ratio of the duration between two successive collisions and the turbulence time-scale. In most cases, these corrections can be neglected.

One must also keep in mind that the force term $\mathbf{F}(\mathbf{v}, \mathbf{u})$ appearing in equations (30) and (31) corresponds to the simplified model (2), however, for larger Reynolds numbers, one should at least use the more precise models (3) and (5) or even take into account the quadratic term in the drag force. This modifies even more radically the structure of the equations as they become fractional differential equations due to the Boussinesq-Basset history term.

It now remains to evaluate the importance of these new effects in real flows. This will be the object of our next work.

Another future extension of the present work is dusty plasmas dynamics. For mesoscopic charged dust particles embedded in a plasma, one would also have a hybrid, macroscopic-kinetic description similar to that derived in this article. This coupling would here also stem from the existence of reaction forces exerted by the dust particles on the plasma. The macroscopic magneto-hydrodynamic equations for the plasma would, thus, be coupled to the kinetic equation governing the evolution of the phase-space distribution of the dust particles. Of course, the nature of the forces and of the collisions would be very different than for neutral particles.

Acknowledgements

The author would like to thank Professor Marian Negrea and his colleagues from the team of the ASSOCIATION EURATOM - MEdC ROMANIA, NUC-INT (FUS) of the Physics Department of the University of Craiova, Rumania, for inviting him to the Colloquium “Anomalous transport in fusion plasmas” they organized in Craiova in October 2008.

This work, supported by the European Communities under the contract of Association between EURATOM and the Belgian State, was carried out within the framework of the European Fusion Development Agreement. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

References

[1] Zaichik, L., Alipchenkov, V. M. & Sinaiski, E. G. (2008). *Particles in Turbulent Flows.* Wiley-VCH Editor.

[2] Stokes, G. G. (1851), Trans.Camb.Phil.Society, **9**, 8–106.

[3] Boussinesq, J. (1885), C.R.Acad.Sci.Paris, **100**, 935–937.

[4] Basset, A. B. (1888). *Treatise on Hydrodynamics.* Deighton Bell Editor. (London).

[5] Abbad, M. (2003). Contributions sur les forces d'histoire exercées sur des inclusions solides ou fluides (Doctoral Dissertation, Institut National Polytechnique de Lorraine (INPL), Nancy, France).

[6] Maxey, M. R., & Riley, J. J. (1983), Phys.Fluid, **26**, 883–889.

[7] Mainardi, F., Pironi, P. & Tampieri, F. (1995). In B. Tabarrok & S. Dost (Ed.), *Proceedings CANCAM 95* (pp. 836–837). (Victoria, B.C., Canada).

[8] Coimbra, C. M. & Rangel, R. H. (1998), J.Fluid.Mech., **370**, 53–72.

[9] Mashayek, F., & Pandya, R. R. (2003),Progress in Energy and Combustion Science, **29**, 329–378.

[10] Brey, J. J., Dufty, J. W. & Santos, A. (1997), J.Stat.Phys., **87**, 1051–1066.

[11] Novikov, E. A. (1964), JETP, **47**, 1919.