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On the model kinetic description of Coulomb plasma

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Abstract

A new form of the collision operator for a Coulomb plasma is proposed. One-component and many-component systems are considered. The proposed collision operator properly takes into account the relaxation of the first 13 hydrodynamic moments. Besides this, it accounts for the non-diagonal component contribution in the quadratic approximation in the expansion of the linearized collision operator with respect to the complete system of Hermite polynomials. It is shown that for a system of charged particles with the Coulomb interaction potential, these contributions are essential and lead to Spitzer corrections to the transport coefficients. An expression for the intensity of the Langevin source in the kinetic equation is obtained in the same approximation. A new form of the model collision operator for a Boltzmann gas of hard spheres is proposed. For a many-component system we have reconstructed a non-linear model collision integral by using the linearized collision integral found. Unlike previous ones, it does not contain complicated exponential dependence and avoids the coefficients ambiguity in the many-component collision integral.

1 Introduction

The most widely used kinetic model equation, especially in the plasma transport and fluctuation studies, is Bhatnagar, Gross and Krook (BGK) model one [1]. Recall that BGK model collision term for the one-component system is the departure of the distribution function (d.f.) from the Maxwellian whose parameters are the moments of the d.f.:

$$I\{f\} = -\nu\left(f - f^0\right), \tag{1}$$

where

$$f^0 = \frac{n}{(2\pi mT)^{3/2}} \exp -\frac{m(v-\mathbf{V})^2}{2T}, \tag{2}$$

and

$$n(\mathbf{r},t) = \int f d\mathbf{p}, \tag{3}$$

$$\mathbf{V}(\mathbf{r},t) = \frac{1}{n}\int v f d\mathbf{p}, \tag{4}$$

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$$T(\mathbf{r},t)=\int\frac{m(v-\mathbf{V})^2}{3n}fd\mathbf{p}, \tag{5}$$

are the local density, mean velocity and temperature in energy units. This model term vanishes at equilibrium and satisfies the conservation laws and the H-theorem. In the problems of linear transport and fluctuations one usually uses its linearized form:

$$\delta\widehat{I}\,|h\rangle\ =-\nu\left(|h\rangle\ -\sum_{\alpha=1}^{5}|\Psi_\alpha\rangle\,\langle\Psi_\alpha|h\rangle\right), \tag{6}$$

where $|h\rangle$ defined by

$$f=f^0\ +\delta f=f^0\ (1+h), \tag{7}$$

and $|\Psi_\alpha\rangle$ are the first five Hermite polynomials.

The advantage of the BGK model is that the solution of the kinetic equation reduces to that of a system of algebraic equations [2]. A weak point is that the model implies that the Prandtl number equals 1. The BGK model was extended to 13 moments for the Maxwell gas [3]. Later, the extension of the BGK kinetic model to include higher order matrix elements was discussed and it was applied to investigate the generalized Enskog equation and the dynamic structure factor for gas and fluids [4]-[11].The approximation consisted in taking into account exactly a finite part of the matrix operator, while the remaining part was represented by the diagonal matrix elements only. In the present paper we do not consider spatial inhomogeneities and assume the wave vector k =0. But we do take into account the non-diagonal components arising in the collision operator expansion with respect to the complete system of polynomials in the quadratic approximation. It is considering these non-diagonal elements that allows us to obtain a new form for the model integral with the Spitzer corrections taken into account. In the case of the Boltzmann gas of hard spheres these corrections are insignificant.

For the model description of kinetic fluctuations using the Langevin method, the Langevin source intensity should be presented to the same order of approximation as the collision operator in the left side of the Langevin kinetic equation.

A graver situation arises for many-component systems. According to the Gross and Krook (GK) model [14], the collision operator has the form of the deviation of the d.f. from a 'mythical' exponent:

$$I_a\{f_a\}=-\sum_b\nu_{ab}[f_a-\frac{n_a}{(2\pi m_aT_{ab})^{3/2}}\exp-\frac{m_a(\mathbf{v}-\mathbf{V}_{ab})^2}{2T_{ab}}], \tag{8}$$

where the parameters V_{ab} and T_{ab} are connected to the moments of d.f. via linear relations:

$$\mathbf{V}_{ab}=\alpha_{aa}\mathbf{V}_a+\alpha_{ab}\mathbf{V}_b;\ T_{ab}=\beta_{aa}T_a+\beta_{ab}T_b. \tag{9}$$

Coefficients $\alpha_{aa};\alpha_{ab};\beta_{aa};\beta_{ab}$ are chosen in such manner that both conservation laws and balance equations for the momenta and energy for each component hold. Since the number of equations that the parameters of the model should satisfy (for a five-moment description of two-component system these equations have 4: 2 for the balance for momenta and 2 for the balance for temperature) is less than the number of unknown parameters (in this approximation there are 5: $\nu_{ab};\alpha_{aa};\alpha_{ab};\beta_{aa};\beta_{ab}$), there is an arbitrariness in the choice of parameters. Therefore there exist various modifications (see, for example, [15]) of the collision model which correctly describe the relaxation of the five moments. But probably the most dubious point of GK model is the complicated exponential dependence on the d.f.

2 Kinetic model construction

2.1 One-component systems

To construct a corrected BGK model, following Sirovich [16] we introduce two projection operators $\widehat{H}$ and $\widehat{N}$ satisfying

$$\widehat{H}\widehat{N} = \widehat{N}\widehat{H} = 0;\ \widehat{H} + \widehat{N} = \widehat{Id}. \tag{10}$$

Here $\widehat{Id}$ is the identity operator, $\widehat{H}$ is the operator of projection onto the 'hydrodynamical subspace' spanned by kets corresponding to the polynomials of the lowest order in the moment variable. In the BGK model these kets are the first five polynomials which correspond to the collisional invariants: density, momentum and kinetic energy. However, one may include in this subspace polynomials of higher order. Their number and order depend on the physical processes that one wishes to treat 'exactly'. Thus one may take into account non-invariant values like those of the pressure tensor and heat flux. The projection operator $\widehat{N}$ maps the state vector onto the remaining 'non-hydrodynamical subspace'. Since we are interested in a model operator describing correctly the first 13 moments, we take the operator $\widehat{H}$ in the following form:

$$\widehat{H} = \sum_{i=1}^{13} |\Psi_i\rangle \langle\Psi_i| . \tag{11}$$

The linearized collision operator is as follows:

$$\delta\widehat{I} = \widehat{H}\delta\widehat{I}\widehat{H} + \widehat{H}\delta\widehat{I}\widehat{N} + \widehat{N}\delta\widehat{I}\widehat{H} + \widehat{N}\delta\widehat{I}\widehat{N}. \tag{12}$$

Since the first five Hermite polynomials are the eigenfunctions of the collision operator for identical particles corresponding to the zero eigenvalue, it follows that for $1 \leq i \leq 5$

$$\widehat{H}\delta\widehat{I}\widehat{H} = \widehat{H}\delta\widehat{I}\widehat{N} = \widehat{N}\delta\widehat{I}\widehat{H} = 0. \tag{13}$$

The higher Hermit polynomials are eigenfunctions only for Maxwell's molecule. In this case, non-diagonal matrix elements $\widehat{H}\delta\widehat{I}\widehat{N}$ and $\widehat{N}\delta\widehat{I}\widehat{H}$ vanish:

$$\widehat{H}\delta\widehat{I}\widehat{N} = \widehat{N}\delta\widehat{I}\widehat{H} = 0. \tag{14}$$

For any other interaction potentials the Hermite polynomials are not the eigenfunctions of the collision operator, the equality (14) does not hold and the collision operator matrix elements contain non-diagonal elements. Our first approximation is that we accept (14) as a valid formula for the Boltzmann gas of hard spheres and Coulomb plasma. However, the first approximation is not sufficient for describing real gas and plasma. In the second approximation we take into account only the non-diagonal terms closest to the diagonal. As we will show below, the corrections for a Boltzmann gas of hard spheres turn out to be small, but for Coulomb systems they are not small and play a major role in the Spitzer corrections to transport coefficients. We can continue this process and take into account in the third approximation, next, non-diagonal terms more distant from the diagonal elements. We performed these calculations and found that the third approximation gives very small corrections (compared to Spitzer one), that can be neglected.

Since for one-component systems the operator is Hermitian and isotropic, Wigner-Ekkart theorem implies

$$\left\langle Y_l^m h_1(u^2)\right| \delta\widehat{I} \left| Y_{l'}^{m'} h_2(u^2)\right\rangle = \delta_{ll'}\delta_{mm'} \left\langle\langle Y_l^m h_1 | \, \delta\widehat{I} \, | Y_l^m h_2\rangle\right\rangle , \tag{15}$$

where $h_1(u^2)$,$h_2(u^2)$ are arbitrary functions , $Y_l^m-$are spherical harmonics and $\langle\langle|\,|\rangle\rangle$ means the reduced matrix element independent of m. From (15) the selection rule follows: the contribution to the non-diagonal matrix elements $\widehat{H}\delta\widehat{I}\widehat{N}$ and $\widehat{N}\delta\widehat{I}\widehat{H}$ is given only by polynomials with identical pairs of orbital numbers l and m. For example, for the polynomial $|\Psi_6\rangle = \frac{\sqrt{3}}{2}(u_x u_x - \frac{1}{3}u^2)$ defining the xx component of the pressure tensor, the non-zero contribution to the non-diagonal matrix elements is given by non-hydrodynamical polynomials of higher order in u^2 but with the same values of l and m ($l = 2; m = 2$). For example, $\left|\Psi_6^{(2)}\right\rangle = \sqrt{\frac{3}{14}}\frac{1}{2}(u^2-7)(u_x u_x - \frac{1}{3}u^2)$˙The main modeling procedure consists of approximating the non-hydrodinamical contribution. If the operator $\widehat{H}$ involves the first 13 Hermite polynomials, then the neglect of the term $\widehat{N}\delta\widehat{I}\widehat{N}$ does not affect calculations for such transport coefficients as viscosity and heat conductivity. Nevertheless the approximation

$$\widehat{N}\delta\widehat{I}\widehat{N} = -\nu\widehat{N} \tag{16}$$

allows one to describe at least qualitatively the 'tails' of neglected 'non-hydrodynamical' terms (ν corresponds to the longest non-hydrodynamical relaxation time). An account of these 'tails' may be important at the kinetic level of fluctuation description. Using this approximation one may rewrite, in a first approximation, that corresponds to the Maxwell's molecule, the model operator as follows:

$$\delta\widehat{I} = -\nu\widehat{Id} + \widehat{H}(\delta\widehat{I} + \nu)\widehat{H}. \tag{17}$$

For a 13 moment basis for $\widehat{H}$ in the first approximation

$$\delta\widehat{I}\,|h\rangle = -\nu\,|h\rangle + \nu\sum_{i=1}^{5}|\Psi_i\rangle\langle\Psi_i|h\rangle + \sum_{i=6}^{13}|\Psi_i\rangle\left(\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_i\rangle + \nu\right)\langle\Psi_i|h\rangle . \tag{18}$$

In the second approximation, in (18) appear the nearest non-diagonal entries:

$$\sum_{i=6}^{13}|\Psi_i\rangle\langle\Psi_i|\,\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle\left\langle\Psi_i^{(2)}|h\right\rangle , \tag{19}$$

where the non-hydrodynamical polynomials, which we take into account, are

$$\left|\Psi_i^{(2)}\right\rangle = \frac{1}{\sqrt{14}}(u^2-7)\,|\Psi_i\rangle\,,\ 6\le i\le 10 \tag{20}$$

$$\left|\Psi_{r+10}^{(2)}\right\rangle = \frac{1}{\sqrt{280}}(u^4-14u^2+35)\,|\Psi_r\rangle\,,\ 1\le r\le 3. \tag{21}$$

To close the terms (19) we use the following equation for the non-hydrodynamical moments in the Fourier presentation

$$(-i\omega - \left\langle\Psi_i^{(2)}\right|\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle)\left\langle\Psi_i^{(2)}|h\right\rangle_\omega = \left\langle\Psi_i^{(2)}\right|\delta\widehat{I}\,|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega \tag{22}$$

Thus in the second approximation, linearized model collision operator has the form

$$\delta\widehat{I}\,|h\rangle_\omega = -\nu\,|h\rangle_\omega + \nu\sum_{i=1}^{5}|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega - \sum_{i=6}^{13}|\Psi_i\rangle\,({}_i\Lambda_i^{(2)}(\omega)-\nu)\,\langle\Psi_i|h\rangle_\omega\,, \tag{23}$$

where

$${}_i\Lambda_i^{(2)}(\omega) = -\,\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_i\rangle - \frac{\langle\Psi_i|\,\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle^2}{-i\omega - \left\langle\Psi_i^{(2)}\right|\,\delta\widehat{I}\left|\Psi_i^{(2)}\right\rangle} \tag{24}$$

contains the square of the non-diagonal entries and the projection of the kinetic equation resolvent to the non-hydrodynamical subspace. Here we take into account non-stationarity of non-hydrodynamical moments. Thus, although the original collision integral is Markovian, the part projected onto the supspace of 13 moments becomes, in the second approximation, a frequency dependent operator. A similar situation occurs in quantum-mechanical perturbation theory. Note that in the Markov approximation the second order corrections in (24)are negative for any interaction potentials.

Calculate now matrix elements of the operator for a concrete interaction potential, namely for Coulomb plasma and a Boltzmann hard sphere gas. For the Coulomb interaction take the linearized collision operator in the Balescu-Lenard form. In this case the matrix elements are of the form

$$\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_j\rangle = -\int d\mathbf{p}d\mathbf{p}'d\mathbf{k}\frac{e^4k_rk_s\delta(\mathbf{kv}-\mathbf{kv}')}{k^4\,|\varepsilon(\mathbf{kv},\mathbf{k})|^2}f^0(\mathbf{p})f^0(\mathbf{p}')$$

$$x(\frac{\partial\Psi_i}{\partial\mathbf{p}_r}-\frac{\partial\Psi_i}{\partial\mathbf{p}'_r})(\frac{\partial\Psi_j}{\partial\mathbf{p}_s}-\frac{\partial\Psi_j}{\partial\mathbf{p}'_s}). \tag{25}$$

For i=j the entries are

$$\langle\Psi_i|\;\delta\widehat{I}\,|\Psi_i\rangle \le 0 \tag{26}$$

for any polynomials. Equality to zero corresponds to the five-time-degenerate zero eigenvalue. It is easy to see that the matrix elements of the operator $\widehat{H}\delta\widehat{I}\widehat{H}$ have following values:

$$\langle\Psi_i|\;\delta\widehat{I}\,|\Psi_j\rangle = -\,\delta_{ij}[\;\Lambda_1\sum_{k=6}^{10}\;\delta_{ik}+\;\Lambda_2\sum_{k=11}^{13}\;\delta_{ik}], \tag{27}$$

where Λ_1,Λ_2 are the relaxation frequencies of the pressure tensor and the heat flux vector which in terms of plasma parameters are as follows:

$$\Lambda_1 = \frac{8}{5}\frac{ne^4\sqrt{\pi}}{m^{1/2}T^{3/2}}Ln;\quad \Lambda_2 = \frac{2}{3}\;\Lambda_1. \tag{28}$$

The matrix elements of the operators $\widehat{N}\delta\widehat{I}\widehat{N}$ and $\widehat{H}\delta\widehat{I}\widehat{N}$ defining 'tails' and the second approximation for the Coulomb plasma are equal to

$$\langle\Psi_{14}|\;\delta\widehat{I}\,|\Psi_{14}\rangle = -\frac{2}{3}\;\Lambda_1;\;\;\Psi_{14} = \frac{1}{\sqrt{120}}(u^4-10u^2+15) \tag{29}$$

$$\langle\Psi_{15}|\;\delta\widehat{I}\,|\Psi_{15}\rangle = ... = \langle\Psi_{21}|\;\delta\widehat{I}\,|\Psi_{21}\rangle = -\frac{3}{2}\;\Lambda_1;\;\;\Psi_{15} = u_xu_yu_z \tag{30}$$

$$\langle\Psi_{22}|\;\delta\widehat{I}\,|\Psi_{22}\rangle = ... = \langle\Psi_{35}|\;\delta\widehat{I}\,|\Psi_{35}\rangle = -\frac{191}{16}\;\Lambda_1; \tag{31}$$

$$\Psi_{22} = \frac{1}{\sqrt{105}}\frac{1}{8}(35u_x^4 - 30u^2u_x^2 + 3u^4); \tag{32}$$

$$\left\langle \Psi_i^{(2)} \right| \delta\widehat{I} \left| \Psi_i^{(2)} \right\rangle = -\frac{201}{168}\Lambda_1;\ \left\langle \Psi_i \right| \delta\widehat{I} \left| \Psi_i^{(2)} \right\rangle = \frac{3}{2\sqrt{14}}\Lambda_1;\ 6 \le i \le 10 \tag{33}$$

$$\left\langle \Psi_{10+r}^{(2)} \right| \delta\widehat{I} \left| \Psi_{10+r'}^{(2)} \right\rangle = -\frac{15}{14}\delta_{rr'}\Lambda_1;\ \left\langle \Psi_{10+r} \right| \delta\widehat{I} \left| \Psi_{10+r'}^{(2)} \right\rangle = \frac{1}{7}\delta_{rr'}\Lambda_1; \quad 1 \le r \le 3. \tag{34}$$

This estimate implies that the higher tensor character of the polynomial leads to higher values of the matrix elements and for polynomials with the same tensor character the matrix elements are greater for polynomials of higher order in u^2. The smallest value of the diagonal matrix elements for the non-hydrodynamical polynomial is achieved for the Ψ_{14} polynomial and equals Λ_2 (the heat flux relaxation frequency). The same holds for Maxwell's molecule and the Boltzmann gas hard sphere at least. Thus in the first approximation the collision operator is of the form:

$$\delta\widehat{I}\,|h\rangle = -\frac{2}{3}\Lambda_1\,|h\rangle + \frac{2}{3}\Lambda_1\sum_{i=1}^{5}|\Psi_i\rangle\,\langle\Psi_i|h\rangle - \frac{1}{3}\Lambda_1\sum_{i=6}^{10}|\Psi_i\rangle\,\langle\Psi_i|h\rangle\,. \tag{35}$$

We see that the polynomials corresponding to the heat flux disappear. In the second approximation

$$\delta\widehat{I}\,|h\rangle_\omega = -\nu(|h\rangle_\omega - \sum_{i=1}^{13}|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega) \tag{36}$$

$$-\sum_{i=6}^{10}\Lambda_1^{(2)}(\omega)\,|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega - \sum_{i=11}^{13}\Lambda_2^{(2)}(\omega)\,|\Psi_i\rangle\,\langle\Psi_i|h\rangle_\omega\,, \tag{37}$$

where

$$\Lambda_1^{(2)}\ (\omega) = \Lambda_1(1 - \frac{\Lambda_1 9/56}{-i\ \omega + \Lambda_1 205/168});\ \Lambda_2^{(2)}\ (\omega) = \Lambda_2(1 - \frac{\Lambda_1 3/14}{-i\ \omega + \Lambda_1 15/14}). \tag{38}$$

In the Markov approximation ($\omega = 0$) this leads to Spitzer values [17] of transport coefficients.

$$\Lambda_1^{(2)} = \Lambda_1(1 - \frac{27}{205});\quad \Lambda_2^{(2)} = \Lambda_2(1 - \frac{1}{5}). \tag{39}$$

This correction are rather significant. In the third approximation the relaxation frequencies vary by no more than one per cent. Thus our model leads to Spitzer kinetic coefficients.

For a Boltzmann gas of hard spheres,

$$\langle\Psi_i|\,\delta\widehat{I}\,|\Psi_j\rangle = -\frac{R^2}{8n}\int|(\mathbf{v}-\mathbf{v_1})\mathbf{e}|\,f^0(\mathbf{p})f^0(\mathbf{p}')[\Psi_i(\mathbf{p}_1') + \Psi_i(\mathbf{p}') - \Psi_i(\mathbf{p}_1) - \Psi_i(\mathbf{p})]$$

$$\times[\Psi_j(\mathbf{p}_1') + \Psi_j(\mathbf{p}') - \Psi_j(\mathbf{p}_1) - \Psi_j(\mathbf{p})]d\mathbf{p}d\mathbf{p}_1, \tag{40}$$

and

$$\langle\Psi_{14}|\ \delta\widehat{I}\,|\Psi_{14}\rangle = -\frac{2}{3}\ \Lambda_1^G;\ \langle\Psi_{15}|\ \delta\widehat{I}\,|\Psi_{15}\rangle = ... = \langle\Psi_{21}|\ \delta\widehat{I}\,|\Psi_{21}\rangle = -\frac{3}{2}\ \Lambda_1^G; \tag{41}$$

$$\left\langle \Psi_i^{(2)} \right| \delta\widehat{I} \left| \Psi_i^{(2)} \right\rangle = -\frac{17}{14}\Lambda_1^G;\ \langle \Psi_i | \delta\widehat{I} \left| \Psi_i^{(2)} \right\rangle = -\frac{1}{2\sqrt{14}}\Lambda_1^G;\ 6 \le i \le 10 \tag{42}$$

$$\left\langle \Psi_{10+r}^{(2)} \right| \delta\widehat{I} \left| \Psi_{10+r'}^{(2)} \right\rangle = -\frac{15}{14}\delta_{rr'}\Lambda_1^G;\ \langle \Psi_{10+r} | \delta\widehat{I} \left| \Psi_{10+r'}^{(2)} \right\rangle = \frac{1}{3\sqrt{7}}\delta_{rr'}\Lambda_1^G;\ 1 \le r \le 3. \tag{43}$$

$$\Lambda_1^G = \frac{16}{5}\frac{\sqrt{\pi}TnR^2}{m}. \tag{44}$$

The second-order corrections are one order less than for Coulomb plasma and one may stop at the first approximation. Thus in the case of a Boltzmann gas of hard sphere the collision operator may be represented in the form [18]

$$I\{f\} = -\nu\{f - f^0(1 - P_{ij}\frac{\delta v_i \delta v_j}{4PT}m)\}. \tag{45}$$

In the equilibrium state, $P_{ij} = 0$ and $I\{f^0\} = 0$. In the equation for the heat flux only the first term contributes. The relaxation of the pressure tensor is determined by both the first and the last terms in this equation. Thus our collision integral in the form (45) possesses all necessary properties and is free from the drawbacks of the one-component model of the BGK model mentioned above. The linearized form of (45) is congruent with the linearized ellipsoidal statistical model [19].

Earlier, another model correctly describing the viscosity and thermal conductivity relaxation was proposed ad hoc [20]:

$$I\{f\} = -\nu\{f - f^0[1 - \Pr\frac{\mathbf{J}\delta\mathbf{v}}{5P}(\frac{\delta v^2}{mT} - 5)]\} \tag{46}$$

But it does not give a correct description of non-hydrodynamic 'tails'.

2.2 Langevin's source

The expression for the the spectral function of the Langevin's source in the kinetic equation [12]

$$(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \mathbf{F}\frac{\partial}{\partial \mathbf{p}} + \delta\widehat{I}_{\mathbf{p}})\delta f(\mathbf{x},t) + \delta\mathbf{F}\frac{\partial}{\partial \mathbf{p}}f(\mathbf{x},t) = y(\mathbf{x},t) \tag{47}$$

has the form

$$(yy)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} = -(\delta\widehat{I}_{\mathbf{p}_1} + \delta\widehat{I}_{\mathbf{p}_2})f^0(\mathbf{p}_1)\delta(\mathbf{p}_1 - \mathbf{p}_2). \tag{48}$$

For the model description of kinetic fluctuations using the Langevin method the Langevin source (48) should be presented to the same order of approximation as the collision operator in the left side of the Langevin kinetic equation (47). For the BGK model the corresponding expression for Langevin's source has the form

$$(yy)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} = 2\nu f^0(\mathbf{p}_1)\left(\delta(\mathbf{p}_1 - \mathbf{p}_2) - f^0(\mathbf{p}_2)\sum_{i=1}^{5}\Psi_i(\mathbf{p}_1)\Psi_i(\mathbf{p}_2)\right). \tag{49}$$

Langevin's source (49) satisfies the laws of conservation

$$\int \Psi(\mathbf{p}_1)\,(yy)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\,d\mathbf{p}_1 = 0 \tag{50}$$

for $\Psi(\mathbf{p}_1) = 1, \mathbf{p}_1, \frac{p_1^2}{2m}$, but does not lead to the correct Landau-Lifshitz formula [13] for Langevin's sources in the hydrodynamic equations. In the 13 moment basis for $\widehat{H}$ in the first approximation, the expression for the Langevin's source takes the form

$$(yy)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} = 2f^0\ (\mathbf{p}_1\ )[\nu\delta(\mathbf{p}_1\ -\mathbf{p}_2\ ) - \nu f^0\ (\mathbf{p}_2\ )\sum_{i=1}^{5}\ \Psi_i\ (\mathbf{p}_1\ )\Psi_i\ (\mathbf{p}_2\ )$$

$$-f^0\ (\mathbf{p}_2\ )\sum_{i=6}^{13}\ \Psi_i\ (\mathbf{p}_1\ )\Psi_i\ (\mathbf{p}_2\ )(\ \langle\Psi_i|\ \delta\widehat{I}\,|\Psi_i\rangle\ +\nu)]. \tag{51}$$

The first two terms correspond to the BGK model. In the second approximation (23) the spectral function of Langevin's source have the form

$$(yy)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2} = 2f^0(\mathbf{p}_1)\{[\nu\delta(\mathbf{p}_1 - \mathbf{p}_2) - \nu f^0(\mathbf{p}_2)\sum_{i=1}^{13}\Psi_i(\mathbf{p}_1)\Psi_i(\mathbf{p}_2)$$

$$+f^0(\mathbf{p}_2)\sum_{i=6}^{13}\Psi_i(\mathbf{p}_1)\Psi_i(\mathbf{p}_2)[\mathrm{Re}\Lambda_1^{(2)}(\omega)\sum_{\mathrm{k}=6}^{10}\delta_{\mathrm{ik}} + \mathrm{Re}\Lambda_2^{(2)}(\omega)\sum_{\mathrm{k}=11}^{13}\delta_{\mathrm{ik}}]\}, \tag{52}$$

where $\Lambda_1^{(2)}\ (\omega)$ and $\Lambda_2^{(2)}\ (\omega)$ are determined by (38). From (52) the non-Markov Langevin sources in the hydrodynamic equations [21] follows.

2.3 Many-component systems

Using the technique described above for a one-component system, one may get the following expression for the linearized model collision operator of a many-component system in the five-moment approximation:

$$\delta\widehat{I}_{a\mathbf{p}}\delta f_a(\mathbf{p}) = -\ \nu_a\delta f_a(\mathbf{p}) + \sum_{j=1}^{5}\nu_a f_a^0(\mathbf{p})\Psi_j^a(\mathbf{p})\int\Psi_j^a(\mathbf{p}')\delta f_a(\mathbf{p}')d\mathbf{p}'$$

$$+\sum_b\sum_{i,j=1}^{5}\nu_a f_a^0(\mathbf{p})\Psi_i^a(\mathbf{p})\,\langle\Psi_i^a|\,\delta\widehat{I}\,\big|\Psi_j^b\rangle\int\Psi_j^b(\mathbf{p}')\delta f_b(\mathbf{p}')d\mathbf{p}', \tag{53}$$

where $f_a^0(\mathbf{p})$ is the local equilibrium distribution function (with different temperatures and mean velocities), ν_a is the inverse time of the heat flux relaxation of the component a, and $\langle\Psi_i^a|\,\delta\widehat{I}\,\big|\Psi_j^b\rangle$ is the matrix elements of the linearized collision integral (the Balescu-Lenard one, for example).

In order to recover the form of the model collision integral from its linearized form it suffices to use conservation of the elastically interacting particles number. This property, as well as total momentum and energy conservation, are valid both for mean and fluctuating quantities. Consequently the expression for Langevin's source intensity in the kinetic equation for the distribution function fluctuation should satisfy the conditions

$$\sum_b\int\Psi_b(\mathbf{p}_2)\,(y_a y_b)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\,d\mathbf{p}_2 = 0 \tag{54}$$

for $\Psi_b(\mathbf{p}_2) = 1, \mathbf{p}_2, \frac{p_2^2}{2m_b}$.

The spectral function of the Langevin source in a non-equilibrium state is given [22], [23] by

$$(y_a y_b)_{\omega,\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}$$

$$= -(\delta\widehat{I}_{a\mathbf{p}_1} + \delta\widehat{I}_{b\mathbf{p}_2})\delta_{ab}\delta(\mathbf{p}_1 - \mathbf{p}_2)f_a(\mathbf{p}_1) - \delta_{ab}\delta(\mathbf{p}_1 - \mathbf{p}_2)I_a(\mathbf{p}_1) - I_{ab}(\mathbf{p}_1, \mathbf{p}_2), \tag{55}$$

where $I_{ab}(\mathbf{p}_1, \mathbf{p}_2)$ is the so-called 'not integrated' collision operator [22]:

$$\sum_b \int d\mathbf{p}_2 I_{ab}(\mathbf{p}_1, \mathbf{p}_2) = I_a(\mathbf{p}_1). \tag{56}$$

Summing (55) over b and integrating over $\mathbf{p}_2$, and taking into account (54), we get

$$I_a(\mathbf{p}_1) = \frac{1}{2}\sum_b \int (\delta\widehat{I}_{a\mathbf{p}_1} + \delta\widehat{I}_{b\mathbf{p}_2})\delta_{ab}\delta(\mathbf{p}_1 - \mathbf{p}_2)f_a(\mathbf{p}_1)d\mathbf{p}_2. \tag{57}$$

Since (54) and (55) are of general character, the relation (57) is valid both for 'exact' and model collision integrals. Substituting (53) in (57), we obtain a simple enough and at the same time sufficiently rigorous form of the model collision integral for many component plasma:

$$I_a(\mathbf{p}) = -\nu_a(f_a - f_a^0) - \sum_b \nu_{ab} f_a^0[\delta\mathbf{v}_a \frac{T_b\mathbf{V}_a - T_a\mathbf{V}_b}{T_aT_b}\frac{m_a}{T_a}\frac{T_am_b + T_bm_a}{m_b + m_a}$$

$$+(\frac{m_a}{T_a}\delta\mathbf{v}_a^2 - 3)\frac{T_a - T_b}{T_aT_b}\frac{m_a}{m_b + m_a}], \tag{58}$$

where ν_{ab} is the momentum relaxation frequency:

$$\nu_{ab} = \frac{4}{3}\sqrt{2\pi}e_a^2e_b^2n_bL\sqrt{\frac{m_b}{m_a}}\frac{(m_a + m_b)^{1/2}}{(m_aT_b + m_bT_a)^{3/2}}, \tag{59}$$

and $\delta\mathbf{v}_a = \mathbf{v} - \mathbf{V}_a$. The first two terms in (58) describe the relaxation to the local equilibrium state, the third term described the momenta relaxation and the last term, the temperature relaxation. Thus the complicated exponential dependency typical for the BGK model appears to be unfounded and does not hold for states remote from the full equilibrium.

3 Conclusion

Using the well-known projection technique, a new form of the collision operator for a Boltzmann gas of hard spheres and Coulomb plasma was elaborated. The proposed collision operator properly takes into account the relaxation of the first 13 hydrodynamic moments and accounts for the non-diagonal components contribution in the expansion of the linearized collision operator with respect to the complete system of Hermite polynomials. The non-diagonal components accounted for in this basis in the quadratic approximation contribute to the diagonal components. It is shown that for a system of charged particles with the Coulomb interaction potential, these contributions are essential and lead to Spitzer corrections to the transport coefficients. In the case of the Boltzmann gas of hard spheres these corrections are insignificant. In the case of a many-component system, the nonlinear model collision integral is constructed by means of the linearized one found. Unlike previous ones, it does not contain complicated exponential dependence and avoids coefficient ambiguity in the many-component collision integral.

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