Zeeman's Theorem in Krein Spaces

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The aim of this note is to argue that the most natural framework for generalizing Zeeman's theorem (see [3]) is that of the real Krein spaces of dimension greater than 3. According to our knowledge, the largest class of spaces in which this theorem acts is that of Hilbert spaces, as we may see in [2], where, more exactly, it is proved on a real Hilbert space endowed with some particular bilinear form. Unfortunately the surjectivity of the Gram operator attached to this form fails in Proposition 1.2 of [2], erroneously being considered as a consequence of the properties postulated for the bilinear form. This last remark is justified by the following

Example 1 *In the Hilbert space* $X = L^2_{\mathbb{R}}([-1,+1])$ *we consider the subspaces:*

$$X^{\ +} = \{f \in X : f(x) = 0 \ \ at \ \ almost \ \ all \ \ x < 0\} \ and$$

$$X^{\ -} = \{f \in X : f(x) = 0 \ \ at \ \ almost \ \ all \ \ x > 0\}.$$

The scalar product of X *is noted* $\langle .,. \rangle$. *Using the function* $a : [-1,+1] \to \mathbb{R}$, *expressed by*

$$a(x) = \begin{cases} -\exp \frac{x+1}{x} \quad if \quad x \in [-1,0) \\ 0 \quad if \quad x = 0 \\ \exp \frac{x-1}{x} \quad if \quad x \in (0,+1], \end{cases}$$

we may construct the functional $(.,.) : X \times X \to \mathbb{R}$ *by the formula*

$$(f,g) = \int_{-1}^{+1} a(x) f(x) g(x) dx.$$

It is easy to see that $(.,.)$ *is an inner product on* X. *Because* $|a(x)| \leq 1$ *for all* $x \in [-1,+1]$, *this inner product is continuous relative to the norm* $\|\cdot\|$ *of* $L^2_{\mathbb{R}}([-1,+1])$. *Further* $(X,\ (.,.))$ *is decomposable, and a decomposition is*

$X = X^{+} \overset{\perp}{\oplus} X^{-}$, where the orthogonality is valid in both senses of $(.,.)$ and $\langle .,. \rangle$. It is remarkable that $(X, (.,.))$ is not a Krein space, because the subspaces X^{+} and X^{-} are not intrinsically complete. For example the sequence $(f_n)_{n\in\mathbb{N}}$, where

$$f_n(x) = \begin{cases} 0 & if \quad x \in [-1, \frac{1}{n}] \\ \exp \frac{1-x}{4x} & if \quad x \in (\frac{1}{n}, 1] \end{cases}$$

is intrinsically Cauchy, but not convergent, since the function

$$f(x) = \begin{cases} 0 & if \quad x \in [-1, 0] \\ \exp \frac{1-x}{4x} & if \quad x \in (0, 1] \end{cases}$$

does not belong to X^{+}.

Finally, we may see that the corresponding Gram operator is not surjective. In fact, there is no $f \in X$ such that $(f, g) = \langle 1, g \rangle$ for all $g \in X$, because $a^{-1} \notin X$.

The above mentioned omission from the Proposition 1.2 of [2] affects the construction of the decomposition (X^{+}, X^{-}) in Proposition 1.6, and further the entire Krein structure of X, developed in Section 1. Proving the existence of a decomposition is still necessary (it was already supposed in Proposition 1.4!) because it is known that there exist non-decomposable inner product spaces (e.g. see I.11.3 in [1]). Happily, the situation is not too serious, being solved by a simple amendment to the main theorem, as follows

Theorem 2 *Let $(X, (.,.))$ be a real Hilbert space with* $\dim X \geq 3$, *and let $Q : X \times X \to \mathbb{R}$ be a form in X, for which the associated Gram operator is surjective. Then each bijection $f : X \to X$, which preserves the null vectors relative to Q, is an affine map.*

Now, let us remark that defining a form Q on a real Hilbert space X such that

(i) $Q(x, y) = Q(y, x)$ for each $x, y \in X$,

(ii) $x \longmapsto Q(x, y)$ is a bounded linear map for each $y \in X$,

(iii) if $Q(x, y) = 0$ for each $x \in X$, then $y = 0$, and

(iv) the associated Gram operator is surjective

is the same thing as considering a Krein space. Although this Krein structure is not explicitly mentioned in [2], Propositions like 1.5, 1.7 - 1.9, etc. represent general properties of the decomposable indefinite inner product spaces (see [1]). Reformulating [2] in terms of Krein spaces leads to very interesting properties of the null vectors preserving maps, but it will be omitted here like routine.

In conclusion, as a reword of the fundamental result of [2], we point out that

Theorem 3 *Zeeman's theorem holds in every real Krein space of dimension greater than 3.*

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