UNIFICATION BY STRUCTURAL DISCRETENESS

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Abstract

The aim of this note is to remedy what Albert Einstein has named "lack of mathematical structures of discontinuum without calling upon continuum space-time as an aid". The main idea is that similarly to continuity, which is studied in topological structures, discreteness has its specific structures, namely *horistologies*. Therefore we analyze several senses of the term "discreteness", including the structural one. Then, we suggest how to interpret the horistologies as an unifying framework of the Special Relativity, Quantum Physics, Cosmology (via Nottale's Scale Theory) and other topics where super-additivity plays a significant role.

Key words: Super-additivity, horistology, discrete sets and functions

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1 Introduction: Structural pairs

In this note we mainly try to connect ideas on existing results concerning continuity and discreteness, but not to prove new facts. It is well known that the study of continuity is always related to topological structures. The huge amount of results based on these structures contains the classical calculus and all theories of continuum. For several technical details on topological structures we limit ourselves to mention [BN] and [PG]. In counterpart, we will show why discreteness is better studied in horistological structures (introduced in [BT_0], published in [BT_1], [BT_2], and later studied as in references).

A. Dual structures. Let us focus on the following **pairs of structures**:

Topology	Horistology
Uniform Topology	Uniform Horistology
Sub-additive Metric	Super-additive Metric
Sub-additive Norm	Super-additive Norm
(Semi-) Definite Inner Product	Indefinite Inner Product

In the left column we recognize the well known structures of continuum; we claim that the right column contains the structures of discreteness. For brevity we recall the (less-known) definitions of the right-column structures only, in purely formal setting (examples, properties and other details can be seen in the corresponding bibliography).

Pair # 1: Topology / Horistology. Most frequently, we prefer to define the topology by specifying the neighborhoods at each point. The peer of the topology takes the form of a function $\chi : W \to \mathcal{P}(\mathcal{P}(W))$, called *horistology* on W, which satisfies the conditions:

[HOR$_1$] $e \notin P$ for all $P \in \chi(e)$

[HOR$_2$] If $P \in \chi(e)$ and $Q \subseteq P$, then $Q \in \chi(e)$

[HOR$_3$] If $P, Q \in \chi(e)$, then $P \cup Q \in \chi(e)$

[HOR$_4$] $\forall\ P \in \chi(e)\ \exists\ T \in \chi(e)$ such that $[\ell \in P \textit{ and } Q \in \chi(\ell)] \Longrightarrow [Q \subseteq T]$.

The elements of $\chi(e)$ are called *perspectives* of e, and (W, χ) defines a *horistological world*. Property [HOR$_1$] means that each event is "separated" from its perspectives, which justifies the term *horistology* ($\chi\omega\rho\iota\sigma\tau o\sigma$ = *separate*). Conditions [HOR$_2$] and [HOR$_3$] show that $\chi(e)$ forms an ideal in $\mathcal{P}(W)$, in opposition to the topological filters of neighborhoods. The last condition has the meaning of "transitivity"; it is essential in constructing the (causal-like) *proper order* of the horistological world (W, χ), noted $K^{=}(\chi) = K(\chi) \cup \delta$, where

$$K(\chi) = \{(e, \ell) \in W^2 :\ \{\ell\} \in \chi(e)\}.$$

Simple examples of different horistologies having the same proper order show that the horistological structures enclose more information than the order relations do. For instance, the horistologies χ_1 and χ_2 on $\mathbb{R}$, defined by

$$\chi_1(x) = \{P \subset \mathbb{R} : \exists r > 0 \textit{ such that } P \subseteq [x + r, \infty)\}, \chi_2(x) = \mathcal{P}\left((x, \infty)\right),$$

are different (namely $\chi_1 \subset \chi_2$), but $K^{=}(\chi_1) = K^{=}(\chi_2) =$ the usual order of $\mathbb{R}$.

Similar remarks remain valid for each of the subsequent structures.

Pair # 2: Uniform Topology / Uniform Horistology. We recall that uniform topologies are defined by filters of *environs*. In counterpart, we define a *uniform horistology* (briefly u.h.) by a family of relations $\mathcal{H} \subseteq \mathcal{P}(W^2)$, which satisfies the conditions:

[uh$_1$] $\pi \cap \delta = \varnothing$ for all $\pi \in \mathcal{H}$ (where δ means *equality*)

[uh$_2$] If $\pi \in \mathcal{H}$ and $\lambda \subseteq \pi$, then $\lambda \in \mathcal{H}$

[uh$_3$] If $\lambda,\ \pi \in \mathcal{H}$, then $\lambda \cup \pi \in \mathcal{H}$

[uh$_4$] $\forall \pi \in \mathcal{H}\ \exists \theta \in \mathcal{H}$ such that $[\omega \in \mathcal{H}] \Longrightarrow [\theta \supseteq \pi \circ \omega$ and $\theta \supseteq \omega \circ \pi]$.

The elements of $\mathcal{H}$ are called *prospects*, and the pair $(W, \mathcal{H})$ represents a *uniform horistological world*. The properties [uh$_2$] and [uh$_3$] show that $\mathcal{H}$ forms an ideal in $\mathcal{P}(W^2)$.

It is easy to see that if $(W, \mathcal{H})$ is a uniform horistological world, then function $\chi_{\mathcal{H}} : W \to \mathcal{P}(\mathcal{P}(W))$ of values

$$\chi_{\mathcal{H}}(e) = \{P \in \mathcal{P}(W) : \exists \pi \in \mathcal{H} \text{ such that } P \subseteq \pi[e]\}$$

is a horistology on W. In addition, for the *strict proper order* of $\mathcal{H}$ we have

$$K(\mathcal{H}) \stackrel{def.}{=} \cup\{\lambda : \lambda \in \mathcal{H}\} = K(\chi_{\mathcal{H}}).$$

Pair # 3: Sub-additive / Super-additive metrics. Perhaps the simplest case where we meet a pair of structures is that of the metrics, which can be either sub-additive (briefly s.a.) or super-additive (briefly S.a.). More exactly, the sub-additivity is the classical property of a metric ρ, meaning

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y),$$

while super-additivity is the opposite inequality. Because the super-additivity is not possible for each side of a triangle, we have to restrain the domain of such a metric to a strict part of $W \times W$, which usually is an order (i.e. reflexive, anti-symmetric and transitive binary relation). More exactly, if $\Pi \subset W \times W$ is an order on W, then function $\rho : \Pi \to \mathbb{R}_+$ is a *super-additive metric* if it satisfies the conditions:

[S.a.m$_1$] $\rho(e_1, e_2) = 0 \Longleftrightarrow e_1 = e_2$

[S.a.m$_2$] $\rho(e_1, e_3) \geq \rho(e_1, e_2) + \rho(e_2, e_3)$ for all $(e_1, e_2), (e_2, e_3) \in \Pi$.

If only " $\Longleftarrow$ " holds in [S.a.m$_1$], we say that ρ is a *S.a. pseudo metric.* The triplet (W, Π, ρ) is called S.a. *metric* (respectively *pseudo metric*) *world.* As a rule, ρ furnishes more information than Π.

Further restrictions to $\Lambda \subset \Pi$ are always possible, but the converse process, that of prolongation, is also significant (see [BT$_1$], Section 4 below, etc.).

Like in topological framework, where the usual (i.e. sub-additive) metrics produce uniform topologies, the S.a. metrics generate u. horistologies. In fact, if $\rho : \Pi \to \mathbb{R}_+$ is a S.a. metric on W, then the ideal $\mathcal{H}_\rho$, generated by hyperbolic prospects π_r of various radii $r > 0$,

$$\pi_r = \{(e_1, e_2) \in \Pi : \rho(e_1, e_2) > r\},$$

represents a u. horistology. More precisely, $\pi \in \mathcal{H}_\rho$ if and only if $\pi \subseteq \pi_r$ for some $r > 0$. In addition, $K^{=}(\mathcal{H}_\rho) = \Pi$.

Differently from the usual (s.a.) ones, the super-additive metrics allow quantifications. Using a real constant $\hbar > 0$, to each (p-) metric $\rho : \Pi \to \mathbb{R}_+$, we may attach the S.a. pseudo metric $\rho_\hbar : \Pi \to \mathbb{R}_+$, of values

$$\rho_\hbar(e_1, e_2) = \begin{cases} 0 & if \quad \rho(e_1, e_2) \leq \hbar \\ \rho(e_1, e_2) & if \quad \rho(e_1, e_2) > \hbar \end{cases}$$

called *$\hbar$-quantification* of ρ. Alternatively,

$$\Pi_\hbar = \{(e_1, e_2) \in \Pi : \rho(e_1, e_2) > \hbar\}$$

is a strict order of W, and $\rho|_{\Pi_\hbar}$ is a S.a. metric, which equals $\rho_\hbar$ on $\Pi_\hbar$.

Pair # 4: Sub-additive / Super-additive norms. Similarly to metrics, the sub- and super-additive norms in (real) linear spaces form a pair of structures. In the classical theory, the term "sub-additive" is usually omitted, and we simply speak of *normed* (in particular *Banach*) spaces. The property of super-additivity of a norm imposes a restriction of its domain to an order cone. So, if $\Pi \subset W \times W$ is an order on W, then $\rfloor \cdot \lfloor : \Pi[0] \to \mathbb{R}_+$ is a *super-additive* (briefly S.a.) *norm* if it satisfies the conditions:

[S.a.n$_1$] $\rfloor e \lfloor = 0 \iff e = 0$
[S.a.n$_2$] $\rfloor \lambda e \lfloor = \lambda \rfloor e \lfloor$ for all $e \in \Pi[0]$ and $\lambda \geq 0$
[S.a.n$_3$] $\rfloor e_1 + e_2 \lfloor \geq \rfloor e_1 \lfloor + \rfloor e_2 \lfloor$ for all $e_1, e_2 \in \Pi[0]$.

If "$\Longrightarrow$" is not valid in [S.a.n$_1$], we say that $\rfloor \cdot \lfloor$ is a S.a. *pseudo norm*. The triplet $(W, \Pi, \rfloor \cdot \lfloor)$ defines a *S.a. normed world* (*space*). It is a particular case of a S.a. metric world, in the sense that if $\rfloor \cdot \lfloor : \Pi[0] \to \mathbb{R}_+$ is a S.a. (pseudo) norm, then $\rho : \Pi \to \mathbb{R}_+$ of values

$$\rho(e_1, e_2) = \rfloor e_2 - e_1 \lfloor$$

is a S.a. (pseudo) metric.

Like the S.a. metrics, the S.a. norms also allow quantifications.

Pair # 5: Semi-definite / Indefinite inner products. Let W be a linear space over Γ, which is either $\mathbb{R}$ or $\mathbb{C}$. Function $\langle .,. \rangle : W \times W \to \Gamma$ is called *inner product* on W (see [BJ], etc.) if it satisfies the conditions:

[I$_1$] $\langle \alpha e_1 + \beta e_2, e_3 \rangle = \alpha \langle e_1, e_3 \rangle + \beta \langle e_2, e_3 \rangle$
[I$_2$] $\langle e_1, e_2 \rangle = \overline{\langle e_2, e_1 \rangle}$.

An immediate consequence of [I$_2$] is $\langle e, e \rangle \in \mathbb{R}$, so that we may distinguish different cases. The semi-definite inner product spaces are characterized by a constant sign of the square $\langle e, e \rangle$, which is either non-negative (as in *scalar product* and *Hilbert* spaces) or non-positive at all points. Otherwise, there exist $e_1, e_2 \in W$ such that $\langle e_1, e_1 \rangle > 0$ and $\langle e_2, e_2 \rangle < 0$, when we say that the inner product is *indefinite*. In particular, if $\langle e, e \rangle \geq 0$ for all $e \in W$, then function $\|\cdot\| : W \to \mathbb{R}_+$, of values $\|e\| = \sqrt{\langle e, e \rangle}$, is a (sub-additive) norm on W. We can realize a similar construction in the peer case of indefinite (real) inner product spaces, by restricting the problem to subspaces of Π_1 Pontrjagin type (as in [B-P$_1$]). In fact, we may organize these subspaces as relativist worlds of events $W = \mathbb{R} \times H$, where $(H, (\cdot|\cdot))$ is a scalar product space. Then the inner product of the events $e_1 = (t_1, s_1)$ and $e_2 = (t_2, s_2)$ takes the form

$$\langle e_1, e_2 \rangle = c^2 t_1 t_2 - (s_1 | s_2),$$

where $c > 0$ usually stands for the speed of light. The function $\rfloor \cdot \lfloor_t : K[0] \to \mathbb{R}_+$, of values

$$\rfloor e \lfloor_t = \sqrt{c^2 t^2 - s^2},$$

where $s^2 = (s \mid s)$, is a S.a. norm.

B. Dual morphisms. The above pairs of structures ask corresponding **pairs of morphisms**. The morphisms of the (u.) topological structures are the well known (u.) *continuous* functions. In counterpart, the morphisms of the horistological structures are *discrete functions* (details concerning this type of functions can be found in [BT$_2$]). More exactly, let (W, χ) and (V, ψ) be horistological worlds, and let $f : W \to V$ be a function. We say that $g \in V$ is a *germ* of f at e_0 (when e *starts from* e_0, etc.), and we write

$$g = \underset{e_0 \to e}{germ}\ f(e)$$

if for every $P \in \chi(e_0)$ we have $f(P) \in \psi(g)$. Usually, the germ is not unique; we may note the set of all germs of f at e_0 by $Germ(f, e_0)$. However, particular types of germs form singletons in the set $Germ$ (see [CI$_1$]). If

$$f(e_0) \in Germ(f, e_0),$$

we say that f is *discrete* at e_0. This means that $f(P) \in \psi(f(e_0))$ whenever $P \in \chi(e_0)$, briefly

$$f(\chi(e_0)) \subseteq \psi(f(e_0)).$$

In other terms, discreteness of a function regards the direct images of the perspectives, while continuity operates with counter-images of neighborhoods. It is easy to see that discreteness of a function extends the condition of boundedness (see [H-NH], subsection C here below, etc.). We mention that the discrete functions preserve the proper orders of the corresponding horistologies.

If $(W, \mathcal{H})$ and $(V, \mathcal{U})$ are uniform horistological worlds, we similarly define the *uniform discreteness* of f, namely

$$f_{II}(\mathcal{H}) \stackrel{def}{=} \{f_{II}(\pi) : \pi \in \mathcal{H}\} \subseteq \mathcal{U},$$

where $f_{II}(\pi) = \{(f(e_1), f(e_2)) : (e_1, e_2) \in \pi\}$. Obviously, the uniform discreteness implies the point-wise discreteness of f on W.

Similarly to convergence, we may transfer discreteness to nets: If $X = D$ is a directed set, and $\bar{D} = D \cup \{\infty\}$, where $\infty \notin D$, then we define the horistology χ^D of $\bar{D}$ by

$$\chi^D(a) = \begin{cases} \{\varnothing\} & if\ a \in D \\ \{P \subset D : \exists b \in D\ such\ that\ P \subseteq (\leftarrow, b)\} & if\ a = \infty \end{cases}.$$

If the prolongation $\overline{f} : \overline{D} \to Y$, defined by

$$\overline{f}(a) = \begin{cases} f(a) & if\ a \in D \\ g & if\ a = \infty, \end{cases}$$

is discrete at ∞, then we say that g is a *germ* of f, respectively the net f is *emergent* from g.

C. Bornologies. The structures devoted to boundedness, known as *bornologies* (see [H-NH], etc.), are similar to horistologies in many aspects. We recall that a family of parts $\mathcal{B} \subseteq \mathcal{P}(S)$ defines a bornology on the non-void set S if the following conditions hold:

[b$_1$] $\cup\{B : B \in \mathcal{B}\} = S$;

[b$_2$] $[(B \in \mathcal{B})\ \&\ (C \subseteq B)] \Longrightarrow (C \in \mathcal{B})$;

[b$_3$] $(B, C \in \mathcal{B}) \Longrightarrow (B \cup C \in \mathcal{B})$.

The pair $(S, \mathcal{B})$ is named *bornological space*, and the elements of $\mathcal{B}$ are called *bounded sets*.

The similarity between [HOR$_2$], [HOR$_3$] and [b$_2$], [b$_3$] suggests a deeper relation between horistologies and bornologies. In fact, let $(S, \mathcal{B})$ be a bornological space, and let Θ be an element outside S. On the set $\overline{S} = S \cup \{\Theta\}$ we define the *bornological horistology* β, such that

$$\beta(x) = \begin{cases} \mathcal{B} & if\ x = \Theta \notin S \\ \{\varnothing\} & if\ x \in S\ . \end{cases}$$

It is easy to see that β fulfils the conditions [HOR_1] and [HOR_4], hence it is a horistology. The corresponding order of β is

$$K(\beta) = \{(\Theta, x) : x \in S\} \cup \delta,$$

for which we have $K(\beta)[\Theta] = \overline{S}$, and $K(\beta)[x] = \{x\}$ at any other $x \in S$. In the nontrivial case $S \neq \varnothing$, according to [HOR_1] we have $\mathcal{B} \neq \{\varnothing\}$, hence the *finite* sets are always bounded. In particular, $\mathcal{B}$ may exactly consist of finite sets, when the bornology is considered *discrete*. Consequently, the study of the horistological spaces includes boundedness, and respectively finiteness.

2 Physical arguments

From the very beginning, discreteness was somehow reduced to finiteness. The oldest example in this respect is the rejection of an endless mass divisibility, which has led to the idea of elementary particles. Later on, when topology has been imposed as the framework of continuity, discreteness has become the opposite of continuity. Here we can speak of continuous / discontinuous functions, discrete topology, isolated points, etc. There are however a lot of physical arguments to consider discrete sets outside any topology, which are rather based on finiteness. From the huge literature involving this aspect, we will limit ourselves to select only three directions:

(i) The analysis of some singularities and divergent integrals have led to the Schild's conjecture (see [SA]) that "It seems likely that a physical theory based on a discrete space-time background will be free of the infinities which trouble contemporary quantum mechanics." His way to a discrete space-time was to reduce the universe of events to a cubic lattice, but this was hardly accepted because of the Lorentz invariance failure. In spite of several improvements of the cubic lattice techniques (see [HL], [AY], etc.), new difficulties remained inevitable. Therefore this variant of discreteness is nowadays abandoned.

(ii) More recently (see [BLMS], [SR], [HJ], etc.), many scientists prefer the technique of endowing the Lorentzian manifolds with discrete systems of points. They realize this type of discreteness by the so-called process of sprinkling, which is a random selection of points such that the number of sprinkled points in a domain depends only on its volume (Poisson distribution). The set of all sprinkled points forms a causal set, which is a partially ordered (transitive and irreflexive) and locally finite set (each order interval contains a finite number of points). So we may conclude that the sprinkling techniques do reduce discreteness to local finiteness by renouncing the regularity of a cubic lattice.

(iii) An interesting way to elude the problem of space and / or time discreteness was proposed by Snyder in [SHS], where he starts by claiming that it is more important to assure discrete results of our measurements than looking for discrete parts of the universe. This type of discreteness can be easily done within the general quantum physics scheme, where we represent the physical quantities (space and time in this case) by Hermitian operators whose spectra consist of possible results of the measurements. Apart from the difficulties relative to the Lorentz invariance of these spectra, the Snyder's theory doesn't solve the problem of space-time discreteness. We have to appreciate his conclusion that "... the usual assumption concerning the continuous nature of space-time are not necessary for Lorentz invariance...", but practically he only transfers the problem to a higher level, namely to the operators on a Hilbert space (see also [G-P], [B-G], [SBG], etc.) .

Besides the reduction of discreteness to (eventually local) finiteness, many physicists have expressed the feeling that the topologies (especially the Euclidean ones) are not adequate to topics like the relativist and quantum physics, and have suggested to look for other structures. In this respect we will mention only the following cases.

(I) One of the most shocking points of the special relativity was the clocks paradox, which expresses the reversed rule of time-like triangles. In spite of the experimental confirmations of this rule, the first tendency was to avoid such "paradoxes" from the theory, so that the universe of events was transformed into an Euclidean $\mathbb{R}^4$ by introducing an "imaginary time" $x_4 = ict$. Based on this Minkowskian structure of the space-time, Einstein was initially convinced that our universe is a four-dimensional continuum, when he said (for example in [EA]) "Und doch ist keine Aussage banaler als die, daß unsere gewohnte Welt ein vierdimensionales zeiträumliches Kontinuum ist" (i.e. "And still there is no more banal statement than saying that our customary universe is a four-dimensional space-time continuum").

However, the development of the general relativity, which includes gravitation in the relativistic framework, has led Einstein himself to doubt about the continuum character of the space-time. In a letter to Walter Dallenbach (1916) we find the opinion that "The problem seems to me how one can formulate statements about a discontinuum without calling upon continuum space-time as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately".

In this paper we try to show that the horistologies fulfill the Einstein's requirement for mathematical structures of discreteness without calling upon continuum. More exactly, we point out a special meaning to the notion "discrete set", which is available in these structures, and offer an intrinsic way to build up a structurally discrete space-time.

(II) A fundamental assumption in the quantum physics is the existence of quanta of energy, of value $\varepsilon = \hbar\omega$, where ω is the angular speed, $\hbar = h/2\pi$, and $h = 6.625 \cdot 10^{-34} J \cdot s$ is the Planck's constant. An immediate consequence of this postulate is the existence of some minimal values for other quantities (e.g. length), which suggests a discrete distribution of values in the corresponding spaces. In addition, the Heisenberg's uncertainty principle $|\Delta x| \cdot |\Delta p| \geq \hbar$ (see

[HW], or any treatise in quantum mechanics) hides a particular type of horistological spaces since the functional of values $\sqrt{|\Delta x| \cdot |\Delta p|}$ is a super-additive norm (see section 4 below). Because similar relations hold for other pairs of quantities (e.g. time and energy in Bohr's inequality $|\Delta t| \cdot |\Delta E| \geq \hbar$, etc.), we may conclude that the horistologies do correspond to the quantum sense of discreteness too.

(III) Based on his remarkable theorem "Causality implies the Lorentz group" (proved in [ZEC]), Zeeman has severely criticized (see [ZEC_1]) the use of the Euclidean topology on the Minkowskian space-times. In fact, the Euclidean topology is locally homogeneous (while the space-time is not), and the group of all homeomorphisms is too wide and has no physical significance (in comparison to the Lorentz group). Therefore, Zeeman proposes the *fine topology* instead of the Euclidean one, and he shows that this new structure allows recovering the light cone at each event, and all of its homeomorphisms are generated by Lorentz transformations and dilatations (see also [NS], [BT_3]).

Unfortunately, Zeeman's fine topology is locally uncountable, hence there is no metric to describe it, including the intrinsic one. The restrained metrics may ameliorate this trouble (see [BT_2], [BT_4]), but the description of discreteness by these topologies remains unsolved.

3 Discrete sets

In a topological space (S, τ), the term "discrete" refers to subsets of S, which have the property that we can *isolate* each of its points by sufficiently small neighborhoods. More exactly, if $x \in M \in \mathcal{P}(S)$, then x is *isolated* in M (M is *discrete at* x, etc.) if there exists $V \in \tau(x)$ such that $M \cap (V \setminus \{x\}) = \varnothing$. Similarly to other types of sets of great importance in topology (e.g. open, closed, see [PG], etc.), the family of discrete sets characterizes the topology τ. For example, in the case when S itself (hence each subset $M \subseteq S$) is discrete, we obtain the finest topology, noted τ_0, called *discrete*, and defined by

$$V \in \tau_0(x) \Longleftrightarrow x \in V.$$

However, except the trivial case of the discrete topology, the topologically discrete sets do not respect some generally expected features of discreteness. In this respect we mention the following objections:

[Ob_1] The finite sets may be not discrete. (For example $M = \{0, 1\}$ in $\mathbb{R}$, endowed with the topology τ_R of unbounded to the right neighborhoods.)

[Ob_2] There is no lower limit for the size of neighborhoods. (In metric spaces, each open sphere is a neighborhood for all its interior points, but the meet of two open spheres generally contains a strictly smaller sphere.)

[Ob_3] Discreteness is not preserved by continuous functions. (If τ_E and τ_R denote the Euclidean, respectively the open to the right topologies on $\mathbb{R}$, then the identical function $\iota : (\mathbb{R}, \tau_E) \to (\mathbb{R}, \tau_R)$ is everywhere continuous; however, $M = \{0, 1\}$ is discrete in τ_E, but $\iota(M) = M$ is non-discrete in τ_R.)

The only explanation is the inadequacy of the topological structures for the study of discreteness. As a matter of fact, within the topological framework, the opposite of "continuum / continuous" is "discontinuum / discontinuous", but not "discrete". We may interpret this situation as a hint to study discreteness in other than topological structures, primarily in the horistological ones.

The horistological discreteness expresses the same idea of "separating" the events of a set. Differently from topological discreteness, the horistological one depends on some previously chosen order relations. More exactly, let us consider that Λ is a strict order on the horistological world (W, χ), such that $\Lambda \subseteq K(\chi)$, and let M be a subset of W. We say that an event $e \in M$ is Λ-*detachable* from M (alternatively, M is Λ-*discrete* at e, etc.) if

$$M \cap \Lambda[e] \in \chi(e).$$

The set of all Λ-detachable points of M is called Λ-*discrete part* of M, and we note it $\partial_\Lambda(M)$. If each point of M is Λ-detachable, i.e. $\partial_\Lambda(M) = M$, then we consider that M is Λ-*discrete*. The function $\partial_\Lambda : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, which extracts the Λ-discrete part $\partial_\Lambda(M)$ of each subset $M \in \mathcal{P}(W)$, is called *operator of* Λ-*discreteness*.

In the case $\Lambda = K(\chi)$, we may omit mentioning the order, and simply speak of detachability, discreteness, etc.

Similarly to the discrete topology τ_0, to each strict order Λ on a set W, there corresponds a horistology, noted χ_Λ, where all $M \subseteq W$ are $\Lambda-$discrete and $K(\chi_\Lambda) = \Lambda$. We define it by $\chi_\Lambda(e) = \mathcal{P}(\Lambda[e])$ at each $e \in W$. Obviously, χ_Λ is the finest horistology on W (in the sense of [BT$_3$]), for which the strict proper order equals Λ. In particular, if $\Lambda = \emptyset$, then $\chi_\emptyset$, defined by $\chi_\emptyset(e) = \{\varnothing\}$ at each $e \in W$, is the coarsest horistology on W, in which all sets are discrete.

Considering discrete sets in horistological structures (including $\chi_\emptyset$) makes the above objections [Ob$_1$], [Ob$_2$] and [Ob$_3$] disappear. Relative to this notion of discreteness we mention the following properties (established in [B-P$_4$]):

$[d_0]$ $\partial_\Lambda(M) \subseteq M$ for all $M \in \mathcal{P}(W)$;
$[d_1]$ *card* $M \in \mathbb{N} \Longrightarrow \partial_\Lambda(M) = M$ (unlike [Ob$_1$]);
$[d_2]$ $L \subseteq M \Longrightarrow L \cap \partial_\Lambda(M) \subseteq \partial_\Lambda(L;)$
$[d_3]$ $\partial_\Lambda(M) \cap \partial_\Lambda(L) \subseteq \partial_\Lambda(M \cup L)$;
$[d_4]$ $e \in \partial_\Lambda(M) \Longleftrightarrow e \in M \cap \partial_\Lambda(\{e\} \cup \Lambda[M \cap \Lambda[e]])$;
$[d_5]$ For all $e \in W$ and $M \subseteq \Lambda[e]$ we have

$$e \in \partial_\Lambda(\{e\} \cup M) \Longleftrightarrow e \in \partial_\Lambda(\{e\} \cup \Lambda[M]);$$

$[d_6]$ $\Pi \subseteq \Lambda \Longrightarrow \partial_\Lambda(M) \subseteq \partial_\Pi(M)$;
$[d_7]$ $\{\partial_\Lambda(M) = M$ and $\Pi \subseteq \Lambda\} \Longrightarrow \partial_\Pi(M) = M$;
$[d_8]$ $\{\partial_\Lambda(M) = M$ and $L \subseteq M\} \Longrightarrow \partial_\Lambda(L) = L$;
$[d_9]$ $\partial_\Lambda(\partial_\Lambda(M)) = \partial_\Lambda(M)$.

Conversely, we may select several properties from above to recover the entire horistological structure. If W is a non-void set, we say that function $\partial : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ is an *abstract operator of discreteness* if it satisfies the following conditions (inspired by $[d_1] - [d_4]$):

$[\partial_1]$ *card* $M \in \mathbb{N} \Longrightarrow \partial(M) = M$;
$[\partial_2]$ $L \subseteq M \Longrightarrow L \cap \partial(M) \subseteq \partial(L)$;
$[\partial_3]$ $\partial(M) \cap \partial(L) \subseteq \partial(M \cup L)$.
In addition, if Λ is a strict order on W, such that the equivalence
$[\partial_4]$ $e \in \partial(M) \Longleftrightarrow e \in M \cap \partial(\{e\} \cup \Lambda[M \cap \Lambda[e]])$
holds for all $M \in \mathcal{P}(W)$, then we say that Λ is *compatible* with ∂.

The triplet (W, ∂, Λ), where $[\partial_1] - [\partial_4]$ hold, is called *discreteness space.* As before, we say that the elements $e \in \partial(M)$ are *detachable* from M, $\partial(M)$ is the *discrete part* of M, and $M \in \mathcal{P}(W)$ is a *discrete set* if $\partial(M) = M$. If (W, ∂, Λ) is a discreteness space, then function $\chi_{(\partial,\Lambda)} : W \to \mathcal{P}(\mathcal{P}(W))$, of values

$$\chi_{(\partial,\Lambda)}(e) = \{P \subseteq \Lambda[e] : e \in \partial(\{e\} \cup P)\},$$

is a horistology on W. In addition, the proper order of $\chi_{(\partial,\Lambda)}$ is $K(\chi_{(\partial,\Lambda)}) = \Lambda$. Recovering the horistology means that if ∂ is the operator of discreteness in (W, χ), relative to $K = K(\chi)$, then K is compatible with ∂, and $\chi_{(\partial,K)} = \chi$.

Relative to $[\mathrm{Ob}_2]$, it is easy to see that in horistological structures we may impose lower bounds to the perspectives. Because speaking of "size" presumes uniform structures, let $(W, \mathcal{H})$ be a uniform horistological world. We may realize the limitation by using an arbitrary prospect. More exactly, if we select and fix a prospect ξ in $\mathcal{H}$, then

$$\mathcal{H}_\xi = \{\pi \cap \xi : \pi \in \mathcal{H}\}$$

is a u.h. too, generally coarser (i.e. containing fewer elements) than $\mathcal{H}$. In the cases where the horistological structures derive from S.a. norms or metrics, the process of quantification has a similar effect.

To invalidate objection $[\mathrm{Ob}_3]$, we mention another result from [B-P$_4$]: Let (W_1, χ_1) and (W_2, χ_2) be horistological spaces, and let function $f : W_1 \to W_2$ be $1:1$ and strictly monotonic relative to the orders $K(\chi_1)$ and $K(\chi_2)$. If in addition $f_{II}(K(\chi_1)) = K(\chi_2)$, then

$$(f \text{ is discrete on } W_1) \Longrightarrow (f \text{ preserves detachability}).$$

4 Unifying by horistologies

Considering horistological structures obviously is a process of generalization. As usually, the utility of a generalization consists in giving unified vision on particular fields previously considered independent. In the sequel we put forward some common horistological features in relativity, quantum physics and other topics. Consequently, this unification by horistological structures offers a common mathematical language to these fields.

1. Special Relativity. As mentioned in Section 2 (I), Minkowski has introduced the "imaginary time" $x^4 = ict$ (which is an incorrect complexification) to transform the Einsteinean universe $W = \mathbb{R} \times \mathbb{R}^3$ of events into an Euclidean space. Nowadays, the structure of real indefinite inner product, which preserves the entire physical significance, is increasingly replacing the Minkowskian structure (see [CJ], [GR], [NG], [C-B], etc.).

A. Intrinsic horistology. The intrinsic inner product of the special relativity is a particular case of Pair #5 in Section 1, where c equals the speed of light in vacuum, $H = \mathbb{R}^n$ with $n = 1, 2$, or 3, and $(\cdot|\cdot)$ is the Euclidean scalar product of $\mathbb{R}^n$. Concretely, the inner product of the events $e_1 = (t_1, x_1, y_1, z_1)$ and $e_2 = (t_2, x_2, y_2, z_2)$ has the form

$$\langle e_1, e_2 \rangle = c^2 t_1 t_2 - x_1\ y_1 - x_2\ y_2 - x_3\ y_3.$$

The corresponding *quadratic form*

$$Q(e) = \langle e, e \rangle = c^2 t^2 - x^2 - y^2 - z^2,$$

where $e = (t, x, y, z)$, allows the construction of the relation of *causality*,

$$K = \{(e_1, e_2) : Q(e_1 - e_2) > 0, t_2 > t_1\}.$$

This inner product generates the *temporal norm* $\rfloor \cdot \lfloor_t : K^{=}[0] \to \mathbb{R}_+$, of values

$$\rfloor e \lfloor_t = \sqrt{c^2 t^2 - x^2 - y^2 - z^2},$$

and the *temporal metric* $\rho : K^{=} \longrightarrow \mathbb{R}_+$, defined by

$$\rho(e_1, e_2) = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}\ ,$$

which measures *proper time*. The *Aczél's Inequality*

$$|\langle e_1, e_2 \rangle| \geq \langle e_1, e_1 \rangle \langle e_2, e_2 \rangle\ ,$$

(see [A-V], [MDS}, [B-P$_1$], [CI], etc.) assures the super-additivity of the temporal norm and metric. As usually, the S.a. metric ρ generates the uniform horistology $\mathcal{H} \subseteq \mathcal{P}(W^2)$, defined by

$$\pi \in \mathcal{H} \Longleftrightarrow \exists r > 0 \textit{ such that } \pi \subseteq \{(e, \ell) \in W^2 : \rho(e, \ell) > r\}.$$

Finally, using the *hyperbolic perspectives* of vertex e and radius r,

$$H(e, r) = \{\ell \in K[e] : \rho(e, \ell) > r\},$$

we obtain the intrinsic horistology $\chi : W \to \mathcal{P}(\mathcal{P}(W))$, of values

$$\chi(e) = \{P \in \mathcal{P}(W) : \exists r > 0 \textit{ such that } P \subseteq H(e, r)\}.$$

It is easy to see that the qualitative structures $\mathcal{H}$ and χ preserve causality, i.e.

$$K(\chi) = K(\mathcal{H}) = K.$$

B. Quantified time. Because the physical meaning of $\rho(e_1, e_2) = \rfloor e_2 - e_1 \lfloor_t$ is proper time, quantifying ρ represents quantification of time corresponding to a generic constant $\hbar$. First, in the extended causality

$$\overline{K} = \{(e_1, e_2) : Q(e_1 - e_2) \geq 0, t_2 \geq t_1\},$$

we identify the strict order

$$K_\hbar = \{(e_1, e_2) : Q(e_1 - e_2) > \hbar^2, t_2 > t_1\},$$

then we define $\rho_\hbar : \overline{K}_\hbar \longrightarrow \mathbb{R}_+$ by

$$\rho_\hbar(e_1, e_2) = \begin{cases} \quad \rho(e_1, e_2) \ if \ \ (e_1, e_2) \in K_\hbar \\ 0 \qquad\quad if \ \ (e_1, e_2) \in \overline{K} \setminus K_\hbar \ . \end{cases}$$

It is easy to see that $\rho_\hbar$ satisfies the condition
[S.a.m$_1^\hbar$] $\rho_\hbar(e_1, e_2) = 0 \Longleftrightarrow (e_1, e_2) \in \overline{K} \setminus K_\hbar$.

C. Discrete functions. The property of a function $f : W \to W$ of being discrete at an event $e \in W$ takes the form

$$\forall \delta > 0 \ \exists \varepsilon > 0 \ such \ that \ [\rho(e, \ell) > \delta \Longrightarrow \rho(f(e), f(\ell)) > \varepsilon].$$

In particular, the Lorentz transformations are discrete functions since they are isometries (hyperbolic rotations).

The discreteness of a set $M \subset W$ expresses a separation in proper time between each event $e \in M$ and its causal consequences in M. In addition, we may speak of a *uniform discreteness*, if the separation in proper time $\rho(e, \ell)$ cannot go under a fixed lower limit $\hbar$, for all $e \in M$ and $\ell \in M \cap K[e]$.

To conclude, the Einsteinean universe of events is structurally discrete in the sense that its intrinsic structure is horistological.

2. Quantum Physics. An important characteristic of the quantum physics is the presence of quanta for physical quantities, e.g. energy, space, time, and so on. The simplest case concerns scalar quantities, for which the result of a measurement is a number $x \in \mathbb{R}$.

A. Scalar quanta. Almost unanimously, $\mathbb{R}$ is considered a pattern of continuum, especially when endowed with its Euclidean topology. However, as mentioned in Section 3, [Ob$_2$], the topological structures do not allow quanta. At this moment it is significant to remind that we may construct $\mathbb{R}$ on a purely horistological way (see [P-C]), hence it equally is a pattern of structural discreteness. In addition, we may endow $\mathbb{R}$ with a "quantified" horistology $\chi_\hbar : \mathbb{R} \to \mathcal{P}(\mathcal{P}(\mathbb{R}))$, which corresponds to the $\hbar$-quantified metric $\rho_\hbar$, where $\rho(x, y) = y - x$ is defined on the usual order of $\mathbb{R}$. More exactly,

$$P \in \chi_\hbar(x) \Longleftrightarrow [\exists r > \hbar \ such \ that \ P \subseteq H(x, r)].$$

Because $\rho_\hbar(x, y)$ is either 0 or greater than $\hbar$, we may use it to measure scalar quantified physical quantities.

B. Heisenberg horistologies. In Section 2 II, we have mentioned the Heisenberg uncertainty principle as another feature of quantum physics, which involves pairs of physical quantities, e.g. position-impulse, time-energy, etc. It is easy to see that the Heisenberg relations deal with super-additive functionals. For example, relation $|\Delta x| \cdot |\Delta p| \geq \hbar$ involves a quadratic functional that measures area, namely $\mathcal{A} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, of values $\mathcal{A}(\Delta x, \Delta p) = \Delta x \ \cdot \ \Delta p$. Obviously, $\mathcal{A}$ is super-additive, i.e.

$$(\Delta x_1 + \Delta x_2)(\Delta p_1 + \Delta p_2) \geq \Delta x_1 \Delta p_1 + \Delta x_2 \Delta p_2.$$

In addition, if Π is the product order on $S = \mathbb{R} \times \mathbb{R}$, for which $\Pi[0] = \mathbb{R}_+ \times \mathbb{R}_+$ represents the positive cone, then the functional $\rfloor \cdot \lfloor \, : \Pi[0] \to \mathbb{R}_+$, of values $\rfloor(\Delta x, \Delta p)\lfloor \, = \sqrt{\Delta x \Delta p}$, is a S-a norm on $\mathbb{R}^2$. In fact, as long as we deal with positive quantities, the super-additivity, i.e.

$$\sqrt{\Delta x_1 \Delta p_1} + \sqrt{\Delta x_2 \Delta p_2} \leq \sqrt{(\Delta x_1 + \Delta x_2)(\Delta p_1 + \Delta p_2)},$$

reduces to the obvious inequality of the arithmetic and geometric means

$$\sqrt{(\Delta x_1 \Delta p_1)(\Delta x_2 \Delta p_2)} \leq \frac{1}{2}(\Delta x_1 \Delta p_2 + \Delta x_2 \Delta p_1).$$

Consequently, the Heisenberg's relation of uncertainness takes the form

$$\rfloor(|\Delta x|, |\Delta p|)\lfloor \, \geq \sqrt{\hbar},$$

i.e. at each measurement of x and p, $(|\Delta x|, |\Delta p|)$ necessarily places in a sufficiently large hyperbolic $\rfloor \cdot \lfloor$- perspective of the origin.

Naturally, we may transfer this property to the S.a. metric $\rho : \Pi \to \mathbb{R}_+$, generated by $\rfloor \cdot \lfloor$, and finally to the corresponding horistology. The $\hbar$-quantified metric $\rho_\hbar$ generates the so called Heisenberg horistology, noted $\chi_\hbar$.

Horistologies of Heisenberg type axist on any product space $S = W_1 \times W_2$, where (W_1, K_1, ρ_1) and (W_2, K_2, ρ_2) are S.a (p-) metric worlds. First, on the product order $\Pi = K_1 \times K_2$, we define the *product* S.a. (p-) metric ρ, of values

$$\rho((x, u), (y, v)) = \sqrt{\rho_1(x, y)\rho_2(u, v)},$$

and finally, we $\hbar$-quantify ρ. The resulting horistology of $(S, \Pi, \rho_\hbar)$ extends the Heisenberg horistology $\chi_\hbar$ of $\mathbb{R} \times \mathbb{R}$, from above.

C. Discrete functions. We may reformulate the Heisenberg's uncertainty principle in terms of discrete functions: Let $\mathfrak{M}$ be the set of all possible measurements of x and p at the quantum physics scale, and let $m^*(\notin \mathfrak{M})$ be the ideal measurement that furnishes the exact values of these quantities, i.e. $\Delta x = 0$ and $\Delta p = 0$. On the space $\mathfrak{M}^* = \mathfrak{M} \cup \{m^*\}$ we consider the horistology $\mathfrak{h}$ of values

$$\mathfrak{h}(\mathfrak{m}) = \begin{cases} \mathcal{P}(\mathfrak{M}) \; if \; \mathfrak{m} = \mathfrak{m}^* \notin \mathfrak{M} \\ \{\varnothing\} \qquad\quad if \; \mathfrak{m} \in \mathfrak{M}. \end{cases}$$

Finally, let us define the *error function* $\mathfrak{E}$: $\mathfrak{M}^* \to \mathbb{R} \times \mathbb{R}$ by $\mathfrak{E}(\mathfrak{m}) = (|\Delta x|, |\Delta p|)$, where Δx and Δp represent errors of measurement for the position and the impulse of the particle. The Heisenberg's uncertainty principle exactly says that function $\mathfrak{E}$ is discrete at m^* relative to the horistology $\mathfrak{h}$ on $\mathfrak{M}^*$, and the Heisenberg horistology $\chi_\hbar$ on $\mathbb{R} \times \mathbb{R}$.

3. Cosmology. Without doubt, many differences between quantum physics, relativity and cosmology are due to the diversity of scales at which these theories operate. According to Laurent Nottale (see [NL], [NL$_1$], etc.), we may connect these theories by using adequate **scale transformations**.

Let us transform the field φ into $\varphi^/$ by the change of scale of ratio $q = \Delta x / \Delta x^/$ according to the *power law* $\varphi^/ = \varphi \cdot q^{\mathfrak{d}}$. In terms of renormalization group, $\mathfrak{d}$ represents the *anomalous dimension* of the field φ, while in fractal interpretation we have $\mathfrak{d} = D - D_T$, where D is the fractal dimension and D_T is the topological dimension. This power law allows the form

$$\ln(\varphi^/ / \varphi_0) = \ln(\varphi \ / \ \varphi_0) + \mathfrak{d} \ \ln(\Delta x / \Delta x^/),$$

which is strongly resembling the Galilean transformation $(x, t) \to (x^/, t^/)$,

$$\begin{cases} x^/ = x + vt \\ t^/ = t \ . \end{cases}$$

A critical analysis of these transformations has led Nottale to adopt a Lorentz-type transformation of a field under a change of scale. More exactly, the transformation $(\ln \varphi, \mathfrak{d}) \to (\ln \varphi^/, \mathfrak{d}^/)$, caused by a change of scale of characteristic $\log_k \sigma$, has the form

$$\begin{cases} \log_k(\varphi^/ / \varphi_0) = \frac{\log_k(\varphi/\varphi_0) + \mathfrak{d} \ \log_k \sigma}{\sqrt{1 - \log_k^2 \sigma}} \\ \mathfrak{d}^/ = \frac{\mathfrak{d} \ + (\log_k \sigma) \log_k(\varphi/\varphi_0)}{\sqrt{1 - \log_k^2 \sigma}} \ . \end{cases}$$

To stress on the analogy with the relativity of motion, we may remark the correspondence

$$x \longleftrightarrow \log_k(\varphi/\varphi_0), \ ct \longleftrightarrow \mathfrak{d}, \text{ and } \frac{v}{c} \longleftrightarrow \log_k \sigma.$$

An immediate consequence of this similarity is the intrinsic structure of the plane of coordinates $\log_k(\varphi/\varphi_0)$ and $\mathfrak{d}$. The specific condition $\log_k \sigma \in (-1, 1)$ shows that the base k of the logarithms should be great enough, to assure $\sigma \in (\frac{1}{k}, k)$ for all physically accepted σ. For each fixed k, the fundamental invariant of the Nottale transformations, which is $\mathfrak{d}^2 - \log_k^2(\varphi/\varphi_0)$, furnishes a "causal" order Λ, and a S.a. norm $\dashv \cdot \vdash : \Lambda[(0,0)] \to \mathbb{R}_+$, of values

$$\dashv (\log_k(\varphi/\varphi_0), \mathfrak{d}) \vdash \ = \sqrt{\mathfrak{d}^2 - \log_k^2(\varphi/\varphi_0)}$$

Finally, the plane $\mathbb{R} \times \mathbb{R}$, of the variables $\log_k(\varphi \ / \ \varphi_0)$ and $\mathfrak{d}$, becomes a horistological space via the S.a. metric generated by $\dashv \cdot \vdash$. In addition, the Nottale change of scale is a discrete transformations of the plane $\mathbb{R} \times \mathbb{R}$ relative to the horistology generated by $\dashv \cdot \vdash$ because the isometries are always discrete functions.

5 Other topics involving S.a. and horistology

Besides discreteness, the horistological structures may properly contribute to the study of other problems. Without going into details, we mention several topics where S.a. and horistology play an important role.

1. Theory of proof in formal logic. The general form of a theorem (see [ME], etc.) is "If H (hypothesis), then C (conclusion)", briefly $H \Longrightarrow C$.

The proof consists in finding a deductive sequence, which consists of at least intermediate fact, say I, such that $H \Longrightarrow I \Longrightarrow C$. If ∂ denotes the difficulty of deduction, then obviously $\partial(H,C) > \partial(H,I) + \partial(I,C)$, which means super-additivity. In other terms, S.a. is the essential justification of every proof.

2. Indefinite inner products. In spite of the fact that many monographs treat the indefinite inner products without mentioning S.a. (e.g. [BJ]), the Aczél's inequality is fundamental in such spaces (see [A-V], [B-P$_1$], etc.). In addition, the indefinite inner product spaces seem to be more adequate framework of problems like the connection between neutral vectors and isometries: For example, Zeeman's Theorem (see [ZEC], [PWF], [BT$_5$], etc.) naturally holds in Krein spaces.

3. Duality theory of L^pspaces. It is well known that for $p \geq 1$, the space of linear and continuous functionals (called dual) on L^p is L^q, where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, L^p is a Banach space with the norm $\|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}}$. If $p < 1$, we face serious difficulties: $\|\cdot\|_p$ is a S.a. norm, the only linear and continuous functional on L^p is identically null, etc. (see [DMM], etc.). In [CB] we see how to obtain results similar to the classical ones by using this S.a. norm, horistology and discreteness.

4. Stability / instability. Most frequently, we express the property of stability of a particular evolution of a dynamical system by the continuity of the function "*initial state* $\rightarrow$ *evolution*". If the space of initial states and that of evolutions allow horistological structures (e.g. the mathematical pendulum), then the discreteness of this function defines the *discrete instability* (see [B-P$_3$], [B-P$_4$], etc.), which is much stronger than non-stability. Paper [PM$_2$] shows how to study the discrete instability by discrete Lyapunov functions.

5. Concave gauge optimization. Initially, the problem of optimization referred to objective functions defined on convex sets. However, there are practical problems asking optimization of a concave gauge functions (see [B-C], etc.). These functions act on convex parts of a cone, which usually are perspectives of S.a. norms.

6. Invariant description of the movement. Using S.a. metrics, indefinite inner products and horistology we may avoid the Minkowskian (false) complexification and express the relativist topics (Kinematics, Kinetics, Electromagnetism, etc.) in real variables (see [CJ], [GR], [B-C], etc.). The same tools allow relativist interpretation of many results in hyperbolic geometry. For example, based on a relativist Frenet referential, [B-P-T] gives an invariant description of the movement of a particle.

7. Relativist dynamical systems. The relativist character of a dynamical system results by the description of its evolution in terms of events. The main advantage of this approach is the direct relieve of the intrinsic properties, which are invariant under changes of observers, e.g. stability (see [B-P$_6$], etc.). Obviously, these representations have to respect the principles of the Einsteinean relativity, including causality (see [J-T], {BT$_6$], [B-P$_5$], etc.). The proper time, proper space and light signals are essential in parallel and distributed systems theory (see [LL], [M-B], etc.)

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