REMARKS ON THE TOPOLOGIES FOR MINKOWSKI SPACE-TIME

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The conditions that should be satisfied by any topology for Minkowski space-time were formulated in [6], and they are:

(1) The causal order can be deduced from the considered topology, and

(2) The group of homeomorphisms of the considered topology is generated by the inhomogeneous Lorentz group and dilations.

Some examples of such topologies are constructed in [6], [3] and [4], using some particular properties of $\mathbb{R}^4 = M$.

The first requirement obviously leads to the more general problem of deducing orders from topological structures. In this note, we will discuss this problem in terms of [1].

The second requirement is usually solved in two steps: One is Zeeman's theorem "*The group of automorphisms of the set M, preserving the partial order, is* G_0 " (see [5]), and the other is the relation between the group of homeomorphisms and that of order preserving automorphisms. In [2] we may find some extensions of Zeeman's theorem, while the last problem constitutes the second subject of our remarks on ordered topological spaces.

To be more concise, we only discuss the case of the quasi-uniform (q.u.) spaces, the case of general topological spaces being similar.

1. Definition. Let Π be a preorder and Q be a q.u. on S. We say that the q.u.

$$Q_\Pi = \{\pi \in \mathscr{P}(S^2) : \exists \lambda \in Q \textit{ such that } \pi \supseteq \lambda \cap \Pi\}$$

is the *restriction* of Q to Π.

We recall (see [1]) that a base $\mathscr{B}$ of the q.u. Q is *transitive* if :

[T] For each $\lambda, \pi \in \mathscr{B}$ there exists $\omega \in \mathscr{B}$ such that $\lambda \circ \pi \subseteq \omega$.

If so, we say that

$$\Pi(\mathscr{B}) = \cup\{\pi : \pi \in \mathscr{B}\}$$

is the *preorder generated* by $\mathscr{B}$.

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2. Proposition. Let Π be a preorder on the q.u. space [S, Q]. Then Q_{Π} is always finer than Q, and the following three properties are equivalent:
(a) $Q_{\Pi} = Q$,
(b) $\Pi \in Q$,
(c) There exists a transitive base $\mathscr{B}$ of Q such that

$$\Pi = \Pi(\mathscr{B}).$$

3. Definition. If [S, Q] is a q.u. space, we say that

$$\Pi(Q) = \cap\{\Pi(\mathscr{B}) : \mathscr{B} \textit{ is a transitive base of } Q\}$$

is the *proper order* of the q.u. Q.

4. Remark. If $\mathscr{B}$ is a transitive base of the q.u. Q on S, and Q generates the topology τ, then there exists a transitive base β of τ such that

$$\Pi(\beta) = \Pi(\mathscr{B}),$$

but between $\Pi(Q)$ and $\Pi(\tau)$ it is possible to have a strict inclusion.

5. Remark. Because restraining a q.u. (or a topology) to a preorder generated by one of its transitive bases does not modify the initial structure (Proposition 2), one could hope to have

$$Q_{\Pi(Q)} = Q \text{ and } \tau_{\Pi(\tau)} = \tau .$$

Simple examples show that it is not generally the case, hence the following definition is consistent:

6. Definition. We say that the q.u. space [S, Q] is *self-refined* if

$$Q_{\Pi(Q)} = Q .$$

7. THEOREM. *Every homeomorphism of a self-refined q.u. Q is also an automorphism of the proper preorder* $\Pi(Q)$.

Proof. If f is a q.u. continuous transformation of S, there exists $\lambda \in Q$ such that

$$f_{\text{II}}(\lambda) \subseteq \Pi(Q).$$

It is not difficult to prove that

$$\mathscr{B}_{\lambda} = \{\lambda^n : n \in \mathbb{N}\} \cup \{\lambda \cap \pi : \pi \in Q\}$$

is a transitive base of Q, and

$$\Pi(\mathscr{B}_{\lambda}) = \cup\{\lambda^n : n \in \mathbb{N}\}.$$

Using induction arguments, we obtain

$$f_{\text{II}}(\lambda^n) \subseteq \Pi(Q),$$

hence

$$f_{\text{II}}(\Pi(\mathscr{B}_{\lambda})) \subseteq \Pi(Q),$$

and finally

$$f_{\text{II}}(\Pi(Q)) \subseteq \Pi(Q).$$

The converse inclusion is a consequence of the q.u. continuity of $f^{\leftarrow}$. ◇

This theorem remains valid for topological spaces.

8. Example. Let us apply this theorem to the Minkowski space-time $S = \mathbb{R}^4$. We note the causal order on S by Π. It is easy to see that the q.u. Q, generated by the restriction of the Euclidean metric of S to Π, is self-refined. Then by Theorem 7 we may conclude that each homeomorphism of Q is a causal automorphisms of S. In accordance to Zeeman's theorem, the group of these automorphisms exactly consists of proper Lorentz transformations, translations and positive dilations.

Conversely, all these transformations are homeomorphisms of Q. In conclusion, Q is another topological structure for Minkowski space-time, which satisfies the conditions (1) and (2).

9. Remark. For practical purposes (as in the above example), it is useful to know sufficient conditions for self-refinement. For example, the q.u. $Q_0 = Q_{\Pi(Q)}$ is self-refined if it is Archimedean relative to $\Pi(Q)$, i.e. for each $(x, y) \in \Pi$ and $\pi \in Q_0$ there exists $n \in \mathbb{N}$ such that $(x, y) \in \pi^n$.

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