THE POLAR OF A SUPER-ADDITIVE NORM

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1. *Generalities*. Let ($\mathbf{H}$, $\langle .,. \rangle$) be a real pre-Hilbert space and let $\mathbf{E} = \mathbb{R}\times\mathbf{H}$. The *temporal projection* $\mathscr{T}$: $\mathbf{E}\to\mathbf{H}$, and the *spatial projection* $\mathscr{S}$: $\mathbf{E}\to\mathbf{H}$, are defined by $e = (t, x) \overset{\mathscr{T}}{\mapsto} t$, respectively $e = \overset{\mathscr{S}}{\mapsto} x$ ($\mathbb{R}$ and $\mathbf{H}$ are identified with the subspaces $\mathbb{R}\times\mathbf{0}$ and $\mathbf{0}\times\mathbf{H}$). The functional $(\, . \, , \, . \,) : \mathbf{E}\times\mathbf{E}\to\mathbb{R}$, expressed by

$$(e , f) = \mathscr{T}(e)\, \mathscr{T}(f) - \langle \mathscr{S}(e), \mathscr{S}(f) \rangle,$$

is an *indefinite inner product* on $\mathbf{E}$, which makes ($\mathbf{E}$, (. , .)) a Π_1 Pontrjagin space. It is well known that each real Π_1 space has such a decomposition (see [3]).

The cone

$$\mathbf{P} = \{e\in\mathbf{E}: \mathscr{T}(e) > \|\mathscr{S}(e)\| \} \cup \{0\}$$

defines the *causal order* of ($\mathbf{E}$, (. , .)). Because $\mathbf{P}\subset \mathfrak{P}^{++}$, we may define the *Minkowskian norm* $\|\cdot\|_t : \mathbf{P} \to\mathbb{R}^+$ by

$$\|e\|_t = (e, e)^{1/2},$$

where the index t comes from the physical signification of the values of this norm, namely *time*. Using the fundamental inequality for indefinite inner product spaces (see [2]), we obtain the Aczél's inequality

$$(e, f) \geq \|e\|_t \, \|f\|_t$$

for every $e, f \in \mathbf{P}$, with equality if and only if e and f are collinear.

2. D*efinition*. A functional $p : \mathbf{P}\to\mathbb{R}^+$, which satisfies the conditions

i) $p(e) = 0$ if and only if $e = 0$

ii) $p(\lambda e) = \lambda p(e)$ for each $\lambda \in \mathbb{R}^+$ and $e\in\mathbf{P}$

iii) $p(e + f) \geq p(e) + p(f)$ for every $e, f \in \mathbf{P}$ (super-additivity)

is named a *super-additive* (s.a.) *norm*.

3. *Examples of s.a. norms*.

a) $p_1 = \|\cdot\|_t$ (the super-additivity follows from the Aczél's inequality).

b) $p_2 = \mathscr{T}\,|_{\mathbf{P}}$ (the equality always holds in iii)).

c) $p_3 = (\mathscr{T} - \|\cdot\| \circ \mathscr{S})|_{\mathbf{P}}$ (the super-additivity follows from the sub-additivity of the usual norm $\|\cdot\|$ of $\mathbf{H}$).

More generally, according to [1], the formula

$$p(e) = \sup \{\lambda \in \mathbb{R}^+ : e \in \lambda A\}$$

defines a s.a. norm on $\mathbf{P}$ if A is a convex part of $\mathbf{P}$, which has the property that for each $e \in \mathbf{P}$ there exists $\rho \in \mathbb{R}^+$ such that $e \in \alpha A$ for some $\alpha \in [0, \rho)$, but $e \notin \beta A$ for all $\beta \geq \rho$.

4. *Remark.* The converse relation between convexity and super-additivity will be useful later, namely if $p : \mathbf{P} \to \mathbb{R}^+$ is a s.a. norm, then

$$A_r = \{e \in \mathbf{P} : p(e) > r\}$$

are convex sets for all $r > 0$. Simple examples show that such a set generally does not admit any support function (in the classical sense, see [5]), i.e.

$$\sup \{(e, f) : f \in A_r \} = \infty$$

for all $0 \neq e \in \mathbf{P}$. Consequently, it is natural to study the existence of

$$\inf \{(e, f) : f \in A_r \} .$$

More precisely, it is important to show that it is strictly positive, as below.

5. *LEMMA.* *The following conditions are equivalent*:

i) $\tau(p) =: \inf \{ \mathscr{T}(e): p(e) \geq 1\} = 0$,

ii) $\theta(p) =: \inf \{ \mathscr{T}(e): p(e) \geq 1, \mathscr{S}(e) = 0\} = 0$,

iii) $H(p) =: \{ e \in \mathbf{P} : p(e) \geq 1\} = \mathbf{P} \setminus \{0\}$.

Proof. i) $\Rightarrow$ ii). Let us consider $e = (\varepsilon, 0)$ for an arbitrary $\varepsilon > 0$. Because $\tau(p) = 0$, there exists $f = (t, x)$ such that $t < \varepsilon / 3$ and $p(f) > 1$. From $f \in \mathbf{P} \setminus \{0\}$ we deduce $t > \|x\|$, hence $2\varepsilon / 3 > \varepsilon / 3 > \|x\|$, which implies $e - f \in \mathbf{P} \setminus \{0\}$. Using the super-additivity of p, we obtain $p(e) \geq p(f) + p(e - f) \geq 1$. Because $\varepsilon > 0$ was arbitrary, it follows that $\theta(p) = 0$.

ii) $\Rightarrow$ iii). We will show that in the presence of $\theta(p) = 0$, it is impossible to have $p(f) < 1$ for some $f \in \mathbf{P} \setminus \{0\}$. In fact, if we suppose that there exists such an element $f = (t, x)$, from the condition $f \in \mathbf{P} \setminus \{0\}$ it follows that $\varepsilon =: t - \|x\|$ is strictly positive. Because $\theta(p) = 0$, for $e = (\varepsilon / 2 , 0)$ we have $p(e) \geq 1$. Let us consider $\tilde{f} = f - e = (t - (\varepsilon/2), x)$. Since $t - (\varepsilon/2) - \|x\| = \varepsilon/2 > 0$, we also have $\tilde{f} \in \mathbf{P} \setminus \{0\}$. The super-additivity of p gives $1 > p(f) \geq p(\tilde{f}) + p(e) \geq p(\tilde{f})$, hence $p(\tilde{f}) < 1$ too. Now let us observe that for $\tilde{g} = [p(\tilde{f})]^{-1} \tilde{f}$ we have $p(\tilde{g}) = 1$ and $\tilde{f}$ appears as a convex combination $\tilde{f} = \lambda \tilde{g} + (1 - \lambda)0$, where $0 < \lambda =: p(\tilde{f}) < 1$. Correspondingly, $f = \lambda g + (1 - \lambda)e$, where $g = \tilde{g} + e$. Obviously, $p(g) \geq p(\tilde{g}) + p(e) > 1$, hence $g \in H(p)$. Finally, using the convexity of $H(p)$, we obtain $p(f) \geq 1$, which contradicts the hypothesis that $p(f) < 1$.

iii) $\Rightarrow$ i) is obvious. ◇

6. *PROPOSITION.* *For each s.a. norm p we have* $\tau(p) > 0$.

Proof. According to the above Lemma, it is sufficient to show that situation $H(p) = \mathbf{P} \setminus \{0\}$ is impossible. In fact, if this equality would take place, then for each $e \in \mathbf{P} \setminus \{0\}$ and $n \in \mathbb{N}$ we may consider $e_n = \frac{1}{n} e$. Because $e_n \in \mathbf{P} \setminus \{0\}$ too, our equality would imply $p(e_n) \geq 1$, hence $p(e) \geq n$ for any large n, which is impossible. ◇

Now it is clear how to introduce the polar of a s.a. norm.

7. *Definition.* The *polar of a s.a. norm* $p:\mathbf{P} \to \mathbb{R}^+$ is the functional $p^*:\mathbf{P} \to \mathbb{R}^+$ defined by

$$p^*(e) = \inf \{(e, f) : p(f) \geq 1\}.$$

If $p = p^*$ we say that p is *self-polar*.

8. *Examples.* a) $p_1 = \|\cdot\|_t$ is a self-polar s.a. norm. In fact, taking $\|f\|_t \geq 1$ in the Aczél's inequality, we obtain $(e, f) \geq \|e\|_t$, hence

$$p_1^*(e) = \inf \{(e, f) : \|f\|_t \geq 1\} \geq \|e\|_t .$$

On the other hand, if $e \in \mathbf{P} \setminus \{0\}$ and $f = [\|e\|_t]^{-1} e$, then we have

$$p_1^*(e) \leq [\|e\|_t]^{-1} (e, e) = p_1(e).$$

b) $p_2^* = p_3$. Considering $f \in \mathbf{H}(p_2)$, we have

$$p_2^*(e) = \inf \{(\mathscr{T}(e)\, \mathscr{T}(f) - \langle \mathscr{S}(e), \mathscr{S}(f) \rangle) : \mathscr{T}(f) \geq 1\} \geq$$
$$\geq \inf \{(\mathscr{T}(e) - \|\mathscr{S}(e)\|\, \|\mathscr{S}(f)\|) : \mathscr{T}(f) = 1\} =$$
$$= \mathscr{T}(e) - \|\mathscr{S}(e)\| \sup \{\|\mathscr{S}(f)\| : \mathscr{T}(f) = 1\} = p_3(e).$$

Conversely, for every $e \in \mathbf{P}$ and $\varepsilon > 0$ we find $f \in \mathbf{P}$, $p_2(f) = 1$, such that $p_2^*(e) < p_3(e) + \varepsilon$. More exactly, if $e = (t, x)$, we consider $0 < \varepsilon < \|x\|$, and then we construct $\delta = \varepsilon \|x\| [\|x\| - \varepsilon]^{-1}$ and $f = (1, [\|x\| - \varepsilon]^{-1} x)$, so that $f \in \mathbf{P} \setminus \{0\}$, $p_2(f) = 1$, and

$$p_2^*(e) \leq (e, f) = t - \|x\|^2 [\|x\| + \delta]^{-1} < t - \|x\| + \varepsilon = p_3(e) + \varepsilon .$$

c) $p_3^* = p_2$. By definition we have

$$p_3^*(e) = \inf \{(e, f) : \mathscr{T}(f) - \|\mathscr{S}(f)\| \geq 1\} \geq$$
$$\geq \inf \{\mathscr{T}(e)(1 + \|\mathscr{S}(f)\|) - \langle \mathscr{S}(e), \mathscr{S}(f) \rangle) : \mathscr{T}(f) \in \mathbf{H}\} =$$
$$= \mathscr{T}(e) + p_3(e) \inf \{\|\mathscr{S}(f)\| : \mathscr{T}(f) \in \mathbf{H}\} = \mathscr{T}(e) = p_2(e).$$

Conversely, we observe that $f = (1, 0) \in \mathbf{P}$, $p_3(f) = 1$, and $(e, f) = \mathscr{T}(e)$, so that $p_3^*(e) \leq p_2(e)$.

9. *PROPOSITION.* *The polar of a s.a. norm is a s.a. norm too*.

Proof. We must verify the conditions of Definition 2 for p^*. Let us show that $e \in \mathbf{P} \setminus \{0\}$ implies $p^*(e) > 0$. In fact, we may interpret Proposition 6 as the inclusion $H(p) \subseteq L(p) =: \{e \in \mathbf{P} : \mathscr{T}(e) \geq \tau(p)\}$, hence

$$p^*(e) \geq \inf \{(e, f) : f \in L(p)\}.$$

Choosing $e \in \mathbf{P} \setminus \{0\}$, there is determined $k =: \mathscr{T}(e) - \|\mathscr{S}(e)\| > 0$. On the other hand, $f \in L(p)$ implies $\mathscr{T}(f) \geq \tau(p) > 0$ and $\mathscr{T}(f) > \|\mathscr{S}(f)\|$, hence $\langle \mathscr{S}(e), \mathscr{S}(f) \rangle \leq \|\mathscr{S}(e)\| \, \|\mathscr{S}(f)\| < \mathscr{T}(f) \, \|\mathscr{S}(e)\|$, and finally,

$$(e, f) > \mathscr{T}(f) \, [\mathscr{T}(e) - \|\mathscr{S}(e)\| \,] \geq k \, \tau(p) > 0.$$

The rest of the proof is routine. ◇

10. *Remark.* Some details concerning the relations between the s.a. norms p_1 , p_2 , p_3 and their polar s.a. norms are explained by the following simple (but general) properties:

i) $p \leq q$ if and only if $H(p) \subseteq H(q)$,

ii) $p \leq q$ implies $p^* \geq q^*$,

iii) $p \, p^* \leq \|\cdot\|_t^2$,

iv) $p \leq \|\cdot\|_t$ if and only if $p^* \geq \|\cdot\|_t$.

The most significant result is the uniqueness of the self-polar s.a. norm.

11. *THEOREM.* $\|\cdot\|_t$ *is the single self-polar s.a. norm of* **E**.

Proof. We already saw that $\|\cdot\|_t$ is a self-polar s.a. norm. It remains to show that it is the unique s.a. norm with this property, i.e. if p is another self-polar s.a. norm, then $p = \|\cdot\|_t$. In fact, for each $e \in \mathbf{P} \setminus \{0\}$ we may consider the vector $f = [p(e)]^{-1} e$, so that $p(f) = 1$. From the hypothesis $p = p^*$ we deduce $p(e) \leq (e, f) = [\, p(e)]^{-1} \|e\|_t^2$, and consequently $p(e) \leq \|e\|_t$.

On the other hand, if we put $\|f\|_t \geq 1$ in the Aczél's inequality, then we obtain $(e, f) \geq \|e\|_t$, hence $p(e) = p^*(e) \geq \inf \{(e, f) : \|f\|_t \geq 1\} \geq \|e\|_t$.

In conclusion, $p(e) = \|e\|_t$ for every $e \in \mathbf{P}$. ◇

12. *Comment.* It is well known that in a pre-Hilbert space the single self-polar norm (in the usual sense) is the Euclidean one, defined by $\|x\| = \sqrt{\langle x, x \rangle}$. On the indefinite inner product spaces there exists no canonical norm, but self-polar norms can still exist (see [4]) in the sense that

$$p(x) = \sup \{ |(x, y)| : p(y) \leq 1\}.$$

As an alternative, our results show that a theory specific to super-additivity is possible, if we find the most adequate notions to be used in the place of the classical ones.

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