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# REMARKS ON SUPER-ADDITIVE FUNCTIONALS

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**Summary.** We first give the complete formulation of the triangle rule in the plane, relative to the metric $\sqrt{(x_2-x_1)^2-(y_2-y_1)^2}$, which is the logic negation of that for the Euclidean one. Then we extend this rule to super-additive metrics in real linear spaces, and we put forward a construction similar to the Minkowskian semi-norms. Finally, we formulate the same rule for symmetric super-additive metrics in arbitrary spaces (not necessarily linear).

**1.** It is easy to see that the functional

$$\rho(z_1,z_2)=\sqrt{(x_2-x_1)^2-(y_2-y_1)^2}\,, \tag{1}$$

defined for pairs of points $z_1$ $(x_1, y_1)$, $z_2$ $(x_2, y_2)$ in the plane does not fulfill the usual triangle inequality, which is

$$\rho(z_1,z_3)\le\rho(z_1,z_2)+\rho(z_2,z_3)\,. \tag{2}$$

Consequently, whenever we need to deal with such functionals (for example [1], [2]), we naturally have to replace (2) by the opposite inequality,

$$\rho(z_1,z_3)\ge\rho(z_1,z_2)+\rho(z_2,z_3)\,. \tag{3}$$

Obviously, inequality (3) cannot hold for all permutations of the sides of a triangle $z_1$, $z_2$, $z_3$ , and in general, as in the case of (1), $\rho$ does not take real values at each pair of points in the space. This situation justifies restrictions on the domain of $\rho$, as well as specifications about the applicability of (3), like in the cited works. Even so, the problem of the rule of a triangle is not completely solved, because relation (3) is valid for only one order of the sides, but not for each side.

We claim that we could surpass such difficulties if we use the complete form of the triangle rule, which is the former aim in this paper.

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Our second purpose is to construct super-additive functionals on arbitrary real linear spaces by means of convex sets. More exactly, we are interested in obtaining $\rho$ from a "norm" $N$ through the formula $\rho\,(z_1, z_2) = N\,(z_1 - z_2)$, with the preservation of (3), and in expressing the values $N(z)$ in connection to some convex sets of the space.

Since we wish to get inequality (3) in arbitrary real linear spaces, the third problem that appears concerns the prolongation of $\rho$, respectively $N$, which allows the complete formulation of the triangle rule.

**2.** Let $N : \mathbb{R}^2 \to \mathbb{C}$ be a functional of values $N(z) = \sqrt{x^2 - y^2}$, which gives the metric $\rho\,(z_1, z_2) = N\,(z_1 - z_2)$ in (1). Evidently, in terms of [3], the set

$$P = \{\, z(x, y) \in \mathbb{R}^2 : x^2 - y^2 \geq 0,\ x \geq 0\}$$

is a sharp and convex cone in $\mathbb{R}^2$, and $N$ takes real (and positive) values on and only on $P \cup (-P)$.

**Theorem 1.** *Functional N has the properties*:

[$N_1$] $z = 0$ *implies* $N(z) = 0$

[$N_2$] $N(\lambda z) = |\lambda|\, N(z)$ *for all* $\lambda \in \mathbb{R}$

[$N_3$] *If* $z_1$, $z_2$ *and* $z_1 + z_2 \in P \cup (-P)$, *then*

$$\text{either } N(z_1 + z_2) \geq N(z_1) + N(z_2),$$
$$\text{or } N(z_1 + z_2) \leq N(z_1) - N(z_2).$$

**Proof.** Properties [$N_1$] and [$N_2$] are immediate. To prove [$N_3$], we consider the expression $E(\lambda) = (x_1 + \lambda x_2)^2 - (y_1 + \lambda y_2)^2$. Writing

$$E(\lambda) = [x_1 + y_1 + \lambda\,(x_2 + y_2)]\,[x_1 - y_1 + \lambda\,(x_2 - y_2)]$$

we see that the only case when $E(\lambda)$ cannot vanish holds for $x_2 + y_2 = 0$ and $x_2 - y_2 = 0$, i.e. $z_2 = 0$. If $z_2 = 0$, condition [$N_3$] is straightly satisfied. If not, there exist $\lambda_1$ and $\lambda_2$ in $\mathbb{R}$ such that $E(\lambda_1) \geq 0$, and $E(\lambda_2) \leq 0$. In these cases, for the trinomial

$$E(\lambda) = \lambda^2\,(x_2^2 - y_2^2) + 2\lambda\,(x_1 x_2 - y_1 y_2) + (x_1^2 - y_1^2)$$

we have the inequality

$$(x_1 x_2 - y_1 y_2)^2 - (x_1^2 - y_1^2)\,(x_2^2 - y_2^2) \geq 0.$$

Because $z_1, z_2 \in P \cup (-P)$, this result takes the form of the disjunction

$$\text{either } x_1 x_2 - y_1 y_2 \geq N(z_1)\,N(z_2),$$
$$\text{or } x_1 x_2 - y_1 y_2 \leq -N(z_1)\,N(z_2).$$

Amplifying by *2* and adding the expression

$$x_1^2 + x_2^2 - y_1^2 - y_2^2 = N^2(z_1) + N^2(z_2),$$

we obtain

$$\text{either } N^2(z_1 + z_2) \geq [N(z_1) + N(z_2)]^2,$$
$$\text{or } N^2(z_1 + z_2) \leq [N(z_1) - N(z_2)]^2.$$

Taking into account that the above values of $N$ are real and positive, we can extract the square root and obtain [$N_3$]. ◇

**Remark 1.** We shall apply the former inequality from the alternative expressed by [$N_3$] in the case $H(z_1, z_2) \overset{not.}{=} x_1 x_2 - y_1 y_2 \geq 0$, and the second one if $H(z_1, z_2) \leq 0$. In particular, we have $H(z_1, z_2) \geq 0$ for all $z_1, z_2 \in P$, since $x_1^2 - y_1^2 \geq 0$ and $x_2^2 - y_2^2 \geq 0$ imply $x_1 \geq |y_1|$ and $x_2 \geq |y_2|$, hence

$$x_1 x_2 \geq |y_1 y_2| \geq y_1 y_2 \ .$$

In this case, the triangle rule takes the form $N(z_1 + z_2) \geq N(z_1) + N(z_2)$, which is the *super-additivity* (briefly s.a.) of the functional $N$. Consequently, the corresponding metric $\rho$, generated by $N$, fulfills the rule specified in [2].

If compared to the complete triangle rule for the Euclidean metric, i.e. "*each side is less than the sum of the other two sides* ***and*** *greater than their difference*", we see that rule [$N_3$] exactly represents its logical negation.

**3.** Further, we extend $\mathbb{R}^2$ to an arbitrary real linear space. In these spaces we construct super-additive functionals following the method of a Minkowskian semi-norm (see [3], etc.).

Let $E$ be a real linear space with elements $x, y, \ldots$, and let $P$ be a sharp and convex cone in this space. The theorems in this section establish several connections between sets $A \subset E$, which fulfill conditions like

[$a_1$] $\forall\ x \in P\ \exists\ \alpha \geq 0$ *such that* $x \in \alpha A$

[$a_2$] $\forall\ x \in P, x \neq 0, \exists \rho > 0$ *such that* $x \notin \beta A$ *holds for all* $\beta \geq \rho$

[$a_3$] *$A$ is convex*

and functionals $S : P \to \mathbb{R}$, which have properties of the form

[$h_1$] $S(x) = 0 \Leftrightarrow x = 0$ *and* $S(x) > 0$ *for all* $x \in P \setminus \{0\}$

[$h_2$] $S(\lambda x) = \lambda S(x)$ *for all* $\lambda \geq 0$

[$h_3$] $S(x + y) \geq S(x) + S(y)$ *for all* $x, y \in P$.

Considering such sets and functionals makes sense since they exist in the previous case of $E = \mathbb{R}^2$ at least.

Condition [$h_2$] is a particular form of [$N_2$] in the case $\lambda \geq 0$, condition [$h_1$] reinforces [$N_1$], and property [$h_3$] represents the *super-additivity*. Let us remark that [$a_1$] and [$a_2$] allow to define a functional $H_A : P \to \mathbb{R}$, attached to the set A according to the formula

$$H_A(x) = \sup \{\alpha : x \in \alpha A\}. \tag{4}$$

**Theorem 2.** *If a set A satisfies the conditions* [a$_1$], [a$_2$] *and* [a$_3$], *then* $H_A$ , *defined by* (4), *has the properties* [h$_1$], [h$_2$] *and* [h$_3$].

**Proof.** To prove [h$_1$], let us suppose that $H_A(x) = 0$. According to [a$_1$] and definition (4), it follows that $\alpha = 0$ is the only number for which $x \in \alpha A$ is valid. But $0 \cdot A = \{0\}$, hence $x = 0$. Conversely, if $x = 0$, then in accordance to [a$_1$], there exists $\alpha \geq 0$ such that $x \in \alpha A$. The point is that $\alpha = 0$ is the only number of this type. In fact, if we accept that $x \in \alpha_0 A$ for some $\alpha_0 > 0$, then $x \in \alpha A$ should be valid for all $\alpha > 0$, which contradicts condition [a$_2$]. So we have a proof of the equivalence "$S(x) = 0 \Leftrightarrow x = 0$". The second part of [h$_1$] is a consequence of the fact that in [a$_1$] we always have $\alpha \geq 0$.

Property [h$_2$] is a direct consequence of the operations with *sup*.

To prove [h$_3$], we start with $x, y \in P$. If $H_A$ vanishes at one of them, say $x$, then $x = 0$ as before, and we obtain equality in [h$_3$], namely

$$H_A(0 + y) = H_A(y) = H_A(x) + H_A(y).$$

Otherwise, if both $H_A(x) > 0$ and $H_A(y) > 0$, then there is some $\varepsilon_0 > 0$ such that $H_A(x) - \varepsilon_0 > 0$ and $H_A(y) - \varepsilon_0 > 0$. In this case, for each $\varepsilon \in (0, \varepsilon_0)$ we can find two real numbers $m$ and $n$ such that

$$H_A(x) - \varepsilon \leq m \leq H_A(x) \quad \text{and } H_A(y) - \varepsilon \leq n \leq H_A(y),$$

where $\frac{1}{m} x \in A$ and $\frac{1}{n} y \in A$. Because of [a$_3$], membership $\frac{\lambda}{m} x + \frac{1-\lambda}{n} y \in A$ is valid for all $\lambda \in (0, 1)$. In particular, $\lambda = \frac{m}{m+n}$ leads to $x + y \in (m + n) A$, so that $m + n \leq H_A(x + y)$. More than this, because

$$H_A(x + y) \geq H_A(x) + H_A(y) - 2\varepsilon$$

holds for arbitrary small $\varepsilon$, we obtain [h$_3$]. ◇

**Theorem 3.** *Let P be a sharp and convex cone in the real linear space E. If the functional* $S : P \to \mathbb{R}$ *satisfies conditions* [h$_1$], [h$_2$] *and* [h$_3$], *then for all* $r > 0$, *the set*

$$A_r = \{x : S(x) \geq r\} \tag{5}$$

*has the properties* [a$_1$], [a$_2$] *and* [a$_3$].

**Proof.** Let us take an arbitrary $x \in P$, for which we note $S(x) = b$, and let us consider a real number $\alpha$ such that $0 \leq \alpha \leq \frac{b}{r}$. To prove [a$_1$] we distinguish two cases, depending on $b$. If $b = 0$, then [h$_1$] implies $x = 0$, hence $\alpha = 0$ too. In this case, [a$_1$] reduces to $0 \in 0 \cdot A_r$. If $b > 0$, then there exists $\alpha > 0$ such that $\frac{1}{\alpha} \geq \frac{r}{b}$. Using [h$_2$], we deduce $S(\frac{1}{\alpha} x) = \frac{1}{\alpha} S(x) \geq r$, hence $x \in \alpha \cdot A_r$.

To prove [a$_2$], let us take $x$ and $b$ as before, and consider $\rho > \frac{b}{r}$ and $\beta \geq \rho$. We may distinguish two cases:

If $x = 0$, then [$h_1$] gives $b = 0$, hence $S(\frac{1}{\beta}x) = 0 < r$, i.e. $x \notin \beta \cdot A_r$.

If $x \neq 0$, then $S(\frac{1}{\beta}x) = \frac{1}{\beta}S(x) < r$, so that $x \notin \beta \cdot A_r$ again.

Property [$a_3$] is immediate, since from $x \in A_r$, $y \in A_r$, and $0 < \lambda < 1$, we deduce $S(x) \geq r$, $S(y) \geq r$, and by [$h_3$], $S[\lambda x + (1 - \lambda)y] \geq r$. According to the definition (5) of $A_r$, this means $\lambda x + (1 - \lambda)y \in A_r$. ◇

**Corollary 1.** *Let P be a sharp and convex cone in the real linear space E. If the functional* $S : P \to \mathbb{R}$ *satisfies conditions* [$h_1$], [$h_2$] *and* [$h_3$], *then the set* $A_S$, *constructed by* (5), *satisfies* [$a_1$], [$a_2$] *and* [$a_3$]; *the functional* $H_{A_S}$, *attached to* $A_S$ *in accordance to* (4), *has the properties* [$h_1$], [$h_2$] *and* [$h_3$]. *In addition,*

$$S = H_{A_S}.$$

In fact, the former two assertions are reformulations of Theorems 3 and 2. To prove the claimed equality, we first remark that it is obvious at $x = 0$. At any $x \neq 0$, we have $S(x) = b > 0$, so $x \in \alpha A_S$ is possible for $\alpha > 0$ only. Now, it remains to evaluate:

$$H_{A_S}(x) = \sup\{\alpha : \tfrac{1}{\alpha}x \in A_S\} = \sup\{\alpha : S(\tfrac{1}{\alpha}x) \geq 1\} = \sup\{\alpha : \tfrac{1}{\alpha}b \geq 1\} = b.$$

**Corollary 2.** *Let P be a sharp and convex cone in the real linear space E. Let A be a set satisfying conditions* [$a_1$] *and* [$a_2$], *and the functional* $H_A$ *be constructed by* (4). *If B is given by* (5), *for* $S = H_A$, *then* $B \supseteq A \cap P$.

The proof is direct. If $x \in A \cap P$, then $x \in \alpha A$ holds for $\alpha = 1$. This means $H_A(x) \geq 1$, which gives $x \in B$.

Simple examples show that the converse inclusion is not generally valid.

**Remark 2.** The results from above are not restricted to finite dimensional real linear spaces, as in [1], where the dimension is 4. For example, we may take $E = C^0_{\mathbb{R}}([0,1])$, $P$ consisting of $f_0 \equiv 0$ and strictly positive functions, and the super-additive norm $S(f) = \inf\{f(x) : x \in [0, 1]\}$.

**4.** We may remark that the metric defined in [2] makes sense only on pairs $(z_1, z_2)$ where $z_1 < z_2$ holds in a particular order, and the triangle rule is given by the condition (3). In [1], even if the metric is defined for arbitrary pairs $(z_1, z_2)$, it obeys the same condition. In the previous section, where the metric is generated by a norm in real linear spaces, (3) also gives the triangle rule. Now we will show how to formulate a complete triangle rule, which is a logical negation of the Euclidean one, in a general framework that includes these situations. The idea is to prolong the metric by symmetry.

As usually, by *order* in a set $X$ we understand a reflexive, anti-symmetric and transitive binary relation.

**Theorem 4.** *Let $X$ be an arbitrary non-void set, and let $\Pi$ be an order on $X$. If a function $\rho : \Pi \to \mathbb{R}_+$ satisfy the conditions*

[$\rho_1$] $x = y \Rightarrow \rho(x, y) = 0$

[$\rho_2$] $(x, y), (y, z) \in \Pi \Rightarrow \rho(x, y) + \rho(y, z) \leq \rho(x, z)$,

*then there exists a function $\rho^* : \Pi \cup \Pi^{-1} \to \mathbb{R}_+$, which prolongs $\rho$ (i.e. coincides with $\rho$ on $\Pi$), and has the properties:*

[$\rho^*_1$] $x = y \Rightarrow \rho^*(x, y) = 0$

[$\rho^*_2$] $\rho^*(x, y) = \rho^*(y, x)$

[$\rho^*_3$] *If* $(x, y), (y, z), (x, z) \in \Pi \cup \Pi^{-1}$, *then*

$$\textit{either } \rho^*(x, y) + \rho^*(y, z) \leq \rho^*(x, z)$$
$$\textit{or } |\rho^*(x, y) - \rho^*(y, z)| \geq \rho^*(x, z).$$

**Proof.** If $(x, y) \in \Pi$, then we define $\rho^*(x, y) = \rho(x, y)$, and if $(x, y) \in \Pi^{-1}$, then $\rho^*(x, y) = \rho(y, x)$, since $(y, x) \in \Pi$.

The proof of [$\rho^*_1$] is based on the reflexivity of $\Pi$. In fact, since $(x, x) \in \Pi$, and [$\rho_1$] gives $\rho(x, x) = 0$, we obtain $\rho^*(x, x) = 0$.

Property [$\rho^*_2$] is significant for $x \neq y$, so we have to consider two disjoint cases: $(x, y) \in \Pi$, respectively $(y, x) \in \Pi$. In the first case, the construction of $\rho^*$ gives $\rho^*(x, y) = \rho(x, y)$, as well as $\rho^*(y, x) = \rho(x, y)$. The other case is similar.

To prove [$\rho^*_3$] we primarily remark that we may define $\rho^*(x, y)$, $\rho^*(y, z)$ and $\rho^*(x, z)$ in the following cases only:

(a) $(x, y) \in \Pi$, $(y, z) \in \Pi$ hence $(x, z) \in \Pi$

(b) $(x, y) \in \Pi^{-1}$, $(y, z) \in \Pi^{-1}$ hence $(x, z) \in \Pi^{-1}$

(c) $(x, y) \in \Pi$, $(y, z) \in \Pi^{-1}$ and $(x, z) \in \Pi$

(d) $(x, y) \in \Pi$, $(y, z) \in \Pi^{-1}$ and $(x, z) \in \Pi^{-1}$

(e) $(x, y) \in \Pi^{-1}$, $(y, z) \in \Pi$ and $(x, z) \in \Pi$

(f) $(x, y) \in \Pi^{-1}$, $(y, z) \in \Pi$ and $(x, z) \in \Pi^{-1}$.

In the case (a), [$\rho_2$] leads to the first inequality in [$\rho^*_3$]. The symmetry of $\rho^*$ assures the same inequality in case (b).

In the case (c) we have $(x, y) \in \Pi$, $(z, y) \in \Pi$ and $(x, z) \in \Pi$, hence [$\rho_2$] gives $\rho(x, z) + \rho(z, y) \leq \rho(x, y)$. In terms of $\rho^*$, this means

$$\rho^*(x, z) \leq \rho^*(x, y) - \rho^*(y, z),$$

which is the second inequality in [$\rho^*_3$].

The remaining cases are similar. ◇

## REFERENCES

[1] Vladimirov I. S., *Axiomatization of the Space-time Properties in General Relativity Theory* (Russian), Gravitation Problems, Tbilisi, 1965, 75 – 80.
[2] Pimenov R. I., *Metric Spaces of Cinematic Type* (Russian), International Congress of Mathematics, Section 9, Moscow, 1966, 40.
[3] Kelley J. L., Namioka I. N., *Linear Topological Spaces*, Van Nostrand, Princeton, 1963