OPERATING WITH HORISTOLOGIES

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Our purpose is to analyze how the horistological structures are to be compared and derived ones from the others when we correspondingly operate with their supporting sets.

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INTRODUCTION

The horistological structures are introduced in [1] with the aim of developing a qualitative analysis of super-additivity on the same way on which the topology generalizes the sub-additivity. Besides the former models consisting of event spaces (e.g. the Minkowskian space-time of special relativity), the horistological structures generated by super-additive norms are useful in concave programming and even in pure mathematical problems like the duality theory (see [2]). Working with such structures naturally leads to the comparison and to the construction of subspaces, products, quotients, etc., which are well known topics for the other mathematical structures (for topologies see [3], etc.); this justifies our interest in finding the general rules to operate with horistologies.

For the beginning, we recall some of the basic notions that we deal with, as they are considered in [1]. A ***horistology*** on the non-void set W is a function

$$\chi : W \to \mathscr{P}(\mathscr{P}(W))$$

for which:

[h$_1$] $x \notin P$ whenever $P \in \chi(x)$

[h$_2$] If $P \in \chi(x)$ and $Q \subseteq P$, then $Q \in \chi(x)$

[h$_3$] If P, $Q \in \chi(x)$, then $P \cup Q \in \chi(x)$

[h$_4$] For each $P \in \chi(x)$ there exists $L \in \chi(x)$ such that for every $y \in P$ and Q $\in \chi(y)$ we have $Q \subseteq L$.

The pair (W, χ) is called ***horistological world*** (or *space*), and the elements P of $\chi(x)$ are named ***perspectives*** of x.

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A ***uniform horistology*** (briefly u.h.) on W is a family $\mathscr{H} \subseteq \mathscr{P}(W^2)$ for which:

[uh_1] $\pi \cap \Delta = \varnothing$ for all $\pi \in \mathscr{H}$, where $\Delta = \{(x, x) : x \in W\}$

[uh_2] $\pi \in \mathscr{H}$ and $\lambda \subseteq \pi$ imply $\lambda \in \mathscr{H}$

[uh_3] If $\lambda, \pi \in \mathscr{H}$, then $\lambda \cup \pi \in \mathscr{H}$

[uh_4] For each $\pi \in \mathscr{H}$ there exists $\lambda \in \mathscr{H}$ such that for all $\omega \in \mathscr{H}$ we have

$\lambda \supseteq \pi \circ \omega$ and $\lambda \supseteq \omega \circ \pi$.

The pair $(W, \mathscr{H})$ is a ***uniform horistological world*** (*space*), briefly u.h., and the elements of $\mathscr{H}$ are called ***prospects***.

Let (V, ψ) and (W, χ) be horistological worlds. The function $f : V \to W$ is ***discrete*** at $x \in V$ iff for every $P \in \psi(x)$ we have $f(P) \in \chi(f(x))$.

If $(V, \mathscr{L})$ and $(W, \mathscr{H})$ are u.h. worlds, we say that the function $f : V \to W$ is ***uniformly discrete*** on V iff for every $\pi \in \mathscr{L}$ we have $f_2(\pi) \in \mathscr{H}$, where

$$f_2(\pi) = \{(f(a), f(b)) : (a, b) \in \pi\}.$$

If (W, χ) is a horistological world, then

$$\mathbf{K}(\chi) = \{(x, y) \in W^2 : \{y\} \in \chi(x)\} \cup \Delta$$

is an order on W, called ***proper order*** of χ. Similarly, if $\mathscr{H}$ is a u.h. on W, then

$$\mathbf{K}(\mathscr{H}) = [\cup\{\lambda \subseteq W^2 : \lambda \in \mathscr{H}\}] \cup \Delta$$

is the ***proper order*** of $\mathscr{H}$.

If $\mathscr{H}$ is a u.h. on W, then the function $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, expressed by

$$\chi(x) = \{\lambda[x] : \lambda \in \mathscr{H}\},$$

defines a horistology on W. We say that χ is ***generated*** by $\mathscr{H}$. Of course, if $\mathscr{H}$ generates χ, then $\mathbf{K}(\mathscr{H}) = \mathbf{K}(\chi)$.

A. The comparison of the horistological structures

The comparison of the horistological structures is based on the comparison of the ideals defining these structures.

1. ***Definition.*** Let χ and φ be horistologies on the same set W. We say that χ is ***finer*** than φ (or φ is ***coarser*** than χ) iff we have $\chi(x) \supseteq \varphi(x)$ at each $x \in W$. In this case we note $\chi \supseteq \varphi$. Similarly, if $\mathscr{H}$ and $\mathscr{L}$ are uniform horistologies on W, we say that $\mathscr{H}$ is ***finer*** than $\mathscr{L}$ (respectively $\mathscr{L}$ is ***coarser*** than $\mathscr{H}$) iff $\mathscr{H} \supseteq \mathscr{L}$.

2. ***Remark.*** The relation "*finer than*" between the (u.) horistologies on W is a partial order. The examples χ_0 in [1] (II) B 2a and $\mathscr{H}_0$ in [1] (II) A 5a show that on any set W there exist the coarsest elements (respectively the coarsest horistology and the coarsest u.h.) in respect to this order.

3. **PROPOSITION.** *If $\mathscr{H}$ and $\mathscr{L}$ are u. horistologies on W in the relation $\mathscr{H} \supseteq \mathscr{L}$ and χ and φ are the generated horistologies, then*:

(i) $\chi \supseteq \varphi$ *and*
(ii) $\mathbf{K}(\chi) \supseteq \mathbf{K}(\varphi)$.

4. *Remark.* If the proper orders of two (u.) horistologies on W are not comparable (by inclusion), then also the (u.) horistologies themselves are not comparable (by fineness). Because the union of two or more orders generally is not contained in other order, it follows that usually we can not speak of an upper bound of two (u.) horistologies. Consequently, the family of all (u.) horistologies on a given set W does not form a lattice. Similarly, we may see that in general there exists no finest (u.) horistology on a fixed set.

5. PROPOSITION. *Let χ and φ be horistologies on W. Then $\chi \supseteq \varphi$ if and only if the identical map $\iota : (W, \varphi) \to (W, \chi)$ is discrete on W.*

A similar property holds for u. horistologies. It follows a characterization of the equivalent (u.) horistological structures in terms of (u.) discreteness.

In the following considerations, according to [1] (I) C3, the term *p-horometric* stands for the longer *pseudo-super-additive metric.*

6. PROPOSITION. *Let $\sigma : \Pi \to \mathbb{R}^+$ and $\rho : \Omega \to \mathbb{R}^+$ be two (p-) horometrics on the same set W. If $\Pi \supseteq \Omega$ and if there exists a number $k > 0$ such that $\sigma(x, y) \geq k\, \rho(x, y)$ for all $(x, y) \in \Omega$, then the u.h. generated by σ is finer than that generated by ρ.*

7. THEOREM. *If $\{\mathscr{H}_i : i \in \mathfrak{I}\}$ is an arbitrary family of u.h. on W, then*

$$\mathscr{H} = \{ \cap\{\pi_i : i \in \mathfrak{I}\} : \pi_i \in \mathscr{H}_i\}$$

is a u.h. too. More than this, we have $\mathscr{H} = \inf\{\mathscr{H}_i : i \in \mathfrak{I}\}$ and

$$\mathbf{K}(\mathscr{H}) = \cap\{\mathbf{K}(\mathscr{H}_i) : i \in \mathfrak{I}\}.$$

Proof. For the beginning, let us remark that

(*) $$\mathscr{H} = \cap\{\mathscr{H}_i : i \in \mathfrak{I}\}.$$

Then it is easy to see that $\mathscr{H}$ satisfies [uh_1] and is an ideal in $\mathscr{P}(W^2)$, i.e. it also verifies [uh_2] and [uh_3]. In order to prove [uh_4] let π be an arbitrary prospect of $\mathscr{H}$. Then $\pi \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$, and because each $\mathscr{H}_i$ satisfies [uh_4], there will exist $\lambda_i \in \mathscr{H}_i$, $i \in \mathfrak{I}$, such that $\lambda_i \supseteq \pi \circ \omega_i$ and $\lambda_i \supseteq \omega_i \circ \pi$ for any $\omega_i \in \mathscr{H}_i$. Let us note $\lambda = \cap\{\lambda_i : i \in \mathfrak{I}\}$; then obviously $\lambda_i \in \mathscr{H}$. If ω is another element of $\mathscr{H}$, it has the form $\omega = \cap\{\omega_i : i \in \mathfrak{I}\}$, where $\omega_i \in \mathscr{H}_i$, and we have $\omega \circ \pi \subseteq \omega_i \circ \pi \subseteq \lambda_i$, and $\pi \circ \omega \subseteq \pi \circ \omega_i \subseteq \lambda_i$. Because this holds for all $i \in \mathfrak{I}$, it follows that $\omega \circ \pi \subseteq \lambda$, and $\pi \circ \omega \subseteq \lambda$, i.e. [$uh_4$] is true for $\mathscr{H}$.

Showing that $\mathscr{H}$ is the greatest lower bound of the family $\{\mathscr{H}_i : i \in \mathfrak{I}\}$ is also based on (*) as routine.

Finally, because $\mathscr{H} \subseteq \mathscr{H}_i$, we have $\mathbf{K}(\mathscr{H}) \subseteq \mathbf{K}(\mathscr{H}_i)$ for each $i \in \mathfrak{I}$, hence $\mathbf{K}(\mathscr{H}) \subseteq \cap\{\mathbf{K}(\mathscr{H}_i) : i \in \mathfrak{I}\}$. Conversely, if $(x, y) \in \cap\{\mathbf{K}(\mathscr{H}_i) : i \in \mathfrak{I}\}$, then either $x = y$, or $\{(x, y)\} \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$. Consequently, $(x, y) \in \mathbf{K}(\mathscr{H})$ because either $(x, y) \in \Delta$, or $\{(x, y)\} \in \mathscr{H}$. ◇

8. THEOREM. *If* $\{\chi_i : i\in \mathfrak{I}\}$ *is an arbitrary family of horistologies on W, then the function* $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, *defined by*

$$\chi(x) = \{P = \cap\{P_i : i\in \mathfrak{I}\} : P_i \in \chi_i(x)\}$$

is a horistology on W. In addition $\chi = inf\,\{\chi_i : i\in \mathfrak{I}\}$ *and*

$$\mathbf{K}(\chi) = \cap\{\mathbf{K}(\chi_i) : i\in \mathfrak{I}\}.$$

As in the proof of the above theorem, we remark that $\chi(x) = \cap\{\chi_i(x) : i\in \mathfrak{I}\}$ at each $x\in W$, so it is to repeat the stages of that proof.

9. COROLLARY. Although the family of all (u.) horistologies on a given set *W* does not form a lattice, each of its upper bounded subfamily has a smallest upper bound. The problem of constructing this upper bound remains open.

10. THEOREM. *Let* $\{\mathscr{H}_i : i\in \mathfrak{I}\}$ *be an arbitrary family of u. horistologies on W, for which* $\mathscr{H}' = inf\,\{\mathscr{H}_i : i\in \mathfrak{I}\}$. *If* χ_i *and* χ' *are the horistologies generated by* $\mathscr{H}_i$ *and respectively* $\mathscr{H}'$, *then* $\chi' = inf\,\{\chi_i : i\in \mathfrak{I}\}$. *Moreover, if we suppose that there exists* $\mathscr{H}'' = sup\{\mathscr{H}_i : i\in \mathfrak{I}\}$, *then there exists* $\chi = sup\,\{\chi_i : i\in \mathfrak{I}\}$ *too, and it is not finer than the horistology* χ'', *which is generated by* $\mathscr{H}''$.

Proof. In order to prove the first part of the theorem one may directly show that χ' is coarser than each χ, and every other horistology having this property is coarser than χ'.

Using Corollary 9, the affirmation concerning the supremum follows from the fact that χ'' is finer than each χ_i, i.e. χ'' is an upper bound of the considered family of horistologies. Thus if $P\in\chi_i(x)$ for some $i\in \mathfrak{I}$, there will exist $\pi\in\mathscr{H}_i$ such that $P = \pi[x]$. Because $\mathscr{H}'' \supseteq \mathscr{H}_i$ for each $i\in \mathfrak{I}$, it follows that $\pi\in \mathscr{H}''$, which leads to $P\in\chi''(x)$. If we remark that *x* is arbitrary, we may conclude that $\chi'' \supseteq \chi_i$ holds for each $i\in \mathfrak{I}$. Consequently, there exists $\chi = sup\,\{\chi_i : i\in \mathfrak{I}\}$, and $\chi'' \supseteq \chi$. ◇

As Remark A 4d in [1] (I) shows, if we restrict the order on which a p-horometric is defined, then we obtain another p-horometric. By the following two propositions we analyze such a restriction in the case of (u.) horistological worlds.

11. PROPOSITION. *Let* $(W, \mathscr{H})$ *be a u.h. space with the proper order* $\mathbf{K}(\mathscr{H})=\Pi$, *and let* $\Omega\subseteq\Pi$ *be another order on W. Then* $\mathscr{H}' = \{\pi \cap\Omega : \pi\in\mathscr{H}\}$ *also is a u.h. on W, and* $\mathbf{K}(\mathscr{H}') = \Omega$.

Proof. We may consider the restriction $\mathscr{H}'$ of $\mathscr{H}$ to Ω as $\mathscr{H}' = inf\,\{\mathscr{H}, \mathscr{H}_1\}$, where $\mathscr{H}_1$ is the u.h. generated by $\Omega \setminus \Delta$ like in [1] (I) A 5a, i.e. it is the principal ideal $\mathscr{P}(\Omega \setminus \Delta)$. ◇

12. PROPOSITION. *Let* (W, χ) *be a horistological world with* $\mathbf{K}(\chi) = \Pi$, *and let* Ω *be another order on W such that* $\Omega \subseteq\Pi$. *Then the function* $\chi' : W \to \mathscr{P}(\mathscr{P}(W))$, *expressed at each* $x\in W$ *by* $\chi'(x) = \{P \cap\Omega\,[x] : P\in\chi(x)\}$ *also is a horistology on W, and* $\mathbf{K}(\chi') = \Omega$.

Proof. It is easy to see that $\chi^{/} = inf\,\{\chi, \chi_1\}$, where χ_1 is the horistology defined by $\chi_1(x) = \mathscr{P}(\Omega\,[x])$ at each $x \in W$. ◇

13. THEOREM. *The family of all* (*u.*) *horistologies on a fixed set W is **inductively ordered*** (*i.e. every totally ordered subfamily of* (*u.*) *horistologies on W has a smallest upper bound*).

Proof. Let us discuss the case of u. horistologies. Thus, let $\mathscr{F} = \{\mathscr{H}_i : i \in \mathfrak{I}\}$ be a family of u.h. on W, for which we note $\mathbf{K}(\mathscr{H}_i) = \Pi_i$, $i \in \mathfrak{I}$. If $\mathscr{F}$ is totally ordered, then the family $\{\Pi_i : i \in \mathfrak{I}\}$ is also totally ordered by inclusion in $\mathscr{P}(W^2)$, hence their union $\Pi = \cup\{\Pi_i : i \in \mathfrak{I}\}$ is an order on W too. In this case it is clear that the u.h. $\mathscr{H} = \mathscr{P}(\Pi \setminus \Delta)$ is finer than each $\mathscr{H}_i$ (since $\Pi_i \subseteq \Pi$, and $\mathscr{H}_i \subseteq \mathscr{P}(\Pi_i \setminus \Delta)$ for each $i \in \mathfrak{I}$). In other words, $\mathscr{F}$ is an upper bounded family of u. horistologies, hence by Corollary A9 it has a smallest upper bound.

The case of (non-u.) horistological spaces is similar. ◇

14. COROLLARY. According to Zorn's Lemma, it follows that for each (uniform) horistology on a fixed set *W* there exists a finer (u.) horistology, which, at the same time, is a maximal element in the family of all (u.) horistologies on *W*.

15. *Remark.* Coming back to the problem of finding (pseudo-) horometrics that generate a given u.h., we may reformulate the construction in Theorem [1] (II) A 9 in terms of comparison between u. horistologies. Thus, if $\mathscr{B}$ is an open base consisting of exhaustive prospects of $\mathscr{H}$, σ is a (p-) horometric generated by an element of $\mathscr{B}$, and $\mathscr{H}_\sigma$ denotes the u. horistology of σ, then

$$\mathscr{H} = sup\,\{\mathscr{H}_\sigma : \sigma \text{ is generated by an element of } \mathscr{B}\}.$$

In fact, $\mathscr{H}$ is finer than each $\mathscr{H}_\sigma$ because $\{(x, y) : \sigma(x, y) \geq 1\}$ is an (open) prospect of $\mathscr{H}$. On the other hand, if another u.h. $\mathscr{H}^*$ is finer than each $\mathscr{H}_\sigma$, then $\mathscr{H} \subseteq \mathscr{H}^*$ because for each $\lambda \in \mathscr{H}$ we may find σ and $\varepsilon > 0$ such that

$$\lambda \subseteq \{(x, y) : \sigma(x, y) \geq \varepsilon\} \in \mathscr{H}_\sigma \subseteq \mathscr{H}^*.$$

In the subsequent proposition, $\mathfrak{I}$ will be an arbitrary family of indexes and *W* will be a fixed (non-void) set; Π_i denotes an order on *W* corresponding to $i \in \mathfrak{I}$, and finally, $\Pi = \cap\{\Pi_i : i \in \mathfrak{I}\}$.

16. PROPOSITION. *If* $\{\sigma_i : i \in \mathfrak{I}\}$ *is a family of p-horometrics* $\sigma_i : \Pi_i \to \mathbb{R}^+$, *then the functional* $\sigma : \Pi \to \mathbb{R}^+$, *defined by* $\sigma(x, y) = inf\,\{\sigma_i(x, y) : i \in \mathfrak{I}\}$ *also is a p-horometric on W. If* $\mathscr{H}$ *and* $\mathscr{H}_i$ *are the u. horistologies generated by* σ *and respectively* σ_i, *then* $\mathscr{H} \subseteq inf\,\{\mathscr{H}_i : i \in \mathfrak{I}\}$, *with equality whenever* $\mathfrak{I}$ *is finite*.

Proof. Obviously, Π is an order on *W*. Showing that σ verifies condition ($i^{/}$) of Definition [1] (I) C3 also is trivial (perhaps it is more interesting to remark that σ is not necessarily a horometric, even if all σ_i are).

In order to prove the super-additivity of σ it is sufficient to note that for each $i \in \mathfrak{I}$ and for each $(x, y), (y, z) \in \Pi$ we have

$$\sigma(x, y) + \sigma(y, z) = \inf\{\sigma_j(x, y) : j \in \mathfrak{I}\} + \inf\{\sigma_k(y, z) : k \in \mathfrak{I}\} \leq$$
$$\leq \sigma_i(x, y) + \sigma_i(y, z) \leq \sigma_i(x, z),$$

because the inequalities $\sigma(x, y) + \sigma(y, z) \leq \sigma_i(x, z)$ for all $i \in \mathfrak{I}$ imply

$$\sigma(x, y) + \sigma(y, z) \leq \sigma(x, z).$$

Now, let us show that $\mathscr{H}$ is coarser than each $\mathscr{H}_i$. In fact, if $\pi \in \mathscr{H}$, then there exists $\varepsilon > 0$ such that $\pi \subseteq \{(x, y) : \sigma(x, y) \geq \varepsilon\}$, hence on account of the expression of σ, we also have $\pi \subseteq \{(x, y) : \sigma_i(x, y) \geq \varepsilon\}$ for all $i \in \mathfrak{I}$. From relation $\mathscr{H} \subseteq \mathscr{H}_i$ for arbitrary $i \in \mathfrak{I}$ it follows that $\mathscr{H} \subseteq \inf\{\mathscr{H}_i : i \in \mathfrak{I}\}$.

Finally, let us consider that $\mathfrak{I}$ is finite, or more precisely, $\mathfrak{I} = \{1, 2, \ldots, n\}$. In this case, if $\mathscr{H}^*$ is coarser than each $\mathscr{H}_i$, it follows that $\mathscr{H}^* \subseteq \mathscr{H}$ too. In fact, if $\pi \in \mathscr{H}^*$, we obtain $\pi \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$, i.e. for each $i \in \mathfrak{I}$ there exists $\varepsilon_i > 0$ such that $\pi \subseteq \{(x, y) : \sigma_i(x, y) \geq \varepsilon_i\}$. Noting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\} > 0$, we have $\pi \subseteq \{(x, y) : \sigma(x, y) \geq \varepsilon\}$, hence $\mathscr{H} = \inf\{\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_n\}$. ◇

17. *Example.* If $\mathfrak{I}$ is an infinite set in the above proposition, then the strict inclusion $\mathscr{H} \subset \inf\{\mathscr{H}_i : i \in \mathfrak{I}\}$ is possible, even if all $\mathscr{H}_i$ are equal. To see this, let σ_0 be the temporal metric of a Minkowskian space-time E (like in [1] (I) B), and let us define $\sigma_n = \frac{1}{n}\sigma_0$ for each $n \in \mathbb{N} \setminus \{0\}$. Then $\inf\{\mathscr{H}_n : n \in \mathbb{N}\}$ is the Minkowskian u. horistology on E, while $\mathscr{H}$ is the coarsest one.

B. Induced horistological structures

In this section we will analyze the construction and some properties of the direct and inverse images of the horistological structures.

1. THEOREM. *Let* $(W, \mathscr{H})$ *be a u.h. space, and let* $f : W \to M$, *be an injective function, where M is an arbitrary set. Then* $f(\mathscr{H}) = \{f_2(\pi) : \pi \in \mathscr{H}\}$ *is a u.h. on M, namely the coarsest u.h. for which f is uniformly discrete.*

Proof. It is to show that $f(\mathscr{H})$ satisfies conditions $[uh_1]$ – $[uh_4]$. In order to prove $[uh_1]$, it is sufficient to recall that for all $(x, y) \in \pi \in \mathscr{H}$ we have $x \neq y$; because f is injective, we obtain $f(x) \neq f(y)$, and consequently $\Delta \cap f_2(\pi) = \varnothing$.

For proving $[uh_2]$ let us consider $f_2(\pi) \in f(\mathscr{H})$ and $\lambda \subseteq f_2(\pi)$. Using again the fact that f is injective, we deduce $f_2^{\leftarrow}(\lambda) \subseteq \pi$, hence $f_2^{\leftarrow}(\lambda) \in \mathscr{H}$. Further, because $\lambda \subseteq f_2(W^2)$, we have $f_2(f_2^{\leftarrow}(\lambda)) = \lambda$, hence $\lambda \in f(\mathscr{H})$.

Condition $[uh_3]$ is a simple consequence of the equality $f_2(\pi) \cup f_2(\lambda) = f_2(\pi \cup \lambda)$, which holds for all $\pi, \lambda \subseteq W^2$.

Because each element of $f(\mathscr{H})$ has the form $f_2(\pi)$ for some $\pi\in\mathscr{H}$, we may formulate condition [uh$_4$] as follows: For each $\pi\in\mathscr{H}$ there exists $\lambda\in\mathscr{H}$, such that for every $\omega\in\mathscr{H}$ we have $f_2(\lambda)\supseteq f_2(\pi)\circ f_2(\omega)$ and $f_2(\lambda)\supseteq f_2(\omega)\circ f_2(\pi)$. Let λ be the prospect which corresponds to π by virtue of the fact that $\mathscr{H}$ satisfies [uh$_4$], i.e. $\lambda\supseteq\pi\circ\omega$ and $\lambda\supseteq\omega\circ\pi$ for all $\omega\in\mathscr{H}$. Under the hypothesis that f is injective, we have $f_2(\omega)\circ f_2(\pi)\subseteq f_2(\omega\circ\pi)$, which achieves the proof of [uh$_4$].

Finally, reasoning like in Proposition A5, we can easily see that f is uniformly discrete relative to the u. horistologies $\mathscr{H}$ on W and $\mathscr{L}$ on M if and only if $\mathscr{L}$ is finer than $f(\mathscr{H})$, i.e. $f(\mathscr{H})\subseteq\mathscr{L}$. ◇

2. THEOREM. *Let (W, χ) be a horistological world, and let function $f: W\to M$, be injective, where M is arbitrary. Then function $\varphi: M\to\mathscr{P}(\mathscr{P}(M))$, defined by*

$$\varphi(x) = \begin{cases}\{f(P): P\in\chi(x')\} & if\ \ x = f(x') \\ \{\varnothing\} & if\ \ x\notin f(W)\end{cases}$$

is a horistology on M, namely the coarsest horistology for which f is discrete on W.

Proof. We verify conditions [h$_1$] – [h$_4$] by going through the same stages as for the above theorem. The single difference consists in considering two cases, namely $x\in f(W)$ or not. ◇

3. *Definition.* We say that the u.h. $f(\mathscr{H})$ in Theorem B1 is the ***direct image*** of the u.h. $\mathscr{H}$ on M (if necessary we may specify "through f"). Similarly, horistology φ in Theorem B2 is the ***direct image*** of the horistology χ on M.

The following theorem gives a relation between the direct images of uniform and non-uniform horistological structures.

4. THEOREM. *Let $\mathscr{H}$ be an u.h. on W, and let χ be the horistology generated by $\mathscr{H}$. If $f: W\to M$ is injective, we note by $f(\mathscr{H})$ and φ the direct images of $\mathscr{H}$ and respectively χ on M. Then $f(\mathscr{H})$ generates φ.*

Proof. According to [1] (II) B2c, we have $\chi(x') = \{\pi[x'] : \pi\in\mathscr{H}\}$ at each $x'\in W$, hence

$$\varphi(x) = \begin{cases}\{f(\pi[x']): \pi\in\mathscr{H}\} & if\ \ x = f(x') \\ \{\varnothing\} & if\ \ x\notin f(W).\end{cases}$$

On the other hand, the horistology ψ, generated by $f(\mathscr{H})$, has the form

$$\psi(x) = \{(f_2(\pi))[x] : \pi\in\mathscr{H}\},$$

where $x\in M$, hence the problem is to prove that $\varphi = \psi$. Now, we shall distinguish two cases, namely either $x\in M\setminus f(W)$, or $x\in f(W)$. In the first case we have $\psi(x) = \{\varnothing\} = \varphi(x)$, because in $f_2(\pi)$ there is no pair with x on the first place. In the second case we have $f(\pi[x']) = (f_2(\pi))[x]$ whenever $x = f(x')$. ◇

5. ***Examples.*** As we easily see, the hypothesis that f is injective is essential in the above construction of the direct images of horistological structures. However, it is possible to obtain horistological structures as direct images through some non-injective functions. For example, function $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, expressed by

$$\chi(x) = \{\mathscr{P}(\Pi[y]) : (x, y) \in \Pi \setminus \Delta\},$$

where Π is a total order on W, defines a horistology on W. In particular, let us take $W = \mathbb{R}^2$, and let Π be the lexical order of the plane. If f_1 is the first projection of $\mathbb{R}^2$, then $f_1(\chi)$ also is a horistology on $\mathbb{R}$, which is generated on the same way on $W = \mathbb{R}$ by its usual order. At the same time, other simple examples (e.g. the second projection f_2 in the above case) show that, in general, the direct images of a horistology on a product space through projections on the components are not horistologies.

6. THEOREM. *Let* $(W, \mathscr{H})$ *be an u.h. space, and let* $f : W \to M$ *be an arbitrary function. Then the family* $f^{\leftarrow}(\mathscr{H}) \subseteq \mathscr{P}(W^2)$, *defined by*

$$f^{\leftarrow}(\mathscr{H}) = \{\pi : \exists \pi' \in \mathscr{H} \text{ such that } \pi \subseteq f_2^{\leftarrow}(\pi')\},$$

is a horistology on W, namely the finest one for which f is u. discrete.

Proof. We shall prove the conditions $[uh_1]$ – $[uh_4]$. Thus, for $[uh_1]$, if Δ and Δ' are the diagonals of W^2 respectively M^2, we have $\pi' \cap \Delta' = \varnothing$ for each $\pi' \in \mathscr{H}$, hence $f_2^{\leftarrow}(\pi') \cap \Delta = \varnothing$ too, and a fortiori $\pi \cap \Delta = \varnothing$ for each $\pi \in f^{\leftarrow}(\mathscr{H})$.

Condition $[uh_2]$ is an immediate consequence of the definition of $f^{\leftarrow}(\mathscr{H})$, and $[uh_3]$ is directly based on the general relation

$$f_2^{\leftarrow}(\pi') \cup f_2^{\leftarrow}(\lambda') = f_2^{\leftarrow}(\pi' \cup \lambda').$$

In order to prove $[uh_4]$, let π be arbitrary in $f^{\leftarrow}(\mathscr{H})$, i.e. $\pi \subseteq f_2^{\leftarrow}(\pi')$ for some $\pi' \in \mathscr{H}$. Because $\mathscr{H}$ satisfies $[uh_4]$, there exists $\lambda' \in \mathscr{H}$, which corresponds to π' in this condition. Let us note $\lambda = f_2^{\leftarrow}(\lambda')$, and let ω be another element of $f^{\leftarrow}(\mathscr{H})$ such that $\omega \subseteq f_2^{\leftarrow}(\omega')$ for some $\omega' \in \mathscr{H}$. From $\pi' \circ \omega' \subseteq \lambda'$ we deduce

$$\pi \circ \omega \subseteq f_2^{\leftarrow}(\pi') \circ f_2^{\leftarrow}(\omega') \subseteq f_2^{\leftarrow}(\pi' \circ \omega') \subseteq f_2^{\leftarrow}(\lambda') = \lambda.$$

Similarly, $\omega' \circ \pi' \subseteq \lambda'$ gives $\omega \circ \pi \subseteq \lambda$.

Finally, we have $f_2(\pi) \in \mathscr{H}$ for each $\pi \in f^{\leftarrow}(\mathscr{H})$, i.e. f is u. discrete relative to $f^{\leftarrow}(\mathscr{H})$ on W and $\mathscr{H}$ on M. On the other hand, if $\mathscr{H}^*$ is another u.h. on W for which f is u. discrete, we have $f_2(\pi) \in \mathscr{H}$ for each $\pi \in \mathscr{H}^*$. Then using the general relation

$$\pi \subseteq f_2^{\leftarrow}(f_2(\pi)),$$

it follows that $\pi \in f^{\leftarrow}(\mathscr{H})$, i.e. $\mathscr{H}^*$ is coarser than $f^{\leftarrow}(\mathscr{H})$. ◇

7. **THEOREM.** *Let $f : W \to M$ be an arbitrary function, and let χ be a horistology on M. Then function $\psi : W \to \mathscr{P}(\mathscr{P}(W))$, whose value at $x \in W$ is defined by*

$$\psi(x) = \{P : \exists P' \in \chi(f(x)) \text{ such that } P \subseteq f^{\leftarrow}(P')\},$$

is a horistology on W; more precisely, $\psi = f^{\leftarrow}(\chi)$ is the finest horistology for which f is discrete on W.

The proof goes through the same stages as those for the above theorem so we will omit it. A terminology similar to that of Definition B3 is natural.

8. ***Definition.*** We say that the u.h. $f^{\leftarrow}(\mathscr{H})$ in Theorem B6, and the horistology $f^{\leftarrow}(\chi)$ in Theorem B7, are the ***inverse images*** of the u.h. $\mathscr{H}$, and respectively of the horistology χ, through function *f*, on *W*.

9. **THEOREM.** *Let $f : W \to M$ be an arbitrary function, and let $\mathscr{H}$ be a u.h. on M. If χ is the horistology generated by $\mathscr{H}$, then $f^{\leftarrow}(\mathscr{H})$ exactly generates $f^{\leftarrow}(\chi)$.*

Proof. The inverse image of the horistology χ has a base consisting of sets of the form $f^{\leftarrow}(\pi[f(x)])$, where $\pi \in \mathscr{H}$ and $x \in W$. On the other hand, at each $x \in W$, the horistology generated by $f^{\leftarrow}(\mathscr{H})$ has an ideal base consisting of sets of the form $f_2^{\leftarrow}(\pi)[x]$. It is easy to see that these two bases coincide. ◇

10. **THEOREM.** *Let W be an arbitrary set, and let $\mathfrak{I}$ be an arbitrary family of indices. We suppose that to each $i \in \mathfrak{I}$ there corresponds a u.h. world $(M_i, \mathscr{H}_i)$ and a function $f_i : W \to M_i$. Then the family $\mathscr{H} \subseteq \mathscr{P}(W^2)$, consisting of all sets π for which there exist $\pi_i \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$ such that*

$$\pi \subseteq \cap\{(f_i)_2^{\leftarrow}(\pi_i) : i \in \mathfrak{I}\},$$

is a u.h. on W, namely the finest one for which all the functions f_i are u. discrete.

Proof. If $f_i^{\leftarrow}(\mathscr{H}_i)$ denotes the inverse image of $\mathscr{H}_i$ through f_i, then according to the above Theorem A7, we may consider that $\mathscr{H} = inf\{f_i^{\leftarrow}(\mathscr{H}_i) : i \in \mathfrak{I}\}$.

In order to prove the discreteness of the functions f_i, we may consider $f_i = \iota \circ g_i$ for each $i \in \mathfrak{I}$, where the identity acts as $\iota : (W, \mathscr{H}) \to (W, f_i^{\leftarrow}(\mathscr{H}_i))$, and

$$g_i : (W, f_i^{\leftarrow}(\mathscr{H}_i)) \to (M_i, \mathscr{H}_i)$$

takes the same values as f_i. Consequently, each f_i appears as a composition of two u. discrete functions.

Finally, if $\mathscr{H}^*$ is another u.h. on *W*, for which all f_i are u. discrete, then $\mathscr{H}^*$ is coarser than $\mathscr{H}$. In fact, if $\pi \in \mathscr{H}^*$, then we find $\pi_i \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$, such that $(f_i)_2(\pi) \subseteq \pi_i$, and consequently $\pi \subseteq (f_i)_2^{\leftarrow}(\pi_i)$. Then $\pi \subseteq \cap\{(f_i)_2^{\leftarrow}(\pi_i) : i \in \mathfrak{I}\}$, hence $\pi \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$. ◇

The same construction may be done with (non-uniform) horistological worlds, so we will describe it without proof.

11. THEOREM. *Let W and* $\mathfrak{I}$ *be arbitrary non-void sets. To each* $i \in \mathfrak{I}$ *we attach a horistological world* (M_i, χ_i) *and an application* $f_i : W \to M_i$. *We claim that function* $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, *which assigns to each* $x \in W$ *the family of all subsets P of W for which there exist* $P_i \in \chi_i(f_i(x))$ *for all* $i \in \mathfrak{I}$, *such that*

$$P \subseteq \cap\{ f_i^{\leftarrow}(P_i) : i \in \mathfrak{I}\},$$

is a horistology on W, namely the finest one for which all f_i *are discrete on W*.

12. COROLLARY. Under the conditions in Theorem B10, if χ_i are horistologies generated by $\mathscr{H}_i$ on M_i, then the horistology χ, constructed as in Theorem B11 for the same family of $\mathfrak{I}$ of indexes, coincides with the horistology generated by $\mathscr{H}$ on *W*.

Proof. In the above conditions, we have

$$\chi = inf\{ f_i^{\leftarrow}(\chi_i) : i \in \mathfrak{I}\}$$

and

$$\mathscr{H} = inf\{ f_i^{\leftarrow}(\mathscr{H}_i) : i \in \mathfrak{I}\}.$$

According to Theorem B9, $f_i^{\leftarrow}(\mathscr{H}_i)$ generates $f_i^{\leftarrow}(\chi_i)$ for each , $i \in \mathfrak{I}$, hence the assertion follows as a consequence of Theorem A10. ◇

13. *Remark.* Like in the case of the topological structures (see [3], etc.), we say that the structures $\mathscr{H}$ and χ in Theorems B10 and B11 are *initial horistological structures*. We may formulate the properties expressed by the mentioned theorems in terms of function composition, as the following two propositions show.

14. PROPOSITION. *Under the conditions of Theorem B10, if* $(L, \mathscr{D})$ *is another u.h. world, then function h*: $L \to W$ *is u. discrete relative to* $\mathscr{D}$ *and* $\mathscr{H}$ *if and only if* $f_i \circ h$ *is u. discrete for each* $i \in \mathfrak{I}$.

Proof. If *h* is u. discrete, then $f_i \circ h$ is u. discrete for each $i \in \mathfrak{I}$ too. Conversely, if π is an arbitrary prospect of $\mathscr{D}$, then $\lambda = h_2(\pi) \in \mathscr{H}$. In fact, because each $f_i \circ h$ is u. discrete, we have $(f_i \circ h)_2(\pi) \in \mathscr{H}_i$ for each $i \in \mathfrak{I}$. Then $(f_i)_2(\lambda) \in \mathscr{H}_i$, and so $\lambda \in f_i^{\leftarrow}(\mathscr{H}_i)$ for each $i \in \mathfrak{I}$. If we recall that $\mathscr{H} = \cap\{ f_i^{\leftarrow}(\mathscr{H}_i) : i \in \mathfrak{I}\}$, we really obtain $\lambda \in \mathscr{H}$. ◇

15. PROPOSITION. *Beyond the conditions in the above Theorem B11, let* (L, ψ) *be another horistological world. Then function h*: $L \to W$ *is discrete at* $x \in L$ *relative to* ψ *and* χ *if and only if all the functions* $f_i \circ h$ *are discrete at* $x \in L$ *in respect to* ψ *and* χ_i.

We may reason like in the proof of the previous proposition, but referring to non-uniformly discrete functions.

C. Horistological subspaces, products and quotients

In essence, this paragraph contains some particular cases when we can apply the methods of deriving horistological structures as in the previous section.

1. THEOREM. *Let* $(M, \mathscr{H})$ *be an u.h. space, and let us consider* $W \subseteq M$. *Then the family* $\mathscr{L} \subset \mathscr{P}(W^2)$, *expressed by*

$$\mathscr{L} = \{\pi : \exists \pi' \in \mathscr{H} \text{ such that } \pi = \pi' \cap W^2\}$$

is a u. horistology on W, namely the finest one for which the embedding of W in M is uniformly discrete.

Similarly, if χ *is a horistology on M, then function* $\psi : W \to \mathscr{P}(\mathscr{P}(W))$, *defined by*

$$\psi(x) = \{P : \exists P' \in \chi(x) \text{ such that } P = P' \cap W\},$$

is a horistology on W, namely the finest one for which the embedding of W in M is discrete on W.

In addition, if $\mathscr{H}$ *generates* χ *on M, then* $\mathscr{L}$ *generates* ψ *on W.*

Proof. If $f : W \to M$ denotes the embedding $f(x) = x$, then $\lambda \cap W^2 = f_2^{\leftarrow}(\lambda)$ for each $\lambda \subseteq M^2$, hence $\mathscr{L} = f^{\leftarrow}(\mathscr{H})$. We only mention that, because f is injective, the family $\{ f_2^{\leftarrow}(\lambda) : \lambda \in \mathscr{H}\}$ is an ideal, but not only an ideal base.

We may similarly see that $\psi = f^{\leftarrow}(\chi)$.

The last sentence is a corollary of Theorem B9. ◇

2. THEOREM. *Let W be a subset of M. Then each u.h.* $\mathscr{H}_W$ *on W, considered as a family of parts of* M^2, *is a u.h. on M too (noted* $\mathscr{H}_M$ *).*

Similarly, if χ_W *is a horistology on W, then function* $\chi_M : M \to \mathscr{P}(\mathscr{P}(M))$, *defined at each* $x \in M$ *by*

$$\chi_M(x) = \begin{cases} \chi_W(x) & \text{if } x \in W \\ \{\varnothing\} & \text{if } x \in M \setminus W \end{cases}$$

is a horistology on M: more exactly, it is the coarsest one for which the embedding of W in M is discrete on W.

In addition, if $\mathscr{H}_W$ *generates* χ_W *on W, then* $\mathscr{H}_M$ *generates* χ_M *on M.*

Proof. The embedding $f(x) = x$ of W in M is injective, so we may consider that $\mathscr{H}_M$ is the direct image of $\mathscr{H}_W$ through f, as in Theorem B1. Similarly, χ_M is the direct image of χ_W through f, in the sense of Theorem B2. The last assertion simply follows from Theorem B4. ◇

3. *Remark.* Like in topology, it is natural to say that $(W, \mathscr{L})$ in Theorem C1 is a *uniform horistological subspace* of $(M, \mathscr{H})$, and (W, ψ) is a *horistological subspace* of (M, χ). Considering over-spaces (as in Theorem C2) seems to be more significant for horistological than for topological structures.

The construction of a *product* of (u.) horistological spaces is a direct issue of Theorems B10 and B11. In fact, if $W = \mathsf{X}\{M_i : i \in \mathfrak{J}\}$, and f_i denotes the projection of W on the component M_i, then $(W, \mathscr{H})$ from Theorem B10 exactly represents the *product* u.h. space of the u.h. spaces $(M_i, \mathscr{H}_i)$, $i \in \mathfrak{J}$. Under similar conditions, (W, χ) of Theorem B11 is the *product* of the horistological spaces (M_i, χ_i), $i \in \mathfrak{J}$.

The study of the *quotient* horistological structures cannot be simply reduces to the constructions described in section B (see for example B5), and some additional conditions are necessary. For the sake of clarity, we start by specifying the usual notations: So, if θ is an equivalence relation on W, then we note the corresponding *equivalence classes* by $\tilde{x} = \theta[x]$. For each part $P \subseteq W$, we note its *equivalence extension* by $\tilde{P} = \{\tilde{x} : x \in P\}$, and for each $\pi \subseteq W^2$ we write $\tilde{\pi} = \{(\tilde{x}, \tilde{y}) : (x, y) \in \pi\}$. Finally, for each family $\mathscr{F} \subseteq \mathscr{P}(W)$ (or $\mathfrak{F} \subseteq \mathscr{P}(W^2)$) we note $\tilde{\mathscr{F}} = \{\tilde{P} : P \in \mathscr{F}\}$ (respectively $\tilde{\mathfrak{F}} = \{\tilde{\pi} : \pi \in \mathfrak{F}\}$).

4**. *Definition.*** We say that a relation π on W is ***stable*** relative to the equivalence θ iff $\theta \circ \pi \subseteq \pi$ and $\pi \circ \theta \subseteq \pi$. Similarly, a family $\mathfrak{F}$ of relations on W is said to be ***stable*** relative to θ iff each of its elements is so.

5. THEOREM. *Let* $(W, \mathscr{H})$ *be an u.h. space, and let* θ *be an equivalence on W. If a base* $\mathscr{B}$ *of* $\mathscr{H}$ *is stable relative to* θ, *then* $\tilde{\mathscr{B}}$ *forms a base of a uniform horistology on* $\tilde{W}$. *This u. horistology* (*noted* $\tilde{\mathscr{H}}$ *and called* ***quotient*** *of* $\mathscr{H}$ *by* θ) *is independent of the choice of the stable base* $\mathscr{B}$, *and it is the coarsest u.h. on* $\tilde{W}$ *for which the canonical function* $x \mapsto \tilde{x}$ *is uniformly discrete*.

Proof. Because the canonical map $x \mapsto \tilde{x}$ is not generally injective, we may not apply Theorem B1, and therefore we must directly verify the conditions that define the ideal base of a u. horistology.

The former condition means $\tilde{\Delta} \cap \tilde{\pi} = \varnothing$ for each $\pi \in \mathscr{B}$. In fact, if we suppose the contrary, from $(\tilde{x}, \tilde{y}) \in \tilde{\Delta} \cap \tilde{\pi}$ it follows $\tilde{x} = \tilde{y}$, hence there exists at least one element $z \in \tilde{x}$ such that $(z, z) \in \pi$, which is impossible.

The property of $\tilde{\mathscr{B}}$ to be an ideal base asks for any $\tilde{\lambda}, \tilde{\pi} \in \tilde{\mathscr{B}}$ to find $\tilde{\omega} \in \tilde{\mathscr{B}}$ such that $\tilde{\lambda} \cup \tilde{\pi} \subseteq \tilde{\omega}$. Obviously, it follows from the same property of $\mathscr{B}$.

The last condition exactly is [uh$_4$]. In order to prove it, let $\tilde{\pi}$ be an arbitrary element of $\tilde{\mathscr{B}}$, and let $\lambda \in \mathscr{B}$ be the prospect for which $\pi \circ \omega \subseteq \lambda$ and $\omega \circ \pi \subseteq \lambda$ for all $\omega \in \mathscr{B}$. It is easy to see that $\tilde{\lambda}$ is the required element, i.e. $\tilde{\pi} \circ \tilde{\omega} \subseteq \tilde{\lambda}$ and $\tilde{\omega} \circ \tilde{\pi} \subseteq \tilde{\lambda}$ hold for all $\tilde{\omega} \in \tilde{\mathscr{B}}$. In fact, because $\mathscr{B}$ is stable base relative to θ, from $(\tilde{x}, \tilde{z}) \in \tilde{\pi}$ and $(\tilde{z}, \tilde{y}) \in \tilde{\omega}$ we deduce that $(x', z') \in \pi$ and $(z', y') \in \omega$ for all $x' \in \tilde{x}$, $y' \in \tilde{y}$ and $z' \in \tilde{z}$. Consequently, $(x', y') \in \pi$, hence $(\tilde{x}, \tilde{y}) \in \tilde{\lambda}$.

We may similarly obtain $\tilde{\omega} \circ \tilde{\pi} \subseteq \tilde{\lambda}$.

Finally, if $\mathscr{B}_1$ and $\mathscr{B}_2$ are stable bases of $\mathscr{H}$, then $\tilde{\mathscr{B}}_1$ and $\tilde{\mathscr{B}}_2$ are equivalent ideal bases, hence $\tilde{\mathscr{H}}$ is unique.

In addition, the canonical application $q(x) = \tilde{x}$ is u. discrete because $\pi \in \mathscr{H}$ gives $q_2(\pi) = \tilde{\pi} \in \tilde{\mathscr{H}}$. It is also clear that every u.h. on $\tilde{W}$, for which function q is u. discrete, must be finer than the quotient u. horistology. ◇

6. THEOREM. *Let* (W, χ) *be a horistological world, and let* θ *be an equivalence relation on* W*. If* χ *has a base* β *such that relation*

$$\pi_{x,P} = \{(x, y) : y \in P\}$$

is stable relative to θ *for each* $P \in \beta(x)$ *and* $x \in W$*, then function*

$$\tilde{\beta} : \tilde{W} \to \mathscr{P}(\mathscr{P}(\tilde{W})),$$

defined by

$$\tilde{\beta}(\tilde{x}) = \{\tilde{P} : P \in \beta(x)\},$$

is a base for a horistology on $\tilde{W}$ *(noted* $\tilde{\chi}$ *and called* ***quotient*** *of* χ *by* θ*). The horistology* $\tilde{\chi}$ *does not depend on the choice of the stable base* β*, and it may be characterized as the coarsest horistology of* $\tilde{W}$ *for which the canonical application* $q(x) = \tilde{x}$ *is discrete on* W*.*

Proof. Under the above stated conditions it is clear that two equivalent elements have the same perspectives in the base β and each perspective of this base includes the whole class $\tilde{x}$ whenever it contains x.

The effective proof consists in showing that $\tilde{\beta}$ is a base of a horistology; being similar to the above one, we will omit the details. ◇

7. COROLLARY. Let $(W, \mathscr{H})$ and θ be like in Theorem C5, and let $\mathscr{H}$ generate the horistology χ. Then χ satisfies the hypotheses of Theorem C6, and the u.h. $\tilde{\mathscr{H}}$ generates $\tilde{\chi}$.

8. *Remark.* The question concerning the proper orders of the derived (uniform) horistologies has the following simple answer:

a) If Π is the proper order of $\mathscr{H}$ (or χ) in Theorem C1, then $\Pi \cap W^2$ is the proper order of $\mathscr{L}$ (respectively of ψ).

b) The construction of an over-space extends the initial proper order by identity on the greater space.

c) The proper order of a product of horistological structures is the product of the corresponding proper orders of the factors.

d) If Π is the proper order of $\mathscr{H}$ in Theorem C5 (or χ in Theorem C6) then $\tilde{\Pi}$ is the proper order of the quotient $\tilde{\mathscr{H}}$ (respectively $\tilde{\chi}$).

The last problem that we will discuss here concerns the metrizability of the derived horistological structures.

9. THEOREM. *Every u. horistological subspace* (*or over-space*) *of a metrizable u.h. space is metrizable too.*

Proof. Using the notations in the first part of Theorem C1, the problem is to show that $\mathscr{L}$ is metrizable (in the sense of [1] (II) A5b, A9, etc., i.e. by (p) horometrics) whenever $\mathscr{H}$ is. Let $\sigma : \Pi \to \mathbb{R}^+$ be a p-horometric that generates $\mathscr{H}$ on *M*. Then $\Omega = \Pi \cap W^2$ is an order on *W*, and the restriction ρ of σ to *W*, i.e. $\rho : \Omega \to \mathbb{R}^+$, is a p-horometric on this set. If π_r and ω_r are the hyperbolical prospects of radius $r > 0$ corresponding to σ and ρ, then $\omega_r = \pi_r \cap W^2$, hence ρ generates $\mathscr{L}$.

The affirmation concerning the extension follows from the fact that we may extend each p-horometric $\sigma : \Pi \to \mathbb{R}^+$ of *M*, to an over-space $L \supseteq M$. In fact, if Δ is the diagonal of *L*, and $\Lambda = \Pi \cup \Delta$, then the p-horometric $\overline{\sigma} : \Lambda \to \mathbb{R}^+$, of the form

$$\overline{\sigma}(x, y) = \begin{cases} 0 & if\ (x,y) \in \Delta \\ \sigma(x,y) & if\ (x,y) \in \Pi , \end{cases}$$

defines the same hyperbolical prospects. ◇

10. THEOREM. *Every finite product of metrizable u.h. spaces is also metrizable.*

Proof. Like in the above Remark C3, let us take $W = \times \{M_i : i \in \mathfrak{I}\}$, and let us note by f_i the projection of *W* on M_i. Using the hypothesis of metrizability, for each $(M_i, \mathscr{H}_i)$ we can find a p-horometric $\sigma_i : \Pi_i \to \mathbb{R}^+$, which generates $\mathscr{H}_i$. If Π denotes the product order of Π_i, then the functionals $\overline{\sigma}_i : \Pi \to \mathbb{R}^+$, defined by

$$\overline{\sigma}_i((x_k)_{k \in \mathfrak{I}}, (y_j)_{j \in \mathfrak{I}}) = \sigma_i(x_i, y_i),$$

are p-horometrics on *W*. It is not difficult to see that each $\overline{\sigma}_i$ generates on *W* the inverse image $f_i^{\leftarrow}(\mathscr{H}_i)$. According to Theorem B10, the product u. horistology $\mathscr{H}$ on *W* allows the form

$$\mathscr{H} = inf\{ f_i^{\leftarrow}(\mathscr{H}_i) : i \in \mathfrak{I}\}.$$

So it remains to apply Proposition A16, which shows that $\sigma : \Pi \to \mathbb{R}^+$, of values

$$\sigma(x, y) = inf\{\overline{\sigma}_i(x, y) : i \in \mathfrak{I}\},$$

exactly generates $\mathscr{H}$ when $\mathfrak{I}$ is finite. ◇

11**.** ***Definition.*** Let (*W*, Π, σ) be a p-horometric space, and θ be an equivalence relation on *W* such that Π is stable relative to θ. We say that the p-horometric σ is ***stable*** relative to θ, iff

$$\sigma(x, y) = \sigma(x', y')$$

whenever $(x, x'), (y, y') \in \theta$. In this case, the functional $\tilde{\sigma} : \tilde{\Pi} \to \mathbb{R}^+$, defined by

$$\tilde{\sigma}(\tilde{x}, \tilde{y}) = \sigma(x, y),$$

where $x \in \tilde{x}$, and $y \in \tilde{y}$, is a p-horometric. We call it ***quotient*** p-horometric of σ relative to θ.

12. **THEOREM.** *Let* (W, Π, σ) *be a p-horometric space for which* σ *is stable relative to an equivalence relation* θ *on W. Then the u.h.* $\mathscr{H}$, *generated by* σ, *has a stable base relative to* θ, *and the quotient p-horometric* $\tilde{\sigma}$ *generates the quotient u.h.* $\tilde{\mathscr{H}}$ *on the quotient space* $\tilde{W}$.

Proof. The family $\mathscr{B}$ of all hyperbolical prospects in respect to σ obviously is stable relative to θ and forms a base of $\mathscr{H}$. If

$$\pi_r = \{(x, y) \in W^2 : \sigma(x, y) \geq r\},$$

where $r > 0$, is an arbitrary element of $\mathscr{B}$, then

$$\tilde{\pi}_r = \{(\tilde{x}, \tilde{y}) \in \tilde{W}^2 : \tilde{\sigma}(\tilde{x}, \tilde{y}) \geq r\},$$

hence $\tilde{\mathscr{H}}$ has a base consisting of all hyperbolical prospects generated by the p-horometric $\tilde{\sigma}$. ◇

Of course, the present study of the operations with spaces (worlds) endowed with horistological structures is far from being complete, but when necessary, making use of similar methods, we may investigate the other cases (e.g. direct sums, function spaces. etc.).

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