GENERALIZING THE MINKOWSKIAN SPACE-TIME (I)

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We organize the set of events, which possibly happen in a scalar product space, a normed space or in a metric space in a way similar to the Minkowskian space-time. Then we generalize the forthcoming structures to worlds of events endowed with super-additive norms and metrics.

A. SPACE-TIME MATHEMATICAL BACKGROUND

It is well known that the special relativity deals with events which happen in $\mathbb{R}$, $\mathbb{R}^2$, or $\mathbb{R}^3$. By "event" we understand a couple $e = (t, \bar{x})$, where t represents the moment when, and $\bar{x} = (x_1, x_2, x_3)$ represents the place where something happens, most frequently emission or reception of light. Consequently, the set $M = \mathbb{R}\times\mathbb{R}^3$ of all possible events constitutes the fundamental framework of the theory of relativity.

The structure of M is properly introduced by the indefinite inner product i.e. $(. , .) : M \times M \to\mathbb{R}$, expressed by

$$(e_1, e_2) = c^2\, t_1\, t_2 - x_1\, x_2 - y_1\, y_2 - z_1\, z_2\,,$$

where $e_i = (t_i;\, x_i,\, y_i,\, z_i)$, $i = 1, 2$, and c is the light's speed in vacuum. The couple $(M, (. , .))$ is the Minkowskian space-time of the special relativity.

Developing the theory, it is often apparent that this feature of M of being a very particular space is inconvenient in the study of the events' specific properties because of the geometric (even Euclidean) and algebraic extra properties. This is the reason for considering some generalizations of M. Such an immediate generalization is based on the remark that in the inner product of M there concur the usual product of $\mathbb{R}$ and the Euclidean scalar product of $\mathbb{R}^3$, $\langle \bar{x}_1, \bar{x}_2\rangle = x_1\, x_2 + y_1\, y_2 + z_1\, z_2$, i.e.

$$(e_1, e_2) = c^2\, t_1\, t_2 - \langle \bar{x}_1, \bar{x}_2\rangle.$$

Replacing $\mathbb{R}^3$ by an arbitrary real scalar product space and taking into account that the constant c is dependent on the units used for space and time, we are led to the following notion:

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1. Definition. Let (H, $\langle .,. \rangle$) be a real scalar product space and $E = \mathbb{R}\times H$. For any $e, f \in E$ of the form $e = (t, x)$ and $f = (s, y)$, we note

(1) $$(e, f) = t\,s - \langle x, y \rangle$$

and we say that (E, (. , .)) is the *world of events in the scalar product space* H, or briefly, it is a *scalar event world*.

2. Remark. The functional (. , .) : $E \times E \to \mathbb{R}$ from above is an indefinite inner product on E, that explains why the scalar event worlds are usually treated as Pontrjagin Π_1 spaces (see [1]). The term "world" has been used by physicists to denote the space-time from the very beginning of the relativity. We will adopt this terminology in order to underline the necessity of new specific structures when we deal with sets of events instead of material points.

From a mathematical point of view, the fact that dim $M \le 4$ is not significant for the structure of the considered world. Consequently, we will present the events' fundamental properties directly in a scalar event world. Primarily we recall the principal notions that derive from the inner product of E.

3. Terminology and notation. Let (E, (. , .)) be a scalar event world and let $e = (t, x)$, $f = (s, y)$, etc. be events (elements of E).

a) The set $K = \{(e, f)\in E^2 : s - t > \|x - y\| \text{ or } e = f\}$ is called *causal relation*, briefly *causality*.

b) The functional $\|\cdot\|_t : K[0] \to \mathbb{R}_+$, expressed by $\|e\|_t = \sqrt{(e,e)}$, is called *temporal norm*.

c) The functional $\sigma : K \to \mathbb{R}_+$, defined by $\sigma(e, f) = \|f - e\|_t$ is called *temporal metric*.

d) If $(e, f) = 0$ we say that e and f are *orthogonal*, and we note $e \perp f$.

e) If $e, f \in K[0] \setminus \{0\}$, then the *hyperbolical angle* between e and f is

$$\angle(e, f) = \{(e', f') : \exists\ \lambda, \mu \in \mathbb{R}_+ \text{ such that } e' = \lambda\, e \text{ and } f' = \mu f\}.$$

The set of all hyperbolical angles is denoted by $\mathfrak{H}$, and the function $m : \mathfrak{H} \to \mathbb{R}$, expressed by

$$m(\angle(e, f)) = \text{arc ch}\,[(e, f) / \|e\|_t \|f\|_t]$$

is called *measure* of the hyperbolical angles.

4. Physical meaning. In the particular case of the Minkowskian space-time all the notions from above (and the list continues) have concrete significance (see any book on relativity, for example [4]). Thus $(e, f)\in K$ expresses the fact that the events e and f belong to some particle's life. The temporal norm, and respectively the temporal metric, measure the proper time of a free particle between two events. If $\omega \in K[0] \setminus \{0\}$, its support line represents an inertial observer that passes through the origin of the space at the moment zero.

The condition $\omega \perp (e - f)$, when $\omega \in K[0] \setminus \{0\}$, means that the events e and f are simultaneous for the observer generated by ω. The number $m(\angle(e, f))$ shows how great is the relative velocity of the inertial observers generated by e and f, and so on.

We mention that most part of the above notions are usually accompanied by their duals: spatial relation, signal relation, spacial norm, spatial metric, spatial orthogonality, circular angles, etc., which may be similarly treated.

In the sequel, we analyze some of the most remarkable mathematical properties involving the considered notions.

5. PROPOSITION. *K is a linear order on the scalar event world E.*

Proof. It is sufficient to show that the set of all positive elements, that is

$$\mathbf{P} \equiv K[0] = \{(t, x) \in E : t > \|x\|\} \cup \{0\}$$

is a sharp cone, i.e. $\mathbf{P} \cap (-\mathbf{P}) = \{0\}$, $\lambda \mathbf{P} \subseteq \mathbf{P}$ for all $\lambda \geq 0$, and $\mathbf{P} + \mathbf{P} \subseteq \mathbf{P}$. The first and the second conditions are obvious. In order to prove the third condition, let us consider $e = (t, x)$, $f = (s, y) \in \mathbf{P}$. From $t > \|x\|$ and $s > \|y\|$ we deduce $t + s > \|x\| + \|y\| \geq \|x + y\|$. The other cases are trivial, hence $e + f \in \mathbf{P}$. It remains to see that $(e, f) \in K$ if and only if $f - e \in \mathbf{P}$. ◇

6. LEMMA. *For each* $e, f \in \mathbf{P} = K[0]$, *the Aczél's inequality holds, i.e.*

$$(e, f) \geq \|e\|_t \, \|f\|_t ,$$

with equality if and only if $e = 0$, $f = 0$, or $f = \lambda e$ *for some* $\lambda \in \mathbb{R}_+$.

Proof. Let us note $e = (t, x)$, $f = (s, y)$, and $T(\lambda) = (e + \lambda f, e + \lambda f)$, where λ is arbitrary in $\mathbb{R}$. The trinomial T must have real roots because $T(0) = (e, e) \geq 0$ and $T(\lambda_0) = -\|x + \lambda_0 y\|^2 \leq 0$, where $\lambda_0 = -t / s$ for $s > 0$ (if $s = 0$ then $f = 0$, and the result is obvious). Consequently, $\Delta \equiv (e, f)^2 - (e, e)(f, f) \geq 0$. It remains to remark that $(e, f) \geq 0$ because $t > \|x\|$ and $s > \|y\|$ imply $ts > \|x\| \, \|y\| \geq \langle x, y \rangle$ (Cauchy – Bunjakowski – Schwartz inequality in H).

The equality $(e, f) = \|e\|_t \, \|f\|_t$ means $\Delta = 0$, hence $T(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. Since $T(\lambda_0) \leq 0$, it follows that $x + \lambda_0 y = 0$.

The cases that remain are trivial. ◇

7. PROPOSITION. *The temporal norm has the following properties:*

i) $\|e\|_t = 0$ *if and only if* $e = 0$

ii) $\|\lambda e\|_t = \lambda \, \|e\|_t$ *for all* $\lambda \in \mathbb{R}_+$ *and* $e \in \mathbf{P}$

iii) $\|e + f\|_t \geq \|e\|_t + \|f\|_t$ *for all* $e, f \in \mathbf{P}$.

Proof. The first property is a consequence of the fact that $\|\cdot\|_t$ is restricted to **P**. A direct calculus gives ii). In order to prove iii), we primarily remark that $e + f \in \mathbf{P}$ (as in the proof of Proposition 5). By adding the expression

$$(e, e) + (f, f) = \|e\|_t^2 + \|f\|_t^2$$

to the Aczél's inequality multiplied by 2, we obtain

$$(e + f, e + f) \geq (\|e\|_t + \|f\|_t)^2.$$

As a completion we mention that the equality in iii) holds if and only if e and f are collinear. ◇

8. LEMMA. *If* $\|e\|_t = \|f\|_t$, *then* $(e, f) \in \delta \cup \complement(K \cup K^{-1})$.

Proof. Supposing $e = (t, x)$, $f = (s, y)$, and $t > s > 0$, we can write the hypothesis in the form $t^2 - s^2 = \|x\|^2 - \|y\|^2$, i.e.

$$(t - s)(t + s) = (\|x\| - \|y\|)\,(\|x\| + \|y\|).$$

From $e, f \in \mathbf{P}$, which, as a rule, is understood from the moment when we speak of $\|e\|_t$ and $\|f\|_t$, it follows that $t + s > \|x\| + \|y\|$, hence

$$t - s < \|x\| - \|y\| \leq \|x - y\|.$$

The other cases are trivial. ◇

9. Remark. The properties involving orthogonality, hyperbolical angles, etc. may not be subject to further generalizations because of their direct dependence upon the inner product. Consequently, the following properties are mentioned without proof.

When the orthogonality is interpreted as simultaneity in the Minkowskian space-time, the Lorentz transformations play an important role. Repeating the reason in the case of a scalar event world, we see that the Lorentz transformations keep up the well-known vectorial form

$$\begin{cases} x' = x - \langle x, v\rangle \|v\|^{-2} v - (1 - \|v\|^2 c^{-2})^{-1/2}(t - \langle x, v\rangle \|v\|^{-2})v \\ t' = (1 - \|v\|^2 c^{-2})^{-1/2}(t - \langle x, v\rangle c^{-2}), \end{cases}$$

where the speed $v = dx/dt$ is understood in the sense of the usual topology of $(H, \langle .,. \rangle)$. If we identify the line $\Omega = \{ \lambda (1, v) : \lambda \in \mathbb{R}\}$ with an observer, then the condition $e \perp \Omega$ is equal to $t' = 0$, i.e. $\Omega^{\perp}$ is the set of all events simultaneous (with 0) relative to this observer. In addition, we have:

10. PROPOSITION. *If* $\omega \in \mathbf{P} \setminus \{0\}$, *then* $\omega^{\perp} \subset [\delta \cup \complement(K \cup K^{-1})]\,[0]$.

11. Remark. It is easy to see that the notions of orthogonality and hyperbolic angle are independent, i.e. $e \perp f$ is not characterized by some hyperbolical angle between e and f any more. The restrictions imposed to the notion of angle are justified by the possibilities of measurement. More precisely, the definition of $m(\angle(e, f))$ is based on the Aczél's inequality.

The hyperbolical angles appear as equivalence classes in the set $(\mathbf{P} \setminus \{0\})^2$. Because $m(\angle(e, f))$ does not depend on representatives, we may take vectors of the same temporal norm, and so we are led to measure hyperbolical angles by arcs of a hyperbola. As usually, the hyperbola of radius $r > 0$ in the plane Lin $\{e, f\}$ is the set $H(r; e, f) = \{ g = \lambda e + \mu f : \|g\|_t = r\}$. According to Lemma 8, we measure the length of an arc of $H(r; e, f)$ by the spatial metric. Organizing the subspace Lin $\{e, f\}$ like the Minkowskian plane, we obtain:

12. PROPOSITION. *If $e, f, g \in \mathbf{P} \setminus \{0\}$, then*

a) $m(\angle(e, f)) = L / r$, *where L is the length of the arc from the hyperbola $H(r; e, f)$ determined by the semi-straights $\{\lambda e : \lambda \in \mathbb{R}_+\}$ and $\{\mu f : \mu \in \mathbb{R}_+\}$.*

b) $m(\angle(e, f)) + m(\angle(f, g)) = m(\angle(e, g))$ *whenever* $\angle(e, f)$ *and* $\angle(f, g)$ *are adjacent*, i.e. $f = \alpha e + \beta g$ *for some* $\alpha, \beta \geq 0$.

B. EVENTS IN NORMED AND METRIC SPACES

In this part of the paper we will show that instead of the scalar product space $(H, \langle .,. \rangle)$ we may consider a real normed space $(X, \|\cdot\|)$, or a metric space (S, d), without losing the principal properties of a scalar event world. The proofs, which essentially remain the same, will be omitted.

1. Definition. Let $(X, \|\cdot\|)$ be a real normed space. We say that $E = \mathbb{R} \times X$ is the world of all events that happen in the normed space X, or briefly, a *normed event world*. The events, i.e. the elements of E, will be noted $e = (t, x)$, $f = (s, y)$, etc., where $t, s \in \mathbb{R}$ and $x, y \in X$ localize the events in time, respectively in space.

The relation between events

$$K = \{((t, x), (s, y)) \in E \times E : s - t > \|x - y\|\} \cup \delta$$

is called *causal relation*, or *causality*.

The functional $\|\cdot\|_t : K[0] \to \mathbb{R}_+$, expressed by

$$\|(t, x)\|_t = \sqrt{t^2 - \|x\|^2} \qquad (2)$$

is called *temporal norm*.

2. PROPOSITION. *K is a linear order on E*.

The proof formally repeats that of the Proposition A5.

3. PROPOSITION. *The temporal norm has the same properties as in the case of scalar event spaces* (*Proposition A7*).

Proof. The properties i) and ii) are immediate. In order to prove the super-additivity, i.e. $\|e+f\|_t \geq \|e\|_t + \|f\|_t$, we start from the following obvious inequality:

$$t^2 \|y\|^2 + s^2 \|x\|^2 \geq 2ts \|x\| \|y\| .$$

By adding here $s^2 t^2 + \|x\|^2 \|y\|^2$, we obtain

$$(t^2 - \|x\|^2)(s^2 - \|y\|^2) \leq (ts - \|x\| \|y\|)^2 .$$

Since $e = (t, x)$, $f = (s, y) \in K[0] \setminus \{0\}$, all the parentheses from above are positive, hence we may write

$$\|e\|_t \|f\|_t \leq ts - \|x\| \|y\|,$$

and farther

$$[\|e\|_t + \|f\|_t]^2 \leq (t+s)^2 - (\|x\| + \|y\|)^2 .$$

The super-additivity of $\|\cdot\|_t$ follows from the sub-additivity of the usual norm $\|\cdot\|$. ◇

4. Remark. If $\|\cdot\|$ has the property

$$\|x+y\| = \|x\| + \|y\| \Leftrightarrow y = \lambda x ,$$

then $\|\cdot\|_t$ has a similar property, namely

$$\|e+f\|_t = \|e\|_t + \|f\|_t \Leftrightarrow f = \lambda e .$$

In fact, considering $\|e\|_t + \|f\|_t = \|e+f\|_t$, from the above relations

$$[\|e\|_t + \|f\|_t]^2 \leq (t+s)^2 - (\|x\| + \|y\|)^2 \leq$$

$$\leq (t+s)^2 - \|x+y\|^2 = \|e+f\|_t^2 ,$$

we obtain $\|x+y\| = \|x\| + \|y\|$, hence $y = \lambda x$ for some $\lambda \geq 0$. A direct calculus reduces the hypothesis to the equality

$$t^2 \|y\|^2 + s^2 \|x\|^2 = 2ts \|x\| \|y\| ,$$

which gives $s = \lambda t$. Consequently, $f = \lambda e$.

Because there exist norms for which $\|x+y\| = \|x\| + \|y\|$ is possible without $y = \lambda x$, e.g. the "sup" norms, it follows that the normed event worlds are essential generalizations of the scalar event worlds.

5. PROPOSITION. *If* $\|e\|_t = \|f\|_t$, *then* $(e, f) \in \delta \cup \complement(K \cup K^{-1})$.

6. Remark. If $g = (u, z) \in \delta \cup \complement(K \cup K^{-1})$ [0], then we have $u^2 - \|z\|^2 \leq 0$, hence we may speak of a *spatial norm* of g, denoted

$$\|g\|_s = \sqrt{\|z\|^2 - u^2}\ .$$

According to the above property, this is the case of $g = e - f$, which tacitly appears in the next result.

7. THEOREM. *A necessary and sufficient condition for a normed event world* $E = \mathbb{R}\times X$ *to be a scalar event world is that for any* $e, f \in K[0]$ *for which* $\|e\|_t = \|f\|_t = r$, *to have*

$$(*) \qquad \|e+f\|_t^2 - \|e-f\|_s^2 = 4\,r^2.$$

Proof. It is easy to see that (*) holds in every scalar event world, whenever $\|e\|_t = \|f\|_t = r$. Conversely, if $x, y \in X$ and $r > 0$, we write $t = (\|x\|^2 + r^2)^{1/2}$ and $s = (\|y\|^2 + r^2)^{1/2}$, such that for the events $e = (t, x)$ and $f = (s, y)$ we have $\|e\|_t = \|f\|_t = r$. According to the Proposition B5, $\|e-f\|_s$ makes sense. If we introduce these elements in (*), then we obtain

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

so it remains to apply the theorem that characterizes the scalar product spaces between the normed spaces (e.g. see [2]). ◇

8. Remark. Since for each pair of points $x, y \in X$ and $e = (t, x) \in \mathbf{P}$ there exists an event $f = (s, y) \neq e$ such that $(e, f) \in K$, we may formulate a variant of the above theorem in terms of temporal norms exclusively. In this case, instead of (*), the following condition appears:

$$\|e+f\|_t^2 - \|f-e\|_t^2 = 2(\|e\|_t^2 + \|f\|_t^2).$$

In order to distinguish the normed event worlds between other worlds, the following so-called “imminence after common causes” property will be later useful: For every $e, f \in E$ for which $\|e\|_t = \|f\|_t = r > 0$, and every $r' \in (0, r)$, there exists $g \in \mathbf{P}$ such that $(g, e), (g, f) \in K$, and $\|g\|_t > r'$. Here we recognize in g a common cause of e and f, which, in the temporal norm, happen arbitrarily late, but before e and f.

9. PROPOSITION. *The normed event worlds do not have the property of imminence after common causes.*

Proof. We will show that for $e = (t, x)$ and $f = (t, -x)$, where $x \neq 0$, there exists $r' \in (0, r)$, where $r = \|e\|_t = \|f\|_t$, such that $\|g\|_t < r'$ whenever $g \in \mathbf{P}$ and $(g, e), (g, f) \in K$. In fact, if we express $g = (s, y)$, then from $t - s > \|y - x\|$ and $t - s > \|y + x\|$, we deduce that $s < t - \|x\|$. The asked constant is $r' = t - \|x\|$, because $0 \leq \|y\| < s < r'$, $\|g\|_t < r'$, and

$$r = (t^2 - \|x\|^2)^{1/2} = [r'(r' + 2\|x\|)]^{1/2} > r'. \qquad \diamondsuit$$

We may generalize the class of normed event worlds by renouncing linearity, as follows:

10. Definition. Let (S, d) be a metric space. We interpret the set $E = \mathbb{R} \times S$ as the world of all events which happen in S, and we call it *metric event world*.

The set

$$K = \{((t, x), (s, y)) \in E \times E : s - t > d(x, y)\} \cup \delta$$

is called *causal relation* on E, or briefly *causality*.

The functional $\sigma : K \to \mathbb{R}_+$, expressed by

$$\sigma(e, f) = \sqrt{(s-t)^2 - d^2(x, y)}, \tag{3}$$

where $e = (t, x)$ and $f = (s, y)$, is called *temporal metric* on E.

11. PROPOSITION. *The causality is an order relation on every metric event world.*

12. PROPOSITION. *The temporal metrics have the properties*:

i) $\sigma(e, f) = 0$ *if and only if* $e = f$

ii) $\sigma(e, f) + \sigma(f, g) \geq \sigma(e, g)$ *whenever* $(e, f), (f, g) \in K$.

Proof. i) is essentially based on the fact that $\sigma(e, f)$ makes sense if and only if $(e, f) \in K$.

In order to prove ii), let us note $e = (t, x)$, $f = (s, y)$, and $g = (r, z)$, so that the enounced inequality is equivalent to

$$2\,\sigma(e, f)\,\sigma(f, g) \leq d^2(x, y) + d^2(y, z) - d^2(x, z) + 2\,(r - s)\,(s - t).$$

This last inequality follows from

$$\sigma(e, f)\,\sigma(f, g) \leq (r - s)\,(s - t) - d(x, y)\,d(y, z),$$

which we may interpret as an Aczél type inequality, and from

$$-2\,d(x, y)\,d(y, z) \leq d^2(x, y) + d^2(y, z) - d^2(x, z),$$

which is an immediate consequence of the sub-additivity of the usual metric of the space (S, d). $\diamondsuit$

We will treat other properties of the metric event worlds in the next section for more general worlds of events.

C. SUPER-ADDITIVE NORMS AND METRICS

Because the uniform, the topological, and other such structures essentially have a qualitative structure in comparison with the scalar product, normed and metric ones, it is hard (perhaps impossible) to continue the generalization of the event worlds in the same sense. For example, even if (S, τ) is a topological space, and $E = \mathbb{R}\times S$ is the set of all events which may happen in S, we cannot combine the real numbers and the neighborhoods of τ in a way similar to the formulas (1), (2) or (3), in order to obtain significant structures on E. At this point naturally arise the question "*What are the specific qualitative structures of the event worlds?*" which, according to our knowledge, is still open.

On the other hand, there are important examples of sets, which are naturally endowed with structures similar to that of the event worlds, without having the form $\mathbb{R}\times S$. So we are led to consider sets whose elements are not events any longer, but their structures generalize that of an event world. For the beginning we mention some examples of this kind, which represent the objects of this section.

1. Examples. **a)** If $E = \mathbb{R}$, and K is the usual order on $\mathbb{R}$, then the corresponding cone is $\mathbf{P} = \mathbb{R}_+$. It is easy to see that the functional

$$|\cdot| : \mathbb{R}_+ \to \mathbb{R}_+,$$

expressed by $|x| = x$, verifies the conditions i), ii), and iii), formulated in the Proposition A7. Dimension arguments show that we cannot represent $\mathbb{R}$ as an event world.

b) On the set $\mathbb{D} = \mathbb{R}^2$ we define the operations:

$$(a, b) + (c, d) = (a + c, b + d) \textit{ for all } (a, b), (c, d) \in \mathbb{D},$$

$$\lambda \cdot (a, b) = (\lambda a, \lambda b) \textit{ for all } \lambda \in \mathbb{R} \textit{ and } (a, b) \in \mathbb{D},$$

$$(a, b) \times (c, d) = (ac + bd, ad + bc) \textit{ for all } (a, b), (c, d) \in \mathbb{D}.$$

Then $(\mathbb{D}, +, \cdot, \times)$ represents the algebra of so-called "double numbers", which, like $\mathbb{C}$, is another extension of $\mathbb{R}$. Using the *conjugate* $\bar{z} = (a, -b)$ of the double number $z = (a, b)$, we may define an order cone

$$\mathbf{P} = \{z \in \mathbb{D} : z\times \bar{z} > 0 \textit{ and } Re\, z > 0\} \cup \{0\}.$$

The resulting order of $\mathbb{D}$ obviously extends the usual order of $\mathbb{R}$. The function $\|\cdot\|_t : \mathbf{P} \to \mathbb{R}_+$, expressed by $\|z\|_t = (z\times \bar{z})^{1/2}$, exactly is the temporal norm of the Minkowskian plane.

Similarly, we may organize the Minkowskian space-time $\mathbb{R}\times\mathbb{R}^3$ like a 4-dimensional Clifford algebra, etc. In this way we see that even the event worlds may be introduced by pure mathematical considerations, when we agree to overlook the physical meaning of the notions.

Relative to the example of double numbers, we mention the possibility to present the same structure in a shape formally different from that of an event world. Thus, $\mathbb{B} = \mathbb{R}^2$, known as "bi-real numbers" algebra when endowed with the operations + and · like in $\mathbb{D}$, and the internal product

$$(x, y) \otimes (u, v) = (xu, yv) \textit{ for all } (x, y), (u, v) \in \mathbb{B},$$

is algebraically isomorphic with $\mathbb{D}$. The cone

$$\mathbf{Q} = \{(x, y)\in\mathbb{B} : x > 0 \textit{ and } y > 0\} \cup \{0\}$$

corresponds to the product (strict) order of $\mathbb{R}^2$, and the functional $|\cdot| : \mathbf{Q}\to\mathbb{R}_+$, expressed by $|(x, y)| = (xy)^{1/2}$, has the same properties as the norm of $\mathbb{D}$.

c) On the set $\mathscr{F}_{\mathbb{R}}(T)$ of all real functions defined on the non-void set T, we define the order by the cone

$$\mathbf{P} = \{f \in \mathscr{F}_{\mathbb{R}}(T) \colon \exists\varepsilon_f > 0 \textit{ such that } f(t) > \varepsilon_f \textit{ for all } t\in T\} \cup \{0\}.$$

The functional $]\cdot[\,: \mathbf{P}\to\mathbb{R}_+$, expressed by $]f[= \inf_{t\in T} f(t)$, verifies conditions i), ii) and iii) from Proposition A7. The condition involving $\varepsilon_f > 0$ for the positive functions, in the definition of **P**, sometimes is naturally assured, as for example in the case of $\boldsymbol{C}_{\mathbb{R}}(M)$, where M is compact. Then the cone

$$\mathbf{P} = \{f \in \boldsymbol{C}_{\mathbb{R}}(M) \colon f(t) > 0 \textit{ for all } t\in M\} \cup \{0\}$$

generates the product (strict) order on $\boldsymbol{C}_{\mathbb{R}}(M)$, and we have $]f[= 0$ if and only if $f = 0$, since $]\cdot[$ is restricted on **P**. We may similarly organize the spaces $C^k_{\mathbb{R}}([a,b])$, $k\in\mathbb{N}$.

Other particular case is $\mathbb{R}^n$, $n\in\mathbb{N}^*$, endowed with the product strict order, and the functional $]x[= \min\{x_i : i = 1,2,...,n\}$, where $x = (x_1, x_2, \ldots, x_n)$. Slightly extending $\mathbb{R}^n$, we obtain another example, namely

$$E = \{x\in\mathbb{R}^{\mathbb{Z}} : \text{card supp } x\in\mathbb{N}\},$$

with the product strict order, and the functional

$$]x[= \begin{cases} \min\{x_i : x_i \neq 0, i \in \mathbb{Z}\} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0. \end{cases}$$

Because all these "inf-type" norms have the property of imminence after common causes, they essentially differ from the temporal norms on event worlds (analyzed in Proposition B9).

d) Let $\mathscr{A}$ be the Boolean algebra of propositions. As usually, $A \Leftrightarrow B$ denotes that propositions A and B have the same truth values, and $A \Rightarrow B$ means that B is true whenever A is. It is well known that $\Leftrightarrow$ stands for equality, and $K = \{(A, B) \in \mathscr{A} \times \mathscr{A} : A \Rightarrow B\}$ is a partial order on $\mathscr{A}$. If for $(A, B) \in K$ we note the intervals by $[A, B] = \{X \in \mathscr{A} : A \Rightarrow X \Rightarrow B\}$, then the functional $\sigma : K \to \mathbb{R}_+$, of values $\sigma(A, B) = \text{card}\,[A, B] - 1$, verifies conditions i) and ii) of Proposition B 12. The proof is routine, since card $[A, B] = 2^{\,b-a}$, where a and b are the numbers of "true" values of A, respectively B.

Obviously, A is neither linear space, nor event world.

2. Definition. Let W be a real linear space, and let K be a linear order on W. We say that the functional $]\cdot[\, : K[0] \to \mathbb{R}_+$ is a *super-additive* (briefly s.a.) norm on W, if it satisfies the conditions:

i) $]e[\, = 0$ if and only if $e = 0$

ii) $]\lambda e[\, = \lambda\,]e[$ for all $\lambda \geq 0$ and $e \in K[0]$

iii) $]e + f[\, \geq\,]e[+]f[$ for all $e, f \in K[0]$.

If $]\cdot[$ verifies only conditions ii) and iii), we say that it is a *pseudo-super-additive norm* (for brevity, p.s.a. norm). We also propose the term *timer* to be used instead of s.a. norm, to remember its original meaning in a Minkowskian space-time. Consequently, the triplet $(W, K,]\cdot[)$ is called *timer world*.

3. Definition. Let K be an order relation on the non-void set W. We say that the functional $\sigma : K \to \mathbb{R}_+$ is a *super-additive metric* on W (restricted to K) if:

i) $\sigma(e, f) = 0$ if and only if $e = f$

ii) $\sigma(e, g) \geq \sigma(e, f) + \sigma(f, g)$ whenever $(e, f), (f, g) \in K$.

If, instead of i), σ verifies only the condition

i$'$) $e = f$ implies $\sigma(e, f) = 0$,

then σ is called *pseudo-super-additive* metric.

In a shorter form, we speak of an s.a. metric, respectively p.s.a. metric. When there is some interest in keeping the physical nuance, it seems adequate to use terms like *clock*, *chronometer*, *horometric*, etc. To make a choice, the triplet (W, K, σ) will be called *horometric world*.

4. Remarks. **a)** The super-additive norms and metrics must always be partially defined. In fact, in the contrary case, when a s.a. metric would be defined on the whole W^2, by permuting e, f, g in ii), Definition C3, and adding the forthcoming inequalities, we would obtain $1 \geq 2$ (!).

b) If $(W, K,]\cdot[)$ is a timer world, and $\sigma : K \to \mathbb{R}_+$ is defined by

$$\sigma(e, f) = \,]f - e[$$

then (W, K, σ) is a horometric world.

c) If (W, K, σ) is a horometric world, and for some $r > 0$ we define

$$\sigma_r(e, f) = \begin{cases} \sigma(e, f) & \text{if } \sigma(e, f) > r \\ 0 & \text{if } \sigma(e, f) \le r\,, \end{cases}$$

then σ_r is a p.s.a. metric.

d) Further restrictions of the (p.)s.a. norms, respectively (p.)s.a. metrics, to smaller orders are always practicable in order to obtain similar structures. For example, let K and L be linear orders on the real linear space W, such that $L \subset K$. If $]\cdot[\,: K[0] \to \mathbb{R}_+$ is a timer, then $]\cdot[\,|_{L[0]}$ is a timer too. Similarly, we may restrict the horometrics, etc.

Alternatively, the problem of prolongation appears to be more significant. We will treat it in the last part of the section, after a study that generalizes the Properties A8 and B5.

5. LEMMA **(clock monotony).** *If* (W, K, σ) *is a p.-horometric world, then*:

a) *from* $(e, f)\in K$, $(f, g)\in K$ *it follows* $\sigma(e, f) \le \sigma(e, g)$ *and* $\sigma(f, g) \le \sigma(e, g)$

b) $\sigma(e, f) = \inf\{\sigma(e, g): g\in K[f]\} = \inf\{\sigma(h, f): h\in K^{-1}[e]\}$.

Proof. a) σ has non-negative values and is super-additive.

b) Because in the condition $(f, g)\in K$ we may consider $f = g$, it follows that

$$\sigma(e, f) \ge \inf\{\sigma(e, g): g\in K[f]\}.$$

On the other hand, according to a), $\sigma(e, f) \le \sigma(e, g)$ for all $g\in K[f]$, hence the contrary inequality also holds.

We may similarly treat the second inequality. ◇

As a consequence, the pseudo-timers are also monotonous:

6. COROLLARY. *If* $(W, K,]\cdot[)$ *is a pseudo-timer world, then*:

a) *from* $e\in K[0]$ *and* $(e, f)\in K$ *it follows that* $]e[\;\le\;]f[$

b) $]e[\;= \inf\{\,]f[\,: (e, f)\in K\}$.

7. PROPOSITION. *Let* (W, K, σ) *be a pseudo-horometric world. Then* σ *is a horometric if and only if it is strictly monotonous, i.e.*

a) $(e, f)\in K$ *and* $(f, g)\in K \setminus \delta$ *imply* $\sigma(e, f) < \sigma(e, g)$, *and*

b) $(e, f)\in K \setminus \delta$ *and* $(f, g)\in K$ *imply* $\sigma(f, g) < \sigma(e, g)$.

Proof. According to the above lemma, we have to show that if σ is a horometric, then the equality $\sigma(e, f) = \sigma(e, g)$ in a) is impossible. In fact, if this equality were true, then from $\sigma(e, g)\ge\sigma(e, f) + \sigma(f, g)$ we would obtain $\sigma(f, g) = 0$, hence $f = g$. This is contrary to the hypothesis $f, g)\in K \setminus \delta$.

Similarly we discuss the case b).

Conversely, if we take $f = g$ in b), then from the hypothesis $e \ne f$ we obtain $\sigma(e, f) > \sigma(f, f) = 0$. ◇

A similar property holds for pseudo-timers. More precisely, we always have $]0[\ = 0$, but property "$]e[\ = 0$ *only if* $e = 0$" holds exactly for monotonous pseudo-timers.

8. PROPOSITION. *Let* (W, K, σ) *be a horometric world. If* $f \neq g$ *and*

$$\sigma(e, f) = \sigma(e, g),$$

then $(f, g) \in \complement(K \cup K^{-1})$.

Proof. In the contrary case we would have, for example, $(f, g) \in K \setminus \delta$. Using the strict monotony of σ from the above proposition, we would obtain

$$\sigma(e, f) < \sigma(e, g),$$

which contradicts the hypothesis. ◇

9. COROLLARY. *If* $(W, K,]\cdot[)$ *is a timer world, then* $]e[\ =\ ]f[$ and $e \neq f$ *imply* $(e, f) \notin K \cup K^{-1}$.

By the following properties we analyze two senses in which we may understand the process of prolongation: the prolongation to the opposite order, and the prolongation to wider orders.

10. PROPOSITION. *If* (W, K, σ) *is a* (*pseudo*-) *horometric world, then the triplet* $(W, K^{-1}, \sigma^{\leftarrow})$, *where the functional* $\sigma^{\leftarrow} : K^{-1} \to \mathbb{R}_+$ *takes the values* $\sigma^{\leftarrow}(e, f) = \sigma(f, e)$, *is a* (*pseudo*-) *horometric world too.*

The proof is direct. We say that σ and $\sigma^{\leftarrow}$ are dual each-other. Similarly, to each (p.-) timer $]\cdot[\ : K[0] \to \mathbb{R}_+$ we may attach its dual $]\cdot[^{\leftarrow} : K^{-1}[0] \to \mathbb{R}_+$, defined by $]e[^{\leftarrow}\ =\ ]-e[$.

11. PROPOSITION **(The complete rule of the triangle).** *If* (W, K, σ) *and* $(W, K^{-1}, \sigma^{\leftarrow})$ *are dual horometric worlds, then* $\sigma^{\leftrightarrow} : K \cup K^{-1} \to \mathbb{R}_+$, *expressed by*

$$\sigma^{\leftrightarrow}(e, f) = \begin{cases} \sigma(e, f) & \textit{if } (e, f) \in K \\ \sigma^{\leftarrow}(e, f) & \textit{if } (e, f) \in K^{-1} \end{cases},$$

has the following properties:

1. $\sigma^{\leftrightarrow}(e, f) = 0$ *if and only if* $e = f$
2. $\sigma^{\leftrightarrow}(e, f) = \sigma^{\leftrightarrow}(f, e)$ *for every* $(e, f) \in K \cup K^{-1}$
3. *either*

$$\sigma^{\leftrightarrow}(e, f) + \sigma^{\leftrightarrow}(f, g) \leq \sigma^{\leftrightarrow}(e, g),$$

when $(e, f), (f, g) \in K$ *or* $(e, f), (f, g) \in K^{-1}$, *or* (*exclusively*)

$$|\sigma^{\leftrightarrow}(e, f) - \sigma^{\leftrightarrow}(f, g)| \geq \sigma^{\leftrightarrow}(e, g),$$

when $(e, g), (g, f) \in K$ *or* $(e, g), (g, f) \in K^{-1}$.

The proof is direct. The functional $\sigma^{\leftrightarrow}$ is the *symmetric prolongation* of σ and $\sigma^{\leftarrow}$, and therefore we call it *symmetric* s.a. metric, horometric, etc.

Rewording the result from above, we may say that the triangle rule for the symmetric super-additive metrics exactly is the logical negation of the corresponding rule for usual (sub-additive) metrics. Similar considerations are valid for timers.

To study the prolongation to wider orders, the techniques of deriving binary relations play an important role. With this purpose, we follow the way inspired from [3].

12. Definition. Let K be a binary relation on the set W. We say that

$$\dot{K} = K \setminus \delta = \{(e, f) \in K : e \neq f\}$$

is the *strict* sub-relation of K. The relation

$$\overline{K} = \{(e, f) \in W^2 : K[e] \supseteq K[f] \text{ and } K^{-1}[e] \subseteq K^{-1}[f]\}$$

is called (*relational*) *closure* of K. The relation

$$\Sigma(K) = \overline{\dot{K}} \setminus \dot{K}$$

is called *signal relation* generated by K, or simply K-*signal*. We say that K is *signal cohesive* if it satisfies the condition

$$\dot{K}[e] \cap \Sigma(K)[f] \neq \varnothing \text{ whenever } (e, f) \in \overline{\dot{K}} \cup (\overline{\dot{K}})^{-1}.$$

We say that K *distinguishes* the elements of W if for every $e, f \in W$ for which $K[e] = K[f]$ and $K^{-1}[e] = K^{-1}[f]$ it follows that $e = f$.

13. LEMMA. *For every binary relation K on W we have*:

a) $\overline{K}$ *is a preorder*

b) $K = \overline{K}$ *if and only if K is a preorder*

c) $\overline{\overline{K}} = \overline{K}$.

Proof. **a)** $\delta \subseteq \overline{K}$ is obvious. The transitivity of $\overline{K}$ reduces to the transitivity of set inclusion.

b) $K = \overline{K}$ and "$\overline{K}$ *is preorder*" imply "*K is preorder*". Conversely, $\delta \subseteq K$ implies $K \supseteq \overline{K}$, while $K \circ K \subseteq K$ implies $K \subseteq \overline{K}$.

c) is a direct consequence of a) and b). ◇

14. LEMMA. *Let K be a binary relation on W. Then*:

a) $\overline{K} \circ K \subseteq K$ *and* $K \circ \overline{K} \subseteq K$

b) *If K is an order, then* $\overline{\dot{K}} = \dot{K} \cup \Sigma(K)$, *where* $\dot{K} \cap \Sigma(K) = \varnothing$

c) *If K is an order, and* $(e, f), (f, g) \in \overline{\dot{K}}$ *such that* $(e, g) \in \Sigma(K)$, *then*

$$(e, f), (f, g) \in \Sigma(K).$$

Proof. **a)** If $(e, f) \in \overline{K}$ and $(f, g) \in K$, we deduce that $g \in K[f] \subseteq K[e]$, hence $(e, g) \in K$.

b) If K is an order, then $\dot{K}$ is transitive, hence $\dot{K} \subseteq \overline{\dot{K}}$. It remains to take into account the definition of $\Sigma(K)$.

c) In the contrary case at least one of (e, f) or (f, g) does not belong to $\Sigma(K)$. If, for example $(e, f) \notin \Sigma(K)$, then from b) it follows that $(e, f) \in \dot{K}$. Then, applying a), we obtain $(e, g) \in \dot{K}$, which contradicts the hypothesis. ◇

15. LEMMA. *Let K be a binary relation on W. Then*:

a) *If K is anti-symmetric, then it distinguishes the elements of W*

b) *K distinguishes the elements of W if and only if $\overline{K}$ is an order.*

Proof. **a)** Let us suppose that $K \cap K^{-1} = \delta$, and let the elements $e, f \in W$ be such that $K[e] = K[f]$ and $K^{-1}[e] = K^{-1}[f]$. It follows that

$$K[e] \cap K^{-1}[e] = (K \cap K^{-1})[e] = \{e\}.$$

Similarly, $K[f] \cap K^{-1}[f] = \{f\}$, and consequently, $e = f$.

b) Let K be a distinguishing binary relation on W. According to Lemma C13, $\overline{K}$ is a preorder, hence it remains to show that $\overline{K}$ is anti-symmetric. In fact, if $(e, f) \in \overline{K}$ and $(f, e) \in \overline{K}$ hold for $e, f \in W$, it follows that $K[e] = K[f]$ and $K^{-1}[e] = K^{-1}[f]$, hence $e = f$.

Conversely, let us suppose that $\overline{K}$ is an order, hence it is anti-symmetric. Now, if $K[e] = K[f]$ and $K^{-1}[e] = K^{-1}[f]$, this means that both $(e, f) \in \overline{K}$ and $(f, e) \in \overline{K}$, hence $e = f$. ◇

16. Remarks. **a)** The properties of K and $\dot{K}$ to distinguish the elements of W are independent. For example, in $W = \mathbb{R}^2$, the relation

$$K = \{((x, y), (u, v)) \in W^2 : x < y\} \cup \delta$$

is distinguishing, while $\dot{K}$ is not. On the other hand, the relation

$$T = \{((x, y), (u, v)) \in W^2 : x \leq u\},$$

on the same set, is not distinguishing, but $\dot{T}$ is.

b) The causality of any Minkowskian space-time is signal cohesive. As we will later see (Corollary C18), this happens in general metric event worlds.

We obtain simple examples of signal non-cohesive relations whenever the corresponding signal relation reduces to the identity. Thus, if $E = \mathbb{R} \times S$, where (S, d) is a metric space and $T = \{((t, x), (s, y)) \in E^2 : s - t \geq d(x, y)\}$, then we have $\overline{\dot{T}} = T$, hence $\Sigma(T) = \delta$.

Before the analysis of the prolongation for general horometrics, it is useful to return oneself to the case of the temporal metrics in metric event worlds (Definition B10). It is easy to see that, in this particular case, K distinguishes the events of the corresponding world, as well as $\dot{K}$ and $\overline{\dot{K}}$ (see also the Example C 23 a).

17. THEOREM. *Let* (S, d) *be a metric space,* $E = \mathbb{R} \times S$, *and let* (E, K, σ) *be the forthcoming metric event world. If we write*

$$\tilde{K} = \{(e, f) \in E^2 : s - t \geq d(x, y)\},$$

where $e = (t, x)$ *and* $f = (s, y)$, *then we have* $\tilde{K} = \overline{\dot{K}}$.

Proof. If we suppose $(e, f) \in \overline{\dot{K}}$, then we have $\dot{K}[e] \supseteq \dot{K}[f]$, i.e. $(f, g) \in \dot{K}$ implies $(e, g) \in \dot{K}$. In particular, if we take $g = (u, z)$ with $u > s$ and $z = y$, then we obtain $u - t > d(x, z)$. In other words, we can say that $u > s$ implies $u > t + d(x, y)$, hence $s \geq t + d(x, y)$. Consequently, $(e, f) \in \tilde{K}$, which proves the inclusion $\overline{\dot{K}} \subseteq \tilde{K}$.

Conversely, if $(e, f) \in \tilde{K}$, then we have $s - t \geq d(x, y)$. In order to prove that $\dot{K}[e] \supseteq \dot{K}[f]$, let us take $g \in \dot{K}[f]$, hence $u - s > d(y, z)$. Adding these inequalities, we obtain

$$u - t > d(x, y) + d(y, z) \geq d(x, z),$$

i.e. $(e, g) \in K$. Similarly, we can prove that $(\dot{K})^{-1}[e] \subseteq (\dot{K})^{-1}[f]$, hence finally, $\tilde{K} \subseteq \overline{\dot{K}}$. ◇

18. COROLLARY. *In every metric event world, the relation of causality is signal cohesive.*

Proof. If we note $e = (t, x)$ and $f = (s, y)$, then, according to the theorem from above, we may express the condition $(e, f) \notin \overline{\dot{K}} \cup (\overline{\dot{K}})^{-1}$ by the double inequality $-d(x, y) < t - s < d(x, y)$. In order to show that K is signal cohesive it is sufficient to find an event $g = (u, x) \in \dot{K}[f] \cap \Sigma(K)[f]$. According to the same theorem, this condition reduces to $u > t$ and $u - s = d(x, y)$, which are obviously fulfilled by $u = s + d(x, y)$. ◇

19. Remarks. **a)** Because $K \subset \tilde{K}$, the above theorem represents a prolongation property of the temporal metric $\sigma : K \to \mathbb{R}_+$ in the metric event world (E, K, σ). In fact, the functional $\tilde{\sigma} : \tilde{K} \to \mathbb{R}_+$, expressed by the same formula, namely

$$\tilde{\sigma}(e, f) = \sqrt{(s - t)^2 - d^2(x, y)},$$

is a temporal pseudo-metric for which $\tilde{\sigma}\big|_K = \sigma$.

b) From the above theorem, we may easily deduce a characterization of the null set of a temporal metric, namely

$$\{(e, f) \in E^2 : \tilde{\sigma}(e, f) = 0\} = \overline{\dot{K}} \setminus \dot{K},$$

i.e. $\overline{\dot{K}} \setminus \dot{K} = \ker \tilde{\sigma}$. In terms of Definition C12, this means $\ker \tilde{\sigma} = \Sigma(K)$.

c) Similar results are valid for "inf-type" horometrics. For example, if we take $W = \boldsymbol{C}([a, b])$, and we define K and σ as in C1c, then

$$\overline{\dot{K}} = \{(f, g)\in W^2 : f(t) \le g(t) \textit{ for all } t \in [a, b]\}$$

In addition, the functional $\tilde{\sigma} : K \to \mathbb{R}_+$, expressed by

$$\tilde{\sigma}\,(f, g) = \inf\{(g - f)(t) : t \in [a, b]\}$$

is a *p*-horometric for which $\ker \tilde{\sigma} = \Sigma(K)$.

Now we introduce another condition that occurs in the problem of prolongation. If compared with clock monotony, we formally obtain this property by replacing *K* with $\dot{K}$ in the formula b) of the Lemma C5.

20. Definition. Let (W, K, σ) be a (p-) horometric world. We say that σ is *smooth* if it verifies the following conditions:

(s_1) $\dot{K}\,[e] \ne \varnothing$ and $(\dot{K})^{-1}[e] \ne \varnothing$ for every $e \in W$

(s_2) $\sigma(e, f) = \inf\{\sigma(e, g) : g \in \dot{K}\,[f]\} = \inf\{\sigma(h, f) : h \in (\dot{K})^{-1}[e]\}$
for every $(e, f) \in K$.

21. Examples. **a)** The smoothness of σ obviously depends on its domain *K*. For example, if $K[e] = \{e\}$ for some $e \in W$, or more gravely $K = \delta$, condition (s_1) is not satisfied, so that all p-horometrics defined on *K* are not smooth.

b) Let *K* be an order relation on *W* such that (s_1) holds. A very simple (even trivial) example of smooth p-horometric is the null one, defined by $\theta(e, g) = 0$ for all $(e, f) \in K$. Obviously, θ cannot be considered a temporal metric.

c) In every metric event world, the temporal metric is smooth. In fact, for each $e = (t, x)$ we may take $f = (s, x)$ with $s > t$, so that $(e, f) \in \dot{K}$, i.e. we have $\dot{K}\,[e] \ne \varnothing$. Similarly, $(\dot{K})^{-1}[e] \ne \varnothing$.

The equalities of the condition (s_2) follow as consequences of some contrary inequalities. Thus, using the clock monotony and the fact that $\dot{K} \subset K$, we obtain $\sigma(e, f) \le \inf\{\sigma(e, g) : g \in \dot{K}[f]\}$.

Conversely, let $e = (t, x)$, $f = (s, x)$ and $g = (r, y)$ be such that $(e, f) \in K$ and $r \in (s, s + 1)$. Obviously, $g \in \dot{K}[f]$. We shall show that for each $\varepsilon > 0$ there exists *g* of this form, such that

$$\sigma(e, g) \le \sigma(e, f) + \varepsilon.$$

In explicit form, this inequality gives

$$(r - s)(r + s - 2t) \le \varepsilon^2 + 2\varepsilon \cdot \sigma(e, f),$$

which is weaker than the inequality

$$(r - s)(2s + 1 - 2t) \le \varepsilon^2 .$$

Because $2s + 1 - 2t$ is a positive constant, fixed together with *e* and *f*, the last inequality is assured if

$$r - s \le \varepsilon^2 (2s + 1 - 2t)^{-1} .$$

Consequently,

$$\inf\{\sigma(e, g) : g \in \dot{K}\,[f]\} \le \sigma(e, f) + \varepsilon,$$

and it remains to remark that ε here is arbitrary.

d) Let (W, K, σ) be a horometric world in which K satisfies (s_1). We choose a pair $(e_0, f_0)\in K$ such that $\sigma(e_0, f_0) = r > 0$, and we define (as in Remark C4c)

$$\sigma_r(e, f) = \begin{cases} \sigma(e, f) & if\ \sigma(e, f) > r \\ 0 & if\ \sigma(e, f) \le r\ . \end{cases}$$

Then σ_r is not smooth, even if σ is. In fact, we have $\sigma_r(e_0, f_0) = 0$, while

$$\inf\{\sigma_r(e_0, g) : (e_0, f_0)\in \dot{K}\} \ge r,$$

since σ is strictly monotonous (Proposition C7), i.e. $g\in \dot{K}[f_0]$ implies

$$\sigma_r(e_0, g) = \sigma(e_0, g) > r = \sigma(e_0, f_0).$$

e) The restrictions to smaller orders do not generally preserve smoothness. Thus, let K be an (s_1)-order on W, and let $\sigma : K\to\mathbb{R}_+$ be a non-null pseudo-horometric. If $r > 0$ is such that there exists a pair $(e_0, f_0)\in K$ for which we have $\sigma(e_0, f_0) = r$, then we may construct the relation

$$L = \{(e, f)\in K : \sigma(e, f) \ge r\} \cup\delta.$$

It is easy to verify that L is an order, $L \subseteq K$, and $\rho = \sigma\big|_L$ is a s.a. metric. Furthermore, ρ is not smooth because either L does not verify (s_1), or ρ does not verify (s_2). In fact, if $e\in(\dot{L})^{-1}[e_0]$ and $\rho(e, e_0) = m$, then we obtain

$$\rho(e, f_0) \ge \rho(e, e_0) + \rho(e_0, f_0) = m + r.$$

Because for any other $f\in \dot{L}[e_0]$ we also have $\rho(e, f) \ge m + r$, it follows that

$$\inf\{\rho(e, f) : f\in \dot{L}[e_0]\} \ge m + r > m.$$

In order to enounce the main prolongation theorem in a shorter form, we introduce the following terminology and notation:

22. Definition. We say that the (p-) horometric world (W, K, σ) is *normal* if it satisfies the conditions:

i) $\dot{K}$ distinguishes the elements of W

ii) K is signal cohesive

iii) σ is smooth.

The functional $\overline{\sigma} : \overline{\dot{K}} \to\mathbb{R}_+$, expressed by

$$\overline{\sigma}(e, f) = \begin{cases} \sigma(e, f) & if\ (e, f)\in \dot{K} \\ 0 & if\ (e, f)\in \Sigma(K)\ , \end{cases}$$

is called *standard* prolongation of σ. The fact that we may consider $\overline{\sigma}$ as a "prolongation" is based on Lemma C14b.

23. Examples. **a)** Every metric event world is normal.

As we already saw, the condition ii) follows from the Corollary C18, and iii) is proved in Example C21c, hence it remains to prove i). If we write $e = (t, x)$ and $f = (s, y)$, then condition $\dot{K}[e] = \dot{K}[f]$ means that for every $g = (u, z)$ we have $u - t > d(x, z)$ if and only if $u - s > d(y, z)$. In particular, for $z = x$, it follows that $u > t$ implies $u > s + d(x, y)$, hence $t - s \geq d(x, y)$. Similarly, taking $z = y$, from $u > s$ it follows that $u > t + d(x, y)$, hence $s \geq t + d(x, y)$. Consequently, $s = t$ and $x = y$, i.e. $e = f$.

b) Let $W = \mathscr{F}_{\mathbb{R}}(T)$, K and σ be defined as in Example C1c. Then the forthcoming horometric world is normal.

In order to prove that

$$\dot{K} = \{(f, g) \in W^2 : \exists\ \varepsilon > 0 \textit{ such that } f(t) + \varepsilon < g(t) \textit{ for all } t \in T\}$$

distinguishes the elements of W, we show that $f \neq g$ implies $\dot{K}[e] \neq \dot{K}[f]$. In fact, if $f \neq g$, then there is at least one $t_0 \in T$, where we have $f(t_0) \neq g(t_0)$, say $f(t_0) < g(t_0)$. If we note $h = [g(t_0) - f(t_0)] / 2$, then we have $f + h \in \dot{K}[f]$, while $f + h \notin \dot{K}[g]$.

In order to prove that K is signal cohesive, we mention that

$$\overline{\dot{K}} = \{(f, g) \in W^2 : f(t) \leq g(t) \textit{ for all } t \in T\},$$

hence $(f, g) \notin \overline{\dot{K}} \cup (\overline{\dot{K}})^{-1}$ means that there exist $t_1, t_2 \in T$ such that some opposite inequalities hold, say $f(t_1) < g(t_1)$ and $f(t_2) > g(t_2)$. Then, for the function $h : T \to \mathbb{R}$, of values

$$h(t) = \max_{t \in T} \{ f(t) + \varepsilon, g(t)\},$$

where $\varepsilon = g(t_1) - f(t_1)$, we obviously have $(f, h) \in \dot{K}$ and $(g, h) \in \overline{\dot{K}}$. Because $h(t_1) = g(t_1)$, it follows that $(g, h) \in \Sigma(K)$.

Condition (s_1) of Definition C20 is obviously satisfied. In order to prove (s_2), we primarily remark that

$$\sigma(e, f) \leq \inf \{\sigma(e, g) : g \in \dot{K}[f]\},$$

since σ is super-additive.

On the other hand, using the functions

$$g_n = \frac{1}{n} + f \in \dot{K}[f],\ n \in \mathbb{N}^*,$$

we obtain

$$\sigma(e, g_n) = \sigma(e, f) + \frac{1}{n},$$

which leads to the contrary inequality.

c) We may obtain examples of non-normal horometric worlds if we start with orders as in Remark C16, or non-smooth horometrics σ, like in Examples C21d, e, etc.

d) If (E, K, σ) is a metric event world, then the p.s.a. metric $\overline{\sigma}$, defined in Remark C19a, is the standard prolongation of σ.

In the case of $W = \mathscr{F}_{\mathbb{R}}(T)$, the p-horometric $\overline{\sigma} : \overline{K} \to \mathbb{R}_+$, expressed by

$$\overline{\sigma}(f, g) = \inf \{(g - f)(t) : t \in T\},$$

is the standard prolongation of σ.

Now, we may discuss the main prolongation theorem.

24. THEOREM. *If* (W, K, σ) *is a normal horometric world, and* $\bar{\sigma}$ *is the standard prolongation of* σ, *then*:

a) $\bar{\sigma}$ *is a pseudo-horometric*

b) $\bar{\sigma}$ *has no proper extension*,

i.e. *there is no pseudo-horometric* $\sigma^{+}: K^{+} \to \mathbb{R}_{+}$, *such that* $K^{+} \supset \overline{\dot{K}}$ *and*

$$\sigma^{+}\Big|_{\overline{\dot{K}}} = \sigma.$$

Proof. **a)** According to Lemma C15b, $\overline{\dot{K}}$ is an order, and because $\delta \subseteq \Sigma(K)$, $\bar{\sigma}$ verifies condition i′) from C3. In order to prove ii) from the same definition, let us consider $(e, f), (f, g) \in \overline{\dot{K}}$. Then $(e, g) \in \overline{\dot{K}}$ according to Lemma C13, but, by virtue of Lemma C14b, two different situations are possible, namely either $(e, g) \in \Sigma(K)$, or $(e, g) \in \dot{K}$.

In the first case, applying Lemma C14c, we obtain that both $(e, f) \in \Sigma(K)$ and $(f, g) \in \Sigma(K)$, hence the condition of super-additivity is trivially verified because all the values of $\bar{\sigma}$ on (e, f), (f, g) and (e, g) are null.

In the second case, we may distinguish two sub-cases: either both (e, f) and (f, g) are in $\dot{K}$, or only one of them is in $\dot{K}$, and the other is in $\Sigma(K)$. In the first sub-case it is also easy to verify the super-additivity, because $\bar{\sigma}$ has the same values as σ. Finally, let us consider $(e, f) \in \dot{K}$ and $(f, g) \in \Sigma(K)$, the remaining situation, when $(e, f) \in \Sigma(K)$ and $(f, g) \in \dot{K}$, being similar. By virtue of Lemma C14a, we have $(e, g) \in \dot{K}$, and because $\bar{\sigma}(f, g) = 0$, it remains to prove that $\sigma(e, f) \leq \sigma(e, g)$. Using property (s_1) concerning the smooth horometric σ, we may consider elements $h \in \dot{K}[g]$. For every such element we have $(f, h) \in K$ and $\sigma(e, f) \leq \sigma(e, h)$. Being smooth, σ also verifies (s_2), hence

$$\sigma(e, f) \leq \inf\{\sigma(e, h) : h \in \dot{K}[g]\} = \sigma(e, g).$$

b) We will show that the contrary case is absurd. Let us suppose that the p-horometric $\sigma^{+}: K^{+} \to \mathbb{R}_{+}$ is a strict prolongation of $\bar{\sigma}$, i.e. besides the pairs of $\dot{K}$, σ^{+} is defined on at least one additional pair, say $(e, f) \in K^{+} \setminus \overline{\dot{K}}$. We claim that for this pair we have

$$(e, f) \notin \overline{\dot{K}} \cup (\overline{\dot{K}})^{-1}.$$

In fact, if we suppose the contrary, we obtain $(e, f) \in (K^{+})^{-1}$, hence on account of the anti-symmetry of K^{+}, we must have $e = f$, which contradicts the supposition $(e, f) \notin \overline{\dot{K}}$. Then, under the hypothesis that K is signal cohesive, there exists an element

$$g \in \dot{K}[f] \cap \Sigma(K)[e].$$

For the triangle of vertices e, f, g we have $(e, f) \in K^+$, $(f, g) \in \dot{K} \subset K^+$, and $(e, g) \in \Sigma(K) \subset K^+$. But in this situation we deduce that $\sigma^+(e, g) = 0$, and $\sigma^+(f, g) = \sigma(f, g) > 0$, hence now matter how $\sigma^+(e, f)$ is defined, the property of super-additivity cannot be satisfied. ◇

25. Remarks. **a)** We have used the fact that K is signal cohesive only in the proof of part b) of the above theorem.

b) According to the above theorem, the standard prolongations discussed in Example C23d, allow no further prolongations. In fact, we have $\overline{[\overline{\dot{K}}]} = \overline{\dot{K}}$, which also explains why repeating the standard prolongation is inefficient.

c) The above theorem concerns a particular type of prolongation, namely the standard one, but generally, this problem admits many other formulations. For example, if (W, K, σ) is a horometric world and $K_0 \subset K$, then we may easily see that $(W, K_0, \sigma|_{K_0})$ also is a horometric world. One may ask how to reconstruct σ from $\sigma|_{K_0}$, but obviously, this is not a standard prolongation.

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