GENERALIZING THE MINKOWSKIAN SPACE-TIME (II)

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We continue the process of generalizing the event worlds discussed in [1], and we study the structures that correspond to uniformities and topologies. We propose these structures to be a mathematical base for the theory of discreteness, as an alternative to the theory of continuity. The discrete functions represent the morphisms of these structures.

A. UNIFORM HORISTOLOGIES

As we already remarked in [1] (introductory considerations to part C), in the process of generalization of the Minkowskian space-time we are faced with the absence of some qualitative structures that should allow us to study those properties which are dual to topological ones. Hereby we propose a variant to answer this question by constructing the “horistological structures” as generalization of the timer and horometric worlds, previously considered in [1]. For the beginning, we will justify the interest for other than topological structures in the study of super-additivity, and formulate the conditions, which we naturally impose to these new structures.

1. What does not suit topologies?

a) The topology is the study of those properties, which are invariant under continuous transformations, hence it represents the mathematical framework of the continuous vision on the universe. The great number of physical problems that need discrete mathematical models is an argument in favor of non-topological structures. Most frequently, the solutions of such problems are limited to the considered aspects, without intention to build a general theory of discreteness. The exact contrary of “continuous” is “discontinuous”, and both make sense in the same kind of structures (namely topological), hence a general theory of discreteness is practically necessary as an alternative to the topology. In other words, we cannot reduce the diversity of the practice to the topological dichotomy continuous – discontinuous.

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b) The topological structures are not concordant with the super-additive metrics, i.e. such metrics do not directly generate topologies. Of course, many studies of the spaces that are naturally endowed with super-additive metrics make use of topological structures, but they were always obtained by other considerations. For example, in the indefinite inner product spaces one uses the fundamental decomposition (when possible), the semi-norms $p_x = |(x, .)|$, etc. (see [2]).

c) Imposing topologies to the Minkowskian space-time leads to various difficulties, and consequently objections, as follows:

(*i*) The topological concepts (e.g. neighborhood, adherence, etc.) are not physically significant. The continuous transformations generally violate the specific properties of the events (like causal relation, signal relation, etc.).

(*ii*) The usual assumptions concerning the continuous nature of the space-time are not necessary for Lorentz invariance. Similarly, the so much desired Euclidean topologies on the temporal, and respectively spatial subspaces, is not necessarily derived from a topological structure on the whole space-time; more naturally we obtain such structures on the mentioned subspaces by restricting the indefinite inner product, or even the indefinite metric.

(*iii*) Let us attempt to construct a topology on the physical space-time: Starting from the causal order, using the finite dimension, the natural decomposition, etc., the most recommendable topology that we obtain is the Euclidean one. It is well known that this topology is locally isotropic, while the space-time is not. Consequently, its group of homeomorphisms has no physical significance.

(*iv*) Because constructing topology on space-time seems to be not successful, let us accept that such a topology exists a priori. The conclusion will be that even the very specialized ones cannot fulfill all the requirements that naturally are to be imposed. For example, one of the most ingenious topology of space-time is that of Zeeman (see [5]). In fact, his “fine” topology is not locally isotropic, and we can deduce the light cone through any event from this topology; the group of all homeomorphisms of the fine topology is generated by the inhomogeneous Lorentz group and dilations; it induces Euclidean topologies on temporal, respectively on spatial subspaces, etc. In spite of these remarkable qualities, the fine topology is far from being a complete solution. Thus, apart from the fact that it is described but not effectively constructed, Zeeman’s topology looks technically complicated; it is not generated by the intrinsic indefinite inner product; it is not countable, hence it may not be subject to measurements (roughly speaking , it is almost of no use for practical purposes); it refers to the single and very particular case of the 4-dimensional event world, etc.

2. Natural requirements

a) The primary condition that we naturally shall impose to the structures of discreteness concerns their generality, which must be comparable with that of continuity. The general theory of discreteness (here called *horistology* from the Greek horistos (i.e. χωριστοσ) = *separate*) should be neither a simple negation of some topological properties, nor a particular study of some exotic topological concepts. From our point of view, the structures of discreteness deserve the same emphasis as the topological ones, hence they should be dual each other at all of the levels of generality.

b) On the simplest models ($\mathbb{R}$, totally ordered sets, etc.), the topological and the horistological structures must be reducible each to other. We expect the same correspondence to hold for real functions, real sequences, etc.

c) The horistology must generalize the theory of horometric worlds (e.g. in [1]), as well as the topology generalizes the classical (i.e. sub-additive) metric spaces. In particular, the event worlds must represent a field of interpretation and application.

d) The horistological structures must form a mathematical category. Their morphisms must be physically significant and, in the case of the Minkowskian space-time, they must contain the Lorentz transformations.

The horistological structures introduced by the present paper pertinently answer all these conditions. Of course, a detailed review of the previous attempts to discrete physical, mathematical, or even philosophical theories, as well as a comparative (eventually historical) analysis of the other possible variants, remain very useful; nevertheless we will develop the theory without more collateral explanation.

3. Definition. We say that the non-void family $\mathscr{H} \subset \mathscr{P}(W^2)$ is a *uniform horistology* (briefly u.h.) on the world *W*, if it satisfies the conditions:

[uh_1] $\pi \cap \delta = \varnothing$ for all $\pi \in \mathscr{H}$, where δ is the relation of identity;

[uh_2] $\pi \in \mathscr{H}$ and $\lambda \subseteq \pi$ imply $\lambda \in \mathscr{H}$;

[uh_3] If $\lambda, \pi \in \mathscr{H}$, then $\lambda \cup \pi \in \mathscr{H}$;

[uh_4] For each $\pi \in \mathscr{H}$ there exists $\lambda \in \mathscr{H}$ such that for all $\omega \in \mathscr{H}$ we have

$$\lambda \supseteq \pi \circ \omega \text{ and } \lambda \supseteq \omega \circ \pi.$$

The elements of $\mathscr{H}$ will be called *prospects*, and the pair $(W, \mathscr{H})$ will be named *uniform horistological world*.

4. Remarks. a) Definition 3 introduces the uniform horistology by a way similar to that of the uniform topology (see [3], etc.). The conditions [uh_2] and [uh_3] show that $\mathscr{H}$ is an ideal in the lattice $\mathscr{P}(W^2)$ (for the classical lattice theory, ideals, bases, etc., see for example [4]).

b) Instead of $\mathscr{H}$ we may define the u.h. by an ideal base $\mathscr{B}$ that satisfies [uh_1] and [uh_4]. More precisely, the conditions [uh_2] and [uh_3] are replaced by
[uh_b] If $\lambda, \pi \in \mathscr{B}$, then there exists $\omega \in \mathscr{B}$, such that $\omega \supseteq \lambda \cup \pi$.

In this case, $\mathscr{H}$ is the ideal generated by $\mathscr{B}$, i.e.

$$\mathscr{H} = \{ \lambda \subset W^2 : \exists\ \pi \in \mathscr{B} \textit{ such that } \lambda \subseteq \pi \}.$$

c) It is well known that if $\mathscr{F} \subseteq \mathscr{P}(S)$ is a filter, then

$$\mathscr{I}(\mathscr{F}) = \{ X \subseteq S : \complement X \in \mathscr{F} \}$$

is an ideal, and vice-versa. This correspondence may not generally be extended to uniform horistologies and uniform topologies, because we cannot correlate condition [uh_4] and its dual in topological uniformities, excepting some very particular cases (as in the example 5c below). This fact naturally generalizes the similar relation between the sub- and super-additive metrics.

5. Examples. a) If K is an order relation on W, then the principal ideal

$$\mathscr{I}(K) = \mathscr{P}(\dot{K})$$

is a uniform horistology on W. Another extreme is $\mathscr{H}_0 = \{\varnothing\}$.

b) If (W, K, σ) is a horometric world, then we define the hyperbolical prospects of radius $r > 0$ as $\pi_r = \{(x, y) \in K : \sigma(x, y) > r\}$. It is easy to verify that $\mathscr{B} = \{ \pi_r : r > 0\}$ is an ideal base for which

$$\pi_r \circ \pi_s \subseteq \pi_{r+s} .$$

The forthcoming u.h. is called horometric, respectively we say that it is generated by the horometer σ.

c) In the case $W = \mathbb{R}$, let $K = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$ be the usual order of $\mathbb{R}$. If for $r > 0$ we write

$$\pi_r = \{(x, y) \in \mathbb{R}^2 : y - x > r\},$$

then the ideal base $\mathscr{B} = \{ \pi_r : r > 0\}$ generates a u.h. on $\mathbb{R}$. At the same time, the family $\mathscr{F} = \{K \setminus \pi_r : r > 0\}$ is a filter base that generates the right uniform topology of $\mathbb{R}$. This is an example of reciprocally determined u. topology and u. horistology, which we may obviously extend to arbitrary totally ordered sets. The correspondence is realized as in Remark 4c.

d) On the set $W = \mathbb{R}^2$ we define the causal order

$$K = \{((t, x), (s, y)) \in W^2 : s - t > |x - y|\} \cup \delta ,$$

and we write $U_r = \{((t, x), (s, y)) \in W^2 : |x - y| < r\}$. Then the ideal $\mathscr{I}$, generated by the ideal base $\mathscr{B} = \{ \dot{K} \cap U_r : r > 0\}$, satisfies conditions [$uh_1$], [$uh_2$] and [$uh_3$], but not [$uh_4$]. It is interesting to remark that this ideal satisfies a weaker than [uh_4] condition, namely
[uh_-] If $\lambda, \pi \in \mathscr{I}$, then $\lambda \circ \pi \in \mathscr{I}$.

6. PROPOSITION. *If* $(W, \mathscr{H})$ *is a u.h. world, then*

$$K(\mathscr{H}) = [\cup\{\lambda : \lambda \in \mathscr{H}\}] \cup \delta$$

is an order on W.

$K(\mathscr{H})$ is called *proper order* of $\mathscr{H}$.

Proof. Inclusion $\delta \subseteq K(\mathscr{H})$ is explicit. For anti-symmetry, let us suppose that $(x, y) \in [K(\mathscr{H}) \cap K(\mathscr{H})^{-1}] \setminus \delta$. Then $(x, y) \in \lambda$ and $(y, x) \in \pi$ for some $\lambda, \pi \in \mathscr{H}$, hence $(x, x) \in \lambda \circ \pi$. According to [$uh_4$] (or to its consequence [uh_-]), we have $\lambda \circ \pi \in \mathscr{H}$, hence by virtue of [$uh_1$], $(x, x) \in \lambda \circ \pi$ is impossible. The contradiction shows that $K(\mathscr{H})$ is anti-symmetric.

In order to prove the transitivity, we assume that $(x, y), (y, z) \in K(\mathscr{H})$ is due to $(x, y) \in \lambda$ and $(y, z) \in \pi$ for some $\lambda, \pi \in \mathscr{H}$. So $(x, z) \in \lambda \circ \pi \in \mathscr{H}$. ◇

7. Definition. Let λ be a binary relation on $(W, \mathscr{H})$, and let $K = K(\mathscr{H})$ be the proper order of $\mathscr{H}$.

a) We say that λ is *left* (*right*) *open* relative to K if $K \circ \lambda \subseteq \lambda$ (respectively $\lambda \circ K \subseteq \lambda$). If λ is both left and right open, we simply say that it is *open*.

b) We say that λ is *exhaustive* if $\cap\{\lambda^n : n \in \mathbb{N}\} = \varnothing$. If each prospect of $\mathscr{H}$ is exhaustive, then we say that $\mathscr{H}$ is an *exhausting* u.h.

c) We say that the u.h. $\mathscr{H}$ on W is generated by the family $\{\sigma_i : i \in I\}$ of (p-) horometrics $\sigma_i : K \to \mathbb{R}_+$, $i \in I$, if for each $\lambda \in \mathscr{H}$ there exists $i \in I$ and $\varepsilon > 0$ such that $\lambda \subseteq \{(x, y) \in K : \sigma_i(x, y) > \varepsilon\}$.

8. LEMMA. *Let* $(W, \mathscr{H})$ *be a u.h. world and let* $K = K(\mathscr{H})$ *be its proper order. Then* $\mathscr{H}$ *admits a base* $\mathscr{B}$ *consisting of open and transitive prospects.*

Proof. According to [uh_4], for each $\pi \in \mathscr{H}$ there exists $\lambda \in \mathscr{H}$ such that for all $\omega \in \mathscr{H}$ we have $\pi \circ \omega \subseteq \lambda$ and $\omega \circ \pi \subseteq \lambda$, hence

$$\pi \circ \dot{K} \subseteq \lambda \text{ and } \dot{K} \circ \pi \subseteq \lambda .$$

Consequently, $\gamma_\pi \overset{not.}{=} K \circ \pi \circ K \in \mathscr{H}$. Because $\delta \subseteq K$, we have $\gamma_\pi \supseteq \pi$, hence

$$\mathscr{B} = \{\gamma_\pi \in \mathscr{H} : \pi \in \mathscr{H}\}$$

is a base of $\mathscr{H}$. In addition, K is transitive, and each γ_π is open because the composition of binary relations is associative.

Because $K(\mathscr{H}) = [\cup\{\lambda : \lambda \in \mathscr{B}\}] \cup \delta$ holds for any base $\mathscr{B}$ of $\mathscr{H}$, we deduce $\gamma_\pi \subseteq K$. The monotony of the inclusion gives

$$\gamma_\pi \circ \gamma_\pi \subseteq \gamma_\pi \circ K \subseteq \gamma_\pi ,$$

which shows that γ_π is transitive. ◇

We may solve the problem of metricizing the u.h. by the following:

9. THEOREM. *The u.h.* $\mathscr{H}$ *on W is generated by a family of* (*pseudo*-) *horometrics if and only if* $\mathscr{H}$ *is exhaustive*.

Proof. If $\{\sigma_i : i \in I\}$ be a family of (p-) horometrics that generates $\mathscr{H}$, then $\mathscr{H}$ has a base $\mathscr{B}$ consisting of hyperbolical prospects,

$$\mathscr{B} = \{H_{i,\varepsilon} : i \in I, \varepsilon > 0\},$$

where

$$H_{i,\varepsilon} = \{(x, y) \in W^2 : \sigma_i (x, y) > \varepsilon\}.$$

It is easy to prove that each $H_{i,\varepsilon}$ is exhaustive. In fact, in the contrary case, there is a pair $(x, y) \in W^2$ such that

$$(x, y) \in \cap \{ H_{i,\varepsilon}^n : n \in \mathbb{N}\}.$$

Explicitly, this means that for each $n \in \mathbb{N}$ there exists a system of elements in W, say $x = x_0, x_1, \ldots, x_n = y$, such that membership $(x_j, x_{j+1}) \in H_{i,\varepsilon}$ holds for all $j = 0, 1, \ldots, n - 1$. The super-additivity of σ_i gives

$$\sigma_i (x, y) \geq \sum_{j=0}^{n-1} \sigma_i (x_j, x_{j+1}) \geq n\,\varepsilon$$

for arbitrary $n \in \mathbb{N}$, which is impossible.

Conversely, let $(W, \mathscr{H})$ be exhausting. According to the above lemma, there exists an open base $\mathscr{B}$ consisting of exhausting prospects. Then for each $\lambda \in \mathscr{B}$ we may construct an order $K_\lambda = \delta \cup \lambda$ and a function $\sigma_\lambda : K_\lambda \to \mathbb{R}_+$, which takes the values

$$\sigma_\lambda (x, y) = \begin{cases} 0 & \text{if } (x, y) \in \delta \\ 2^{p-1} & \text{if } (x, y) \in \lambda^p \setminus \lambda^{p+1} \text{ for some } p \in \mathbb{N}^*. \end{cases}$$

This construction of σ_λ is based on the fact that the sequence $\{\lambda^p\}_{p \in \mathbb{N}}$ is decreasing since λ is transitive and exhausting. In order to prove that σ_λ is a horometer, we primarily remark that, by definition, $\sigma_\lambda (x, y) = 0$ holds if and only if $x = y$. After that, from $\sigma_\lambda (x, y) = 2^{p-1}$ and $\sigma_\lambda (y, z) = 2^{q-1}$ we deduce $(x, z) \in \lambda^{p+q}$, hence $\sigma_\lambda (x, z) \geq 2^{p+q-1}$. Consequently, the super-additivity of σ_λ follows from the general inequality

$$2^{p-1} + 2^{q-1} \leq 2^{p+q-1},$$

which is valid for all $p, q \in \mathbb{N}^*$.

It remains to prove that the family $\{\sigma_\lambda : \lambda \in \mathscr{B}\}$ generates $\mathscr{H}$. This follows from the fact that each $\lambda \in \mathscr{B}$ allows the representation

$$\lambda = H_{\lambda,1} = \{(x, y) \in K : \sigma_\lambda (x, y) \geq 1\},$$

and $\mathscr{B}$ is a base of $\mathscr{H}$. ◇

10. Remarks. a) As a completion to the second part of the above proof, we may show that $\mathscr{H}$ is also generated by the family $\{\sigma'_\lambda : \lambda \in \mathscr{B}\}$ of pseudo-horometrics $\sigma'_\lambda : K(\mathscr{H}) \to \mathbb{R}_+$, expressed by

$$\sigma'_\lambda(x, y) = \begin{cases} 0 & \text{if } (x,y) \in K(\mathscr{H}) \setminus \lambda \\ 2^{p-1} & \text{if } (x,y) \in \lambda^p \setminus \lambda^{p+1} \text{ for some } p \in \mathbb{N}^* . \end{cases}$$

In order to prove the super-additivity of σ'_λ we shall consider more cases. Primarily, the case $\sigma'_\lambda(x, y) = 0 = \sigma'_\lambda(y, z)$ is trivial. Another new case is $\sigma'_\lambda(x, y) = 0$ and $\sigma'_\lambda(y, z) = 2^{p-1}$. Because λ^p is (left) open, like λ, we have $(x, z) \in \lambda^p$, hence $\sigma'_\lambda(x, z) \geq 2^{p-1}$.

We may similarly discuss the case $\sigma'_\lambda(x, y) = 2^{p-1}$, and $\sigma'_\lambda(y, z) = 0$.

b) Using the above theorem, we may easily construct uniform horistologies that are not generated by a (p-) horometer in the sense of the example 5b. For such a purpose let K be an order relation on W such that W is *dense* relative to $\dot{K}$, i.e. $\dot{K} \subseteq \dot{K} \circ \dot{K}$ (or, equivalently, each $\dot{K}$ interval is non-void). We cannot generate the u.h. $\mathscr{H} = \mathscr{P}(\dot{K})$ by a horometer $\sigma : K \to \mathbb{R}_+$, since $\dot{K}$ is not exhaustive.

By analogy to dual (p-) horometrics, we may speak of *dual* u.h., which look like $\mathscr{H}$ and $\mathscr{H}'$ in the following proposition:

11. PROPOSITION. *If* $(W, \mathscr{H})$ *is a u.h. world, then*

a) $\mathscr{H}' = \{\lambda \subseteq W^2 : \lambda^{-1} \in \mathscr{H}\}$ *also is a u.h. on W, and*

b) $K(\mathscr{H}') = [K(\mathscr{H})]^{-1}$.

The proof is routine and will be omitted.

12. COROLLARY. *The following properties hold for dual u.h.*:

a) *Dual* (*p-*) *horometers generate dual u.h.*

b) *If* $\lambda \in \mathscr{H}$ *and* $\pi \in \mathscr{H}'$, *then* $\lambda \cap \pi = \emptyset$

c) *The dual u.h. are disjoint* (*in the sense that* $\mathscr{H} \cap \mathscr{H}' = \{\emptyset\}$).

Proof. a) In terms of [1], the dual of a (p-) horometric $\sigma : K \to \mathbb{R}_+$ is a function $\sigma^{\leftarrow} : K^{-1} \to \mathbb{R}_+$, which takes the values $\sigma^{\leftarrow}(x, y) = \sigma(y, x)$. We may easily see that the following inclusions are concomitant, i.e.

$$\lambda \subseteq \{(x, y) \in K : \sigma(x, y) > r\} \Leftrightarrow \lambda^{-1} \subseteq \{(y, x) \in K^{-1} : \sigma^{\leftarrow}(y, x) > r\}.$$

b) By Proposition 11, $\lambda \subseteq K(\mathscr{H}) \setminus \delta$, and $\pi \subseteq [K(\mathscr{H})]^{-1} \setminus \delta$.

c) Because of b), $\pi \in \mathscr{H} \cap \mathscr{H}'$ holds only if $\pi = \pi^{-1}$. However, taking $\omega = \pi^{-1}$ in [uh_4], we contradict [uh_1], hence $\pi = \emptyset$. ◇

B. HORISTOLOGICAL WORLDS

In the theory of discreteness, the horistological structures, introduced in this section, play a similar role to that of topological structures in continuity.

1. Definition. Let W be an arbitrary non-void set. We name *horistology* on W any function $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, which satisfies the conditions:

[h_1] $x \notin P$ for all $x \in W$ and $P \in \chi(x)$

[h_2] If $P \in \chi(x)$ and $Q \subseteq P$, then $Q \in \chi(x)$

[h_3] If $P, Q \in \chi(x)$, then $P \cup Q \in \chi(x)$

[h_4] For each $P \in \chi(x)$ there exists $L \in \chi(x)$ such that $Q \subseteq L$ holds for every $y \in P$ and $Q \in \chi(y)$.

The pair (W, χ) will be called *horistological world*. The elements of $\chi(x)$ are named *perspectives* of x, respectively x is a *premise* of each $P \in \chi(x)$.

Obviously, instead of ideals $\chi(x)$, we may define the horistology by ideal bases $\beta(x)$, specified at each $x \in W$.

2. Examples. a) The function χ_0, defined by $\chi_0(x) = \{\varnothing\}$ at each $x \in W$, is a horistology on W.

b) If K be an order relation on W, then function χ, defined at each $x \in W$ by $\chi(x) = \mathscr{P}(\dot{K}[x])$, is a horistology on W.

c) If $(W, \mathscr{H})$ is a u.h. on W, then the function $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, of values

$$\chi(x) = \{\lambda[x] : \lambda \in \mathscr{H}\},$$

is a horistology on W. We say that χ is *generated* by $\mathscr{H}$. In the particular case when $\mathscr{H}$ derives from a horometer $\sigma : K \to \mathbb{R}_+$, each ideal $\chi(x)$ admits a base consisting of *hyperbolical perspectives*

$$H(x, r) = \{y \in W : \sigma(x, y) > r\},\ r \in \mathbb{R}_+^*.$$

d) Let K be an order relation on W, for which $\dot{K}$ is *locally directed*, i.e. for every $(x, y), (x, z) \in \dot{K}$ there exists $u \in W$ such that $(x, u), (u, y), (u, z) \in \dot{K}$. Then the function $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, expressed by

$$\chi(x) = \cup\{\mathscr{P}(\dot{K}[y]) : y \in \dot{K}[x]\},$$

defines a horistology on W. In particular, if $(D, \leq)$ is a directed set (towards greater elements), and we note $\overline{D} = D \cup \{\infty\}$, where we suppose that $x < \infty$ holds for all $x \in D$, then the function $\varphi : \overline{D} \to \mathscr{P}(\mathscr{P}(D))$, of values

$$\varphi(\alpha) = \begin{cases} \cup\{\mathscr{P}(\leftarrow, x) : x \in D\} & if \quad \alpha = \infty \notin D \\ \{\varnothing\} & if \quad \alpha = x \in D \end{cases}$$

is a horistology on $\overline{D}$, called *natural horistology of* $\overline{D}$.

More particularly, if $D = \mathbb{N}$, then for each perspective $P \in \varphi(\infty)$, $\complement P$ is a neighborhood of ∞ in the topology τ, of values

$$\tau(\nu) = \begin{cases} \{V \subseteq \overline{\mathbb{N}} : V \supseteq [n, \infty], n \in \mathbb{N}\} & if\ \nu = \infty \notin \mathbb{N} \\ \{V \subseteq \overline{\mathbb{N}} : V \ni n\} & if\ \nu = n \in \mathbb{N}. \end{cases}$$

3. PROPOSITION. *If* (W, χ) *is a horistological world, then*

$$K(\chi) = \{(x, y) \in W^2 : \text{either } y = x \text{ or } \{y\} \in \chi(x)\}$$

is an order relation on W.

We say that $K(\chi)$ is the *proper order* of χ.

Proof. The reflexivity, i.e. $K(\chi) \supseteq \delta$, is evident. In order to prove the anti-symmetry, let us consider $(x, y) \in K(\chi) \cap [K(\chi)]^{-1}$. If $x \neq y$, it follows that both $y \in P \in \chi(x)$ and $x \in Q \in \chi(y)$ hold for some P and Q. According to [h_4], we have $Q \in \chi(x)$, but $x \in Q$ is impossible by virtue of [h_1]. Consequently, $x = y$.

The transitivity of $K(\chi)$ is another consequence of [h_4]. In fact, if we start with $(x, y) \in K(\chi)$ and $(y, z) \in K(\chi)$, then the essential case is $x \neq y \neq z$, i.e. $y \in P$ and $z \in Q$ for some $P \in \chi(x)$ and $Q \in \chi(y)$. Condition [h_4], gives $Q \in \chi(x)$, hence $(x, z) \in K(\chi)$. ◇

4. Remarks. a) If β is a base of the horistology χ, then we may express the proper order of χ by the formula

$$K(\chi) = \{(x, y) \in W^2 : \exists P \in \beta(x) \text{ such that } y \in P\} \cup \delta.$$

b) If the horistology χ derives from the u.h. $\mathscr{H}$, then

$$K(\chi) = K(\mathscr{H}).$$

c) If (W, K, σ) is a p-horometric world, and χ is the horistology generated by σ, then $K \supseteq K(\chi)$. The equality holds if and only if σ is a (non p-) horometer.

d) Let (W, K, σ) be a p-horometric world, in which σ is additive, i.e.

$$\sigma(x, y) + \sigma(y, z) = \sigma(x, z)$$

holds for all (x, y), $(y, z) \in K$. If χ denotes the (u) horistology generated by σ, then the function $\tau : W \to \mathscr{P}(\mathscr{P}(W))$, defined by

$$\tau(x) = \{K[x] \setminus P : P \in \chi(x)\},$$

is a filter base of neighborhoods for a topology on W. Particularly, this is the case of $W = \mathbb{R}$ in the example A5c (respectively C1a in [1]), when τ generates the "right semi-interval" topology of $\mathbb{R}$.

Similarly, the function $\theta : \mathbb{R} \to \mathscr{P}(\mathscr{P}(\mathbb{R}))$, expressed by

$$\theta(x) = \{\complement P : P \in \chi(x)\},$$

defines the "left unbounded" topology of $\mathbb{R}$. However, passing from ideal to filters through $\complement$ does not generally carry horistologies to topologies.

5. Definition. Let K be an order relation on W, and $A \subseteq W$. We say that

$$K[A] = \cup\{K[x] : x \in A\}$$

is the *positive prolongation* of A relative to K. In the case $K[A] = A$ we say that A is *positively conical* relative to K.

Similarly, we speak of *negative prolongation* of A relative to K, which is

$$K^{-1}[A] = \cup\{ K^{-1}[x] : x \in A\},$$

and of *negatively conical* sets, relative to K, when $K^{-1}[A] = A$.

6. LEMMA. *For every horistology χ on W there exists a base consisting of positively conical prospects relative to the proper order $K = K(\chi)$.*

Proof. The reflexivity, $\delta \subseteq K$, gives $A \subseteq K[A]$ for all $A \subseteq W$. Similarly, the transitivity, $K \circ K \subseteq K$, implies $K[K[A]] = K[A]$ for all $A \subseteq W$. Consequently, it is sufficient to show that for each $P \in \chi(x)$, we have $K[P] \in \chi(x)$. In fact, according to [h_4], for each $P \in \chi(x)$ there exists $L \in \chi(x)$, such that $K[P] \subseteq L$. The conclusion $K[P] \in \chi(x)$ follows from [h_2]. ◇

In the subsequent part of this section, we will discuss several horistological operators, which correspond to the well-known "interior", "adherence", etc., in the general topology. We will see that such operators offer equivalent ways to introduce horistological structures.

7. Definition. Let (W, χ) be a horistological world, and $A \subseteq W$. We say that

$$\mathbf{p}(A) = \{ x \in W : A \in \chi(x)\}$$

is the *premise set* of A. The function $\mathbf{p} : \mathscr{P}(W) \to \mathscr{P}(W)$, which attaches to each set A its premise set $\mathbf{p}(A)$, will be called *premise operator* relative to the horistology χ.

8. Remarks. a) Besides $\mathbf{p}$, we mention the operator $\mathbf{q} : \mathscr{P}(W) \to \mathscr{P}(W)$, for which the value at $B \subseteq W$ is

$$\mathbf{q}(B) = \{ x \in W : P \cup B \neq W \textit{ for all } P \in \chi(x)\}.$$

It is easy to see that $\mathbf{q}$ strongly relates with $\mathbf{p}$, in the sense of the equality

$$\complement\, \mathbf{p}(A) = \mathbf{q}(\complement A),$$

which holds for every $A \subseteq W$.

b) There exist set operators expressed in terms of perspectives, which are only apparently horistological. In fact, we may fully determine them by the proper order of the considered horistology, hence they are order operators. For example, the operator $\boldsymbol{\omega} : \mathscr{P}(W) \to \mathscr{P}(W)$, defined by

$$\boldsymbol{\omega}(A) = \{ x \in W : P \cap A \neq \emptyset \textit{ for some } P \in \chi(x)\},$$

is essentially determined by the order $K = K(\chi)$, in the sense that, equivalently,

$$\boldsymbol{\omega}(A) = \{ x \in W : A \cap \dot{K}[x] \neq \emptyset \}$$

holds at every $A \subseteq W$, without reference to χ.

c) Simple examples show that different horistologies having the same proper order, on W, generally give different premise operators. Consequently, the premise operator essentially is horistological, not an order operator. However, $\mathbf{p}$ has many properties relative to the proper order K, e.g. $\mathbf{p}(A) \subseteq \dot{K}^{-1}[A]$ and $\mathbf{p}(A) = K^{-1}[\mathbf{p}(A)]$, i.e. $\mathbf{p}(A)$ is negatively conical.

Similarly, $\mathbf{q}$ is horistological, not order operator.

9. THEOREM. *Let* (W, χ) *be a horistological world and let* $K = K(\chi)$ *be its proper order. The premise operator* $\mathbf{p}$, *relative to* χ, *satisfies the conditions*:

[p_1] $A \cap \mathbf{p}(A) = \varnothing$ *for all* $A \subseteq W$;

[p_2] $A \subseteq B$ *implies* $\mathbf{p}(A) \supseteq \mathbf{p}(B)$ *for all* $A, B \subseteq W$;

[p_3] $\mathbf{p}(A \cup B) = \mathbf{p}(A) \cap \mathbf{p}(B)$ *for all* $A, B \subseteq W$;

[p_4] $\mathbf{p}(A) \subseteq \mathbf{p}(K[A])$ *for all* $A \subseteq W$.

In addition, we may directly derive K *from* $\mathbf{p}$, i.e.

$$K = \{(x, y) \in W^2 : \textit{either } x = y \textit{ or } x \in \mathbf{p}(\{y\})\}$$

Proof. If, contrarily to [p_1], we suppose that there exists $x \in A \cap \mathbf{p}(A)$, then we obtain $x \in A \in \chi(x)$, which contradicts [h_1].

If $x \in \mathbf{p}(B)$, we deduce $B \in \chi(x)$. Using [h_2], from $A \subseteq B$ we obtain $A \in \chi(x)$, i.e. $x \in \mathbf{p}(A)$, which proves [p_2].

In order to prove [p_3], we primarily remark that, according to [p_2], from the general inclusions $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows that $\mathbf{p}(A) \supseteq \mathbf{p}(A \cup B)$ and $\mathbf{p}(B) \supseteq \mathbf{p}(A \cup B)$, hence $\mathbf{p}(A) \cap \mathbf{p}(B) \supseteq \mathbf{p}(A \cup B)$. The converse inclusion is based on [h_3], which states that $A \in \chi(x)$ and $B \in \chi(x)$ imply $A \cup B \in \chi(x)$.

Property [p_4] is a simple rewording of Lemma 6.

The proof of the formula for K is direct. ◇

The subsequent theorem shows that the properties [p_1] – [p_4] from above are sufficient for defining a horistology on W.

10. THEOREM. *Let* W *be a non-void set. If* $\mathbf{p} : \mathscr{P}(W) \to \mathscr{P}(W)$ *is a set operator that satisfies conditions* [p_1] – [p_4], *then function* $\chi : W \to \mathscr{P}(\mathscr{P}(W))$, *defined at each* $x \in W$ *by*

$$\chi(x) = \{P \subseteq W : x \in \mathbf{p}(P)\},$$

is a horistology on W. *In addition,*

$$K = \{(x, y) \in W^2 : \ x \in \mathbf{p}(\{y\})\} \cup \delta$$

is an order relation on W, *and* $K = K(\chi)$.

Proof. We have to show that χ satisfies conditions [h_1] – [h_4]. Thus, [h_1] is a direct consequence of [p_1].

In order to prove [h_2], we start with $P \in \chi(x)$ and $Q \subseteq P$. According to [p_2], we have $x \in \mathbf{p}(P) \subseteq \mathbf{p}(Q)$, hence $Q \in \chi(x)$.

Condition [h_3] is based on [p_3]. If P, $Q \in \chi(x)$, then $x \in \mathbf{p}(P)$ and $x \in \mathbf{p}(Q)$. Consequently $x \in \mathbf{p}(P) \cap \mathbf{p}(Q) = \mathbf{p}(P \cup Q)$, and so $P \cup Q \in \chi(x)$.

Before proving [h_4], we shall analyze relation K. Obviously, K is reflexive, hence $A \subseteq K[A]$ for all $A \subseteq W$. It is also easy to see that $K = K(\chi)$. According to [p_4], each $P \in \chi(x)$ gives $K[P] \in \chi(x)$. Now, if $y \in P$ and $Q \in \chi(y)$, it follows that $Q \subseteq K[y] \subseteq K[P]$, hence $K[P]$ is the asked perspective L in [h_4]. So we conclude that χ is a horistology on W.

The remaining properties of K follow from $K = K(\chi)$. ◇

11. THEOREM. *The functions* $\chi : W \to \mathscr{P}(\mathscr{P}(W))$ *and* $\mathbf{p} : \mathscr{P}(W) \to \mathscr{P}(W)$ *give equivalent definitions of the horistological structures on W. More exactly, if* $\mathbf{p}_\chi$ *denotes the premise operator relative to* χ, *and* $\chi_\mathbf{p}$ *is the horistology that derives from* $\mathbf{p}$ *as in Theorem 10, then we have*

$$\chi_{\mathbf{p}_\chi} = \chi \text{ and } \mathbf{p}_{\chi_\mathbf{p}} = \mathbf{p}.$$

Proof. We tacitly use the above Theorems 9 and 10 when we refer to $\chi_{\mathbf{p}_\chi}$ as a horistology, and to $\mathbf{p}_{\chi_\mathbf{p}}$ as a premise operator.

The rest of the proof is routine; we shall replace $A \in \chi(x)$ by $x \in \mathbf{p}_\chi(A)$, and $x \in \mathbf{p}(A)$ by $A \in \chi_\mathbf{p}(x)$. ◇

Besides the properties expressed in the above theorems, the premise operator has many other significant properties, as for example:

12. PROPOSITION. *If* (W, χ) *is a horistological world and* $K = K(\chi)$, *then*:

(i) $K^{-1}[\mathbf{p}(K[A])] = \mathbf{p}(A)$ *for all* $A \subseteq W$

(ii) $\mathbf{p}(\emptyset) = W$ *and* $\mathbf{p}(W) = \emptyset$

(iii) *If* $A = K^{-1}[A]$, *then* $\mathbf{p}(A) = \emptyset$

(iv) $\mathbf{p}(\mathbf{p}(A))$ *is either* $\emptyset$ *or* W.

Proof. (i) Since K is reflexive, i.e. $A \subseteq K[A]$, [p_2] gives $\mathbf{p}(A) \supseteq \mathbf{p}(K[A])$. Because $\mathbf{p}(K[A])$ is negatively conical, we deduce $\mathbf{p}(A) \supseteq K^{-1}[\mathbf{p}(K[A])]$. The contrary inclusion holds by [h_4].

(ii) $\emptyset \in \chi(x)$ holds for all $x \in W$, while $W \in \chi(x)$ contradicts [h_1].

(iii) If we suppose that there exists $x \in \mathbf{p}(A)$, where A is negatively conical, it follows that $x \in K^{-1}[A]$, but $x \in A$ contradicts [h_1].

(iv) If $\mathbf{p}(A) \neq \emptyset$, then $\mathbf{p}(\mathbf{p}(A)) = \emptyset$ follows from (iii), since $\mathbf{p}(A)$ is negatively conical. If $\mathbf{p}(A) = \emptyset$, we may use ($ii$). ◇

The proof from above reduces properties (i) – (iv) to the axioms [h_1] – [h_4] of a horistological world, but according to Theorem 11, we may alternatively base it on the properties [p_1] – [p_4] of the premise operator.

C. DISCRETE FUNCTIONS

In this section, we consider functions whose domains and ranges are included in (u) horistological worlds. The purpose is to identify those properties of such functions, which are dual to the continuity in the case of general topology. In order to satisfy the "natural requirements" formulated at the very beginning (especially A2d), we will analyze two types of horistological properties of the functions, which seem to play a role acceptably similar to continuity (a more detailed comparison reminds the history of Cauchy and Darboux conditions).

The former attempt is to start with cases of well connected horistologies and topologies, and "translate" the notion of continuity in terms of perspectives.

Our second suggestion is to consider an independent and specific concept of discreteness, which results after the criticism of the first one.

The starting point of the argumentation will be the classical notions of convergence and continuity.

1. Remarks. a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and let ℓ be its limit relative to some topology on $\mathbb{R}$ (e.g. the Euclidean one). Independently of this topology, one of the simplest, but general, formulations of the fact that $\ell = lim\ x_n$ is the following: *Outside of an arbitrary neighborhood of ℓ, there exists a finite number of terms x_n of the sequence*". It is easy to see that such a condition allows a convenient reformulation in terms of perspectives, which is always possible when the involved topological and horistological structures are reciprocally determined. In particular, the family of finite parts of $\mathbb{N}$ forms an ideal, which represents $\varphi(\infty)$ in the natural horistology of $\mathbb{N}$ (introduced in Example B2d). On the other hand, in circumstances like in Remark B4d, the perspective $P\in\chi(\ell)$ has the expression $P = \complement\, V$, where V is a neighborhood of the limit ℓ. Consequently, relative to the horistological structures defined by φ and χ, the condition $\ell = lim\ x_n$ takes the form: *For each perspective $P\in\chi(\ell)$, there exists a perspective $Q\in\varphi(\infty)$, such that $x_n \in P$ implies $n\in Q$*".

b) It is well known that the convergence of the sequence $f : \mathbb{N}\to\mathbb{R}$, of terms $f(n) = x_n$, to $\ell = lim\ x_n$, is equivalent to the continuity at ∞ of its prolongation to $\overline{\mathbb{N}}$, noted $\overline{f} : \overline{\mathbb{N}} \to\mathbb{R}$, and defined by

$$\overline{f}(\nu) = \begin{cases} x_n & if\ \ \nu = n\in\mathbb{N} \\ \ell & if\ \ \nu = \infty\ . \end{cases}$$

Obviously, this property is generally valid for nets. More than this, it suggests an extension from convergence to the general notions of limit and continuity.

Thus, we may reformulate the condition $\ell = \lim_{x \to x_0} f(x)$ in horistological terms as follows: *For any perspective* $P \in \chi(\ell)$ *there exists* $Q \in \varphi(x_0)$ *such that*

$$f^{\leftarrow}(P) \subseteq Q.$$

Like before, the essential condition that allows us to give such an expression is the complete reciprocal determination of the horistologies φ and χ by the corresponding topologies, which makes neighborhoods and perspectives to be complementary each other. Because such a determination is not generally possible, on this way we practically extrapolate the topological notion of continuity to horistological structures. By the subsequent study (up to C12) we will explore this variant for morphisms of the horistological structures.

2. Definition. Let (V, φ) and (W, χ) be horistological worlds and let $D \subseteq V$, $x_0 \in D$, and a function $f : D \to W$. We say that f is *h-continuous* at x_0 if for each $P \in \chi(f(x_0))$ we have $f^{\leftarrow}(P) \in \varphi(x_0)$.

If f is h-continuous at each $x_0 \in D$, then we say that f is *h-continuous on D.*

We say that $\ell \in W$ is the *h-limit* of f at x_0, and we note

$$\ell = h - \lim_{x \to x_0} f(x),$$

if for every $P \in \chi(\ell)$ we have $f^{\leftarrow}(P) \in \varphi(x_0)$.

Let $(V, \mathscr{L})$ and $(W, \mathscr{H})$ be uniform horistological worlds. We say that the function $f : V \to W$ is *uniformly h-continuous* on V (briefly u.h-continuous), if for every $\pi \in \mathscr{H}$ we have $f_{II}^{\leftarrow}(\pi) \in \mathscr{H}$, where $f_{II}(x, y) = (f(x), f(y))$, and

$$f_{II}^{\leftarrow}(\pi) = \{(x, y) \in V^2 : f_{II}(x, y) \in \pi\}.$$

3. PROPOSITION. *A necessary and sufficient condition for* f (*as in the above definition*) *to be h-continuous at* x_0 *is to have the h-limit* ℓ *at* x_0 *and*

$$\ell = f(x_0).$$

The proof is immediate.

4. PROPOSITION. *Let* (V, φ) *and* (W, χ) *be horistological worlds, and let* $\mathbf{p}_{\varphi}$ *and* $\mathbf{p}_{\chi}$ *be the corresponding premise operators. The function* $f : D \to W$, *where* $D \subseteq V$, *is h-continuous on D if and only if for every* $A \subseteq W$, *we have*

$$(*) \qquad f^{\leftarrow}(\mathbf{p}_{\chi}(A)) \subseteq \mathbf{p}_{\varphi}(f^{\leftarrow}(A)).$$

Proof. Let us suppose that f is h-continuous on D and let A be an arbitrary subset of W. In order to prove the inclusion (*), from $x \in f^{\leftarrow}(\mathbf{p}_{\chi}(A))$ we deduce $f(x) \in \mathbf{p}_{\chi}(A)$, hence $A \in \chi(f(x))$. Using the h-continuity of f at x, it follows that $f^{\leftarrow}(A) \in \varphi(x)$, i.e. $x \in \mathbf{p}_{\varphi}(f^{\leftarrow}(A))$.

Conversely, let us suppose that (*) holds for every $A \subseteq W$. In particular, if we take $A \in \chi\,(f(x))$ for some $x \in D$, then $x \in f^{\leftarrow}(\mathbf{p}_{\chi}(A))$. Using (*), it follows that $x \in \mathbf{p}_{\varphi}\,(f^{\leftarrow}(A))$, or, equivalently, $f^{\leftarrow}(A) \in \varphi\,(x)$. Consequently, f is h-continuous at x, which is arbitrary in D. ◇

5. PROPOSITION. *Let* (V, φ) *and* (W, χ) *be horistological worlds. If the function* $f : V \to W$ *is bijective and h-continuous on V, then* f^{-1} *is strictly monotonous relative to the proper orders of* χ *and* φ.

Proof. If $(x, y) \in \dot{K}(\chi)$, then $\{y\} \in \chi(x)$. Using the h-continuity of f at x, we obtain $\{f^{-1}(y)\} \in \varphi\,(f^{-1}(x))$, hence $(f^{-1}(x)),\ f^{-1}(y)) \in \dot{K}(\varphi)$. ◇

6. PROPOSITION. *Let* (U, φ), (V, ψ) *and* (W, χ) *be horistological worlds. If the function* $f : U \to V$ *is h-continuous at* $x \in U$, *and* $g : V \to W$ *is h-continuous at* $y = f(x) \in V$, *then* $g \circ f$ *is h-continuous at x.*

The proof is direct. Similar properties hold if f and g are h-continuous on the whole U and respectively V, as well as if they are uniformly h-continuous.

7. PROPOSITION. *Let* $(V, \mathscr{L})$ *and* $(W, \mathscr{H})$ *be uniform horistological worlds, and let* φ *and* χ *be the horistologies generated by* $\mathscr{L}$ *and respectively* $\mathscr{H}$. *If* $f : V \to W$ *is u.h-continuous, then f also is h-continuous on V relative to* φ *and* χ.

Proof. If $x \in V$ and $P \in \chi(f(x))$, then there exists $\pi \in \mathscr{H}$ such that $P \subseteq \pi[f(x)]$, hence $(f(x), y) \in \pi$ for all $y \in P$. Because f is u.h-continuous, there exists $\lambda \in \mathscr{L}$ such that $\lambda \supseteq f_{II}^{\leftarrow}(\pi)$, and a fortiori

$$\lambda \supseteq \{(x, x') \in V^2 : y = f(x') \in P\}.$$

This gives $\lambda\,[x] \supseteq f^{\leftarrow}(P)$, hence $f^{\leftarrow}(P) \in \varphi\,(x)$. To conclude,

$$P \in \chi(f(x)) \Rightarrow f^{\leftarrow}(P) \in \varphi\,(x),$$

i.e. f is h-continuous at x. ◇

8. Remark. Let (V, L, ρ) *and* (W, K, σ) be (p-) horometric worlds, and let $\mathscr{L}$ and $\mathscr{H}$ be the uniform horistologies generated by ρ and respectively σ. The function $f : V \to W$ is uniformly h-continuous on V if and only

$$\forall \varepsilon > 0\ \ \exists \delta > 0\ \ \textit{such that}\ [\ \sigma(f(x), f(y)) > \varepsilon\ \Rightarrow\ \rho\,(x, y) > \delta].$$

In particular, the isometric injections (e.g. the Lorentz transformations of the Minkowski space-time) are u. h-continuous. Similar considerations are valid for (non-) uniform h-continuous functions.

In the sequel, we will shortly analyze the notion of convergence that derives from this type of continuity.

9. Definition. Let (W, χ) be a horistological world and let $(D, \leq)$ be a set directed towards greater elements. We say that the element $\ell \in W$ is a *h-limit* of the net $x : D \to W$, and we note $\ell = h - \lim x$, if

$$\forall\ P \in \chi(\ell)\quad \exists\ a \in D\ \textit{ such that } [\ x(b) \in P \Rightarrow (b \ngeq a)\ ]$$

If the net x has a h-limit, we say that x is *h-convergent*.

In particular, we similarly discuss the h-convergence of the sequences.

10. LEMMA. *Let the set $(D, \leq)$ be directed towards greater elements, and let us note $\overline{D} = D \cup \{\infty\}$, where $\infty \notin D$ and $a < \infty$ for all $a \in D$. Then*:

a) *The function $\gamma : \overline{D} \to \mathscr{P}(\mathscr{P}(\overline{D}))$*, defined by

$$\gamma(\alpha) = \begin{cases} \{\varnothing\} & \textit{if } \alpha = a \in D \\ \{P \subset \overline{D} : \complement P \supseteq (a, \infty] \textit{ for some } a \in D\} & \textit{if } \alpha = \infty \in \overline{D} \setminus D, \end{cases}$$

is a horistology on $\overline{D}$ (called complementary horistology of $\overline{D}$).

b) *If (W, χ) is a horistological world, then the net $x : D \to W$ is h-convergent to $\ell \in W$ if and only if its prolongation $\overline{x} : \overline{D} \to W$, defined by*

$$\overline{x}(\nu) = \begin{cases} x(n) & \textit{if } \nu = n \in D \\ \ell & \textit{if } \nu = \infty \in \overline{D} \setminus D, \end{cases}$$

is h-continuous at ∞ relative to the complementary horistology of $\overline{D}$.

The proof is direct and we will omit it.

11. PROPOSITION. *Let (V, ψ) and (W, χ) be horistological worlds. The function $f : V \to W$ is h-continuous at a point $x_0 \in V$ if and only if for each net $x : D \to V$, which is h-convergent to x_0, it follows that the net $f \circ x : D \to W$ is h-convergent to $f(x_0)$.*

Proof. Let us consider that $f : V \to W$ is h-continuous at x_0. If the net x is h-convergent to x_0, then according to the above lemma, the function $\overline{x} : \overline{D} \to V$ is h-continuous at ∞. Using Proposition 6, it follows that the sequence $f \circ x$ is h-convergent to $f(x_0)$.

Conversely, let us suppose that f is not h-continuous at x_0, i.e. there exists a perspective $P \in \chi(f(x_0))$ such that $f^{\leftarrow}(P) \nsubseteq Q$ for all $Q \in \psi(x_0)$. In other words, for every $Q \in \psi(x_0)$ there exists an element $q \in V$ such that $q \notin Q$, while $f(q) \in P$. Let us consider the set $D = \{(q, Q) : q \notin Q \in \psi(x_0), f(q) \in P\}$.

Obviously, D is directed by the relation $(q, Q) \leq (s, S)$ defined by $Q \subseteq S$. Now we may remark that the net $x : D \to V$, of values $x(s, S) = s$, is h-convergent to x_0, since for every $Q \in \psi(x_0)$ there exists $a = (q, Q) \in D$, such that $x(s, S) = s \in Q$ implies $(s, S) \ngeq (q, Q)$ (i.e. $S \nsupseteq Q$). On the other hand, $f \circ x : D \to W$ is not h-convergent to $f(x_0)$ since $(f \circ x)^{\leftarrow}(P) = D$. ◇

12. Remark. The above established properties show how adequate is the h-convergence, respectively the complementary horistology of $\overline{D}$, in the study of h-continuity. Alternatively, we may replace the complementary horistology by the natural one, but then, for the forthcoming type of convergence, the "if" part of the above proposition seems to give up. In the particular case when D is totally ordered (e.g. $D = \mathbb{N}$), the complementary and natural horistologies coincide. Then we may express the h-convergence of the corresponding net (e.g. sequence) in terms of natural horistology. Here, we will not develop a more detailed analysis of h-continuity (respectively h-convergence, etc.) since the failure of some properties required in principle is already transparent.

13. Criticism of *h*-continuity. a) Let (V, ψ) and (W, χ) be horistological worlds and let $f : V \to W$ be a function of range $R \subseteq W$. If

$$R \cap \dot{K}(\chi)\,[f(x_0)] = \varnothing,$$

i.e. the values of f fail from the cone of vertex $f(x_0)$, then the condition of h-continuity is still trivially fulfilled. In particular, the h-convergence turns out to be a condition satisfied by too wide classes of nets, for which this property reduces to an order one.

b) In the most simple linear spaces, which do naturally carry horistological structures (e.g. $\mathbb{R}$, $\mathbb{D}$, $\mathbb{B}$, Minkowskian worlds, etc.), the function of addition is not h-continuous. This incompatibility between the algebraic structure and the horistological property of h-continuity stops the usual course of the analysis.

c) The h-continuity relates with monotony, i.e. respects the proper orders of the involved horistologies in the sense of Proposition 5, only for 1:1 functions. Such a restriction is not convenient as long as in practice we meet a lot of remarkable examples of directly order preserving functions not 1:1.

d) The constant functions, which are always h-continuous, do not preserve discreteness (in its wide sense), hence we cannot accept them to be morphisms of the horistological structures. In particular, the identification of different kinds of events in a Minkowskian space-time is not physically significant.

e) Many simple and frequently used functions, especially acting in $\mathbb{R}$, as for example the polynomial functions (e.g. $f(x) = x^2$), $g(x) = |x|$, etc., are not h-continuous. The same phenomena occur in the algebras $\mathbb{B}$, $\mathbb{D}$, etc. In inner product spaces, the plus-operators are not generally h-continuous (see [2]).

Similar objections are valid for h-convergence.

In conclusion, the idea of continuity, which is so suitable to the study of the topological structures, is considerably less fruitful in horistology. Therefore, in spite of some remarkable properties, we will remove the h-continuity in favor of another type of morphisms of the horistological structures.

14. Definition. Let (V, ψ) and (W, χ) be horistological worlds, $f : A \to W$ be a function, where $A \subseteq V$, and $x_0 \in A$. We say that $g \in W$ is a *germ of f , when x starts from* x_0, and we write

$$g = \underset{x_0 \to x}{germ}\ f(x),$$

if for every $P \in \psi(x_0)$ we have $f(P) \in \chi(g)$.

We say that f is *discrete at* $x_0 \in A$ if $\underset{x_0 \to x}{germ}\ f(x)$ exists and

$$\underset{x_0 \to x}{germ}\ f(x) = f(x_0),$$

i.e. for every $P \in \psi(x_0)$ we have $f(P) \in \chi(f(x_0))$.

If f is discrete at each $x_0 \in A$, we say that f is *discrete on A*.

15. PROPOSITION. *Let* (V, ψ) *and* (W, χ) *be horistological worlds,* $x_0 \in A$, *where* $A \subseteq V$, *and let* $f : A \to W$ *be a function*.

a) *If f is discrete at* x_0, *then it is strictly monotonous at* x_0 *relative to the proper orders* $K(\psi)$ and $K(\chi)$, i..e.

$$(x_0, x) \in \dot{K}(\psi) \Rightarrow (f(x_0), f(x)) \in \dot{K}(\chi).$$

b) *If f is discrete on A, then it is strictly monotonous on A relative to these proper orders*.

Proof. a) From $(x_0, \dot{K}(\psi)$ it follows that $\{x\} \in \psi(x_0)$, and discreteness of f gives $\{f(x)\} \in \chi(f(x_0))$.

b) We similarly reason for all $(x_0, x) \in A^2 \cap \dot{K}(\psi)$. ◇

16. THEOREM. *Let* (V, ψ) *and* (W, χ) *be horistological worlds, for which* $\mathbf{p}_\psi$ *and* $\mathbf{p}_\chi$ *are the corresponding premise operators. The function* $f : A \to W$, *where* $A \subseteq V$, *is discrete on A if and only if for every* $B \subseteq A$ *we have*

(**) $$f(\mathbf{p}_\psi(B)) \subseteq \mathbf{p}_\chi(f(B)).$$

Proof. In order to prove (**), let us take an arbitrary $y \in f(\mathbf{p}_\psi(B))$. Then there exists $x \in \mathbf{p}_\psi(B)$ such that $f(x) = y$. In other words, $B \in \psi(x)$, and since f is discrete at x, we have $f(B) \in \chi(y)$. Consequently, $y \in \mathbf{p}_\chi(f(B))$.

Conversely, if $x \in A$ and $B \in \psi(x) \cap \mathscr{P}(A)$, then we have $x \in \mathbf{p}_\psi(B)$. From (**) we deduce that $f(x) \in \mathbf{p}_\chi(f(B))$, i.e. $f(B) \in \chi(f(x))$. Using Definition 14, this means that f is discrete at x. ◇

17. PROPOSITION. *Let* (U, φ), (V, ψ) *and* (W, χ) *be horistological worlds. If the function* $f : U \to V$ *is discrete at* $x \in U$, *and the function* $g : V \to W$ *is discrete at* $y = f(x) \in V$, *then* $g \circ f$ *is discrete at x.*

The proof is direct.

We may easily extend Proposition 17 to functions discrete on sets.

18. Definition. Let (V, $\mathscr{L}$) and (W, $\mathscr{H}$) be u.h. worlds. We say that the function $f : V \to W$ is *uniformly discrete on V* if for every prospect $\pi \in \mathscr{L}$ we have $f_{II}(\pi) \in \mathscr{H}$, where $f_{II}(\pi) = \{ f_{II}(a, b) : (a, b) \in \pi\}$.

19. PROPOSITION. Every uniformly discrete function $f : V \to W$ is also discrete at each point of V, relative to the horistologies generated by $\mathscr{L}$ on V and $\mathscr{H}$ on W.

Proof. Let ψ and χ be the horistologies generated by $\mathscr{L}$ and $\mathscr{H}$. If $x \in V$ and $P \in \psi(x)$, then there exists $\pi \in \mathscr{L}$ such that $P = \pi[x]$, i.e. $(x, y) \in \pi$ for all $y \in P$. Because f is uniformly discrete, there exists $\lambda \in \mathscr{H}$ such that $f_{II}(\pi) \subseteq \lambda$, hence $(f(x), f(y)) \in \lambda$ for all $y \in P$. Consequently, $f(P) \subseteq \lambda[f(x)]$, i.e. $f(P) \in \chi(f(x))$, which completes the proof. ◇

The converse of the above implication is not generally true, as the following example shows.

20. Example. Let $f : \mathbb{R} \to \mathbb{R}_+^*$ be a strictly positive function relative to the usual order of $\mathbb{R}$. For each $r > 0$ we write

$$B_f(r) = \{(x, y) \in \mathbb{R}^2 : y - x > r \cdot f(x)\}.$$

Obviously, the family $\mathscr{B}_f = \{ B_f(r) : r > 0\}$ is an ideal base in $\mathscr{P}(\mathbb{R}^2)$. The ideal generated by $\mathscr{B}_f$ is a uniform horistology, which we note $\mathscr{H}_f$. If $\mathscr{H}_f$ and $\mathscr{H}_g$ are the uniform horistologies, which correspond in this sense to the functions $f(x) = e^x$ and $g(x) = e^{-x}$, then the identity of $\mathbb{R}$ is not uniformly discrete as a function from ($\mathbb{R}$, $\mathscr{H}_f$) onto ($\mathbb{R}$, $\mathscr{H}_g$). However, it is discrete on $\mathbb{R}$ relative to the derived horistologies.

21. PROPOSITION. *The composition of uniformly discrete functions is a uniformly discrete function too*.

The proof is an immediate consequence of the definition. This property shows that the uniformly discrete functions are the morphisms of the uniform horistological worlds.

22. Remark. Let (V, L, ρ) *and* (W, K, σ) be (p-) horometric worlds, and let $\mathscr{L}$ and $\mathscr{H}$ be the corresponding uniform horistologies. The function $f : V \to W$ is uniformly discrete on V if and only

$$\forall \delta > 0 \ \ \exists \varepsilon > 0 \ \ \textit{such that} \ [\rho(x, y) > \delta \Rightarrow \sigma(f(x), f(y)) > \varepsilon].$$

For example, if there exists a constant $k > 0$ such that

$$\sigma(f(x), f(y)) > k \, \rho(x, y)$$

holds at all $(x, y) \in L$, then f is uniformly discrete. Particularly, the isometries between the (p-) horometric worlds are uniformly discrete functions (e.g. the Lorentz transformations of the Minkowskian worlds).

23. Definition. Let $(D,\le)$ be a directed set, and let (W, χ) be a horistological world. We say that $g\in W$ is a *germ* of the net $x : D \to W$, and we note

$$g = \underset{D}{germ}\ x\,,$$

if for every $a\in D$ there exists $P \in \chi(g)$ such that $b \le a$ implies $x(b)\in P$.

If the net $x : D \to W$ has a germ, we say that it is *emergent*, respectively x emerges from the germ g.

In the case $D = \mathbb{N}$, condition $g = \underset{\mathbb{N}}{germ}\ x$ means that for every $n\in\mathbb{N}$ we have

$$\{x(i) : \mathrm{i} = 1, 2, \ldots, n\} \in \chi(g).$$

The following two properties show that the notion of emergence is suitable for the study of discreteness.

24. LEMMA. *Let $(D,\le)$ be a directed set, and let (W, χ) be a horistological world. The net $x : D\to W$ is emergent from $g\in W$ if and only if its prolongation $\bar{x} : \bar{D} \to W$, defined by*

$$\bar{x}(\alpha) = \begin{cases} x(a) & \text{if } \alpha = a \in D \\ g & \text{if } \alpha = \infty \in \bar{D} \setminus D, \end{cases}$$

is discrete at ∞ relative to the natural horistology of $\bar{D}$.

We may directly base the proof on definitions, so we will omit it.

25. THEOREM. *Let (V, ψ) and (W, χ) be horistological worlds, $f : A \to W$ be a function, where $A \subseteq V$, and $x_0\in A$. Function f is discrete at x if and only if for every net $z : D \to V$, which emerges from x_0, it follows that the composed net $f \circ z : D\to W$ emerges from $f(x_0)$.*

Proof. Let us suppose that f is discrete at x_0. If the net $z : D \to V$ is emergent from x_0, according to the above lemma, $\bar{z}$ is discrete at ∞, hence applying Proposition 17, function $f \circ \bar{z} : \bar{D} \to W$ is discrete at ∞ too. Newly taking into account the lemma, we deduce that the net $f \circ z$ emerges from $f(x_0)$.

Conversely, let us consider that for each net $z : D \to V$, emergent from x_0, it follows that the net $f \circ z : D\to W$ emerges from $f(x_0)$, and let us construct a particular net of this type. Thus, we may remark that the set

$$D = \{(x, P) : x\in P \in \psi(x_0)\}$$

is directed by the relation $(y, Q) \le (x, P)$ defined by $Q \subseteq P$. Obviously, the net $z : D \to V$, of values $z(x, P) = x$, emerges from x_0, since $(y, Q) \le (x, P)$ gives $z(y, Q) = y \in P$. By hypothesis, $f \circ z$ emerges from $f(x_0)$, which means that for every $a = (x, P) \in D$ there exists $R\in \chi(f(x_0))$ such that

$$b = (y, Q) \le (x, P) = a \;\Rightarrow\; (f \circ z)(b) = f(y) \in R\,,$$

where the essential case is $P \ne \varnothing$.

Using the fact that $(y, \{y\}) \leq (x, P)$ holds for all $y \in P$, we easily deduce that for every $P \in \psi(x_0)$ and the mentioned $R \in \chi(f(x_0))$, we have $f(P) \subseteq R$. This proves that f is discrete at x_0 . ◇

The general conclusion is that starting from the Minkowskian space-time, we may construct new structures, namely the (u.) horistologies, such that the morphisms of the (u.) horistological worlds are the (u.) discrete functions. Of course, the practical utility of these mathematical structures depends on the future development of both abstract and applied sciences. For the moment, we may only hope that hereby we have satisfactorily realized the requirement of generalizing super-additivity up to some qualitative structures.

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