DISCRETE SETS OF EVENTS

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Abstract

The horistological structures turn out to be a natural framework for the study of discrete sets, which extends the finiteness. The knowledge of the family of discrete sets allows recovering the horistological structure, and reformulating the discreteness of a function. Similarly to the role of topology in the study of continuum, the horistological discreteness has a purely theoretic motivation; however, this topic is strongly connected to special relativity, discrete event systems, and other practical fields.

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In the sequel we note by S an arbitrary non-void set. The binary relation

$$\delta = \{(x, x) : x \in S\}$$

represents the *equality* on S. If K is a binary relation on S, then $\overset{0}{K} = K \backslash \delta$ is said to be the *strict* K. We use the term *order* in its strict sense, which is more suitable to *causality*. More exactly, $K \subseteq S \times S$ is said to be an *order* on S if it is *reflexive*, i.e. $\delta \subseteq K$, and *sensu stricto* (briefly *s.s.*) *transitive*, which means

$$K \circ \overset{0}{K} \subseteq \overset{0}{K} \text{ and } \overset{0}{K} \circ K \subseteq \overset{0}{K}. \qquad (0)$$

Condition (0) matter-of-factly assures the anti-symmetry ($K \cap K^{-1} = \delta$).

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The *section* of K at a point x is defined by

$$K[x] = \{y \in S : (x, y) \in K\},$$

and the *section* of K at a set $P \in \mathcal{P}(S)$ means

$$K[P] \overset{def.}{=} \{z \in S : \exists y \in P \text{ such that } (y, z) \in K\} = \cup\{K[y] : y \in P\}.$$

As usually, to each function $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ we attach the relation

$$K_\chi = \{(x, y) : \exists P \in \chi(x) \text{ such that } y \in P\} \cup \delta.$$

Taking $[BT]$ as a starting point, we remind the axioms of a horistology:

Definition 1 *Let S be a non-void set. The function $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ is called* ***horistology*** *on the set S if it satisfies the conditions:*
$[h_1]$ *$x \notin P$ for all $x \in S$ and $P \in \chi(x)$;*
$[h_2]$ *$P \in \chi(x), Q \subseteq P \Longrightarrow Q \in \chi(x)$;*
$[h_3]$ *$P, Q \in \chi(x) \Longrightarrow P \cup Q \in \chi(x)$;*
$[h_4]$ *$\forall P \in \chi(x), \exists H \in \chi(x)$ such that $[y \in P$ and $Q \in \chi(y)] \Longrightarrow [Q \subseteq H]$.*

We say that the pair (S, χ) is a *horistological space*. The conditions $[h_2]$ and $[h_3]$ show that $\chi(x)$ forms an ideal at each $x \in S$, which consists of so-called *perspectives* of x.

Example 2 *The relativist space-times represent the simplest cases of real horistological spaces. For example, $S = \mathbb{R}^2$ is a Minkowskian space-time if we endow it with the causal order*

$$K = \{((t_1, s_1), (t_2, s_2)) : t_2 - t_1 > |s_2 - s_1|\} \cup \delta,$$

and with the super-additive metric $\sigma : K \to \mathbb{R}_+$, of values

$$\sigma((t_1, s_1), (t_2, s_2)) = \sqrt{(t_2 - t_1)^2 - (s_2 - s_1)^2}.$$

Using σ we first define the hyperbolic perspectives

$$H((t_0, s_0), r) = \{(t, s) \in K[(t_0, s_0)] : \sigma((t_0, s_0), (t, s)) > r\}$$

of center (t_0, s_0) and radius $r > 0$, and finally

$$\chi((t_0, s_0)) = \{P \in \mathcal{P}(S) : \exists r > 0 \text{ such that } P \subseteq H((t_0, s_0), r)\}.$$

Similar constructions are possible in $\mathbb{R}^4$ and more general worlds of events (see $[BT]$, $[CB]$, $[B - P]$, etc.).

Proposition 3 *If (S,χ) is a horistological space, then K_χ is an order.*

The proof is immediate; condition (0) follows from $[h_4]$. In particular, in the above Minkowskian space-time we have $K_\chi = K$.

Proposition 4 *Let $\Lambda \subseteq S \times S$ be an order on S. If $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ is a horistology on S, then the restriction $\chi|_\Lambda : S \to \mathcal{P}(\mathcal{P}(S))$, of values*

$$\chi|_\Lambda(x) = \{P \cap \Lambda[x] : P \in \chi(x)\}$$

is a horistology too. In addition, $K_{\chi|_\Lambda} = \Lambda \cap K_\chi$.

Lemma 5 *If the function $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ satisfies the condition $[h_2]$, then $[h_4]$ is equivalent to:*
$[h_4^*]$ $P \in \chi(x) \Longleftrightarrow K_\chi[P] \in \chi(x)$.

Proof. Let us show that in the presence of $[h_2]$ and $[h_4]$, $P \in \chi(x)$ implies $K_\chi[P] \in \chi(x)$. In fact, using $[h_2]$, this membership follows from $K_\chi[P] \subseteq H$, where $H \in \chi(x)$ is that of $[h_4]$. To prove this inclusion, if $z \in K_\chi[P]$, then we may take $Q = \{z\}$ in $[h_4]$, and we obtain $z \in H$.

The opposite implication in $[h_4]^*$ follows from $[h_2]$ and the reflexivity of K_χ, which assures the inclusion $P \subseteq K_\chi[P]$. Consequently, $[h_4]^*$ holds.

Conversely, if we suppose the conditions $[h_2]$ and $[h_4]^*$, then $[h_4]$ holds with $H = K_\chi[P]$. In fact, from $y \in P$, $Q \in \chi(y)$, and $z \in Q$ it follows that $z \in H$, hence $Q \subseteq H$. ■

Definition 6 *Let Λ be an order on the horistological space (S,χ), such that $\Lambda \subseteq K_\chi$, and let M be a subset of S. We say that a point $x \in M$ is Λ-**detachable** from M (alternatively, M is Λ-discrete at x, etc.) if*

$$M \cap \overset{0}{\Lambda}[x] \in \chi(x).$$

*The set of all Λ-detachable points of M is called Λ-**discrete part** of M; it is noted $\partial_\Lambda(M)$. If each point of M is Λ-detachable, i.e. $\partial_\Lambda(M) = M$, then we consider that M is Λ-**discrete**. The function*

$$\partial_\Lambda : \mathcal{P}(S) \longrightarrow \mathcal{P}(S),$$

*which extracts the Λ-discrete part $\partial_\Lambda(M)$ of each subset $M \in \mathcal{P}(S)$, is called **operator of** Λ-**discreteness**.*

In the case $\Lambda = K_\chi$, we may omit mentioning Λ, and simply speak of detachability, discreteness, etc. Alternatively, we may interpret the Λ-discreteness as discreteness relative to $\chi|_\Lambda$.

Relative to this notion of discreteness we mention:

Proposition 7 *Let $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ be a horistology and let Λ be an order on S, such that $\Lambda \subseteq K_\chi$. The operator ∂_Λ has the properties:*

$[d_0]$ $\partial_\Lambda(M) \subseteq M$ *for all* $M \in \mathcal{P}(S)$*;*

$[d_1]$ *card* $M \in \mathbb{N} \Longrightarrow \partial_\Lambda(M) = M$*;*

$[d_2]$ $L \subseteq M \Longrightarrow L \cap \partial_\Lambda(M) \subseteq \partial_\Lambda(L;)$

$[d_3]$ $\partial_\Lambda(M) \cap \partial_\Lambda(L) \subseteq \partial_\Lambda(M \cup L)$*;*

$[d_4]$ $x \in \partial_\Lambda(M) \Longleftrightarrow x \in M \cap \partial_\Lambda\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right)$*;*

$[d_5]$ *For all $x \in S$ and $M \subseteq \overset{0}{\Lambda}[x]$ we have*

$$x \in \partial_\Lambda(\{x\} \cup M) \Longleftrightarrow x \in \partial_\Lambda\left(\{x\} \cup \Lambda\left[M\right]\right);$$

$[d_6]$ $\Pi \subseteq \Lambda \Longrightarrow \partial_\Lambda(M) \subseteq \partial_\Pi(M)$*;*

$[d_7]$ $\{\partial_\Lambda(M) = M$ *and* $\Pi \subseteq \Lambda\} \Longrightarrow \partial_\Pi(M) = M$*;*

$[d_8]$ $\{\partial_\Lambda(M) = M$ *and* $L \subseteq M\} \Longrightarrow \partial_\Lambda(L) = L$*;*

$[d_9]$ $\partial_\Lambda\left(\partial_\Lambda(M)\right) = \partial_\Lambda(M)$*.*

Proof. $[d_0]$ follows from the very definitions of ∂_Λ; it says that the operator ∂_Λ is contractive.

$[d_1]$ shows that the finite sets are Λ-discrete for all $\Lambda \subseteq K_\chi$. In fact, for each $y \in \overset{0}{K}_\chi$ we have $\{y\} \in \chi(x)$. Consequently, according to $[h_3]$, every finite set in $\overset{0}{K}_\chi[x]$ is also a perspective of x. In our case, M is finite, and $M \cap \overset{0}{K}_\chi[x]$ is a fortiori finite, hence $M \cap \overset{0}{K}_\chi[x] \in \chi(x)$. In addition, from $\Lambda \subseteq K_\chi$ it follows that $M \cap \overset{0}{\Lambda}[x] \subseteq M \cap \overset{0}{K}_\chi[x]$, so that according to $[h_2]$, $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$ too. By using this result at arbitrary x in M, we may conclude that M is Λ-discrete at each of its points, i.e. $\partial_\Lambda(M) = M$.

$[d_2]$ establishes that passing to smaller sets preserves the property of being detachable. It follows from the inclusion $L \cap \overset{0}{\Lambda}[x] \subseteq M \cap \overset{0}{\Lambda}[x]$ and $[h_2]$.

$[d_3]$ is a consequence of $[h_3]$. However, we mention that this inclusion is frequently strict, e.g. the case when $M = \{x\}$, and $x \notin L$.

$[d_4]$ is a characterization of the Λ-detachability. First off, let us remark that both implications in $[d_4]$ make use of a general set theoretical relation, which holds for *sensu stricto* orders, namely

$$\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right) \cap \overset{0}{\Lambda}[x] = \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]. \tag{1}$$

To justify this equality we may take into account that $x \notin \overset{0}{\Lambda}[x]$, so that a De Morgan transformation of the left hand term gives

$$\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right) \cap \overset{0}{\Lambda}[x] = \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right] \cap \overset{0}{\Lambda}[x].$$

On the other hand, it is easy to see that

$$\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right] \cap \overset{0}{\Lambda}[x] = \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right],$$

i.e. $\Lambda[M \cap \overset{0}{\Lambda}[x]] \subseteq \overset{0}{\Lambda}[x]$. In fact, if $z \in \Lambda[M \cap \overset{0}{\Lambda}[x]]$, then there is an $y \in M \cap \overset{0}{\Lambda}[x]$ such that $(y, z) \in \Lambda$. But $y \in \overset{0}{\Lambda}[x]$ means $(x, y) \in \overset{0}{\Lambda}$, so that the strict transitivity of Λ, expressed by (0), leads to $(x, z) \in \overset{0}{\Lambda}$. Equivalently, $z \in \overset{0}{\Lambda}[x]$, hence (1) is correct.

To prove "$\Longrightarrow$" we start with $x \in M$ and $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$. Using $[h_4]^*$ we obtain $K_\chi[M \cap \overset{0}{\Lambda}[x]] \in \chi(x)$, and by $[h_2]$, $\Lambda[M \cap \overset{0}{\Lambda}[x]] \in \chi(x)$. Consequently, according to definition 2, from (1) we deduce

$$x \in \partial_\Lambda\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right).$$

Finally, $x \in M$ holds by hypothesis, hence "$\Longrightarrow$" is proved.

Conversely, let us take $x \in M$ and $x \in \partial_\Lambda\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right)$. By definition, the latter membership means

$$\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right) \cap \overset{0}{\Lambda}[x] \in \chi(x),$$

or equivalently, via (1), $\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right] \in \chi(x)$.

Finally, because Λ is reflexive, we have $M \cap \overset{0}{\Lambda}[x] \subseteq \Lambda \left[M \cap \overset{0}{\Lambda}[x] \right]$, hence according to $[h_2]$ we obtain $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$. Together with $x \in M$, this means $x \in \partial_\Lambda(M)$.

$[d_5]$ essentially follows from $[h_4]^*$. By hypothesis, we have

$$(\{x\} \cup M) \cap \overset{0}{\Lambda}[x] = M \cap \overset{0}{\Lambda}[x] = M \in \chi(x).$$

According to $[h_4]^*$ and $[h_2]$, $\Lambda[M] \in \chi(x)$ too. Because Λ is s.s. transitive, we have $\Lambda[M] \subseteq \overset{0}{\Lambda}[x]$, and

$$(\{x\} \cup \Lambda[M]) \cap \overset{0}{\Lambda}[x] = \Lambda[M] \cap \overset{0}{\Lambda}[x] = \Lambda[M] \in \chi(x).$$

Consequently, $x \in \partial_\Lambda(\{x\} \cup \Lambda[M])$.

The converse implication is valid because of "$\Longleftarrow$" in $[h_4]^*$.

$[d_6]$ means that smaller orders lead to more detachable points. This fact is based on the inclusion $M \cap \overset{0}{\Pi}[x] \subseteq M \cap \overset{0}{\Lambda}[x]$ and $[h_2]$.

$[d_7]$ is a consequence of $[d_6]$, expressed in terms of discreteness.

$[d_8]$ follows from $[d_2]$, and shows that the subsets of a Λ-discrete set are also Λ-discrete.

$[d_9]$ means that ∂_Λ is idempotent. The inclusion $\partial_\Lambda(\partial_\Lambda(M)) \subseteq \partial_\Lambda(M)$ is a consequence of $[d_0]$. To obtain the opposite inclusion we may replace L by $\partial_\Lambda(M)$ in $[d_2]$, which is possible because of $[d_0]$. ■

Remark 8 *The role of the restriction $\Lambda \subseteq K_\chi$ is somehow hidden, but very important. For example, it is visible in $[d_1]$, while in the final part of the proof of $[d_4]$ it seems useless; however, intermediate results like*

$$\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \in \chi(x)$$

could lose their sense if we accept $\Lambda \nsubseteq K_\chi$.

The extreme case $\Lambda = \delta$ is still acceptable, but $\overset{0}{\delta}[x] = \varnothing$ leads to a trivial situation when all subsets of S are δ-discrete. On the other hand, simple examples show how inappropriate can be assuming $\Lambda \supset K_\chi$. In particular, let Λ be strictly wider than K in the Minkowskian space-time $(\mathbb{R}^2, \chi)$, i.e. there is some $t', s' \in \mathbb{R}$ such that $0 < t' < s'$ and

$$\Lambda[(0,0)] \supset K_\chi[(0,0)] \cup \{(t', s')\}.$$

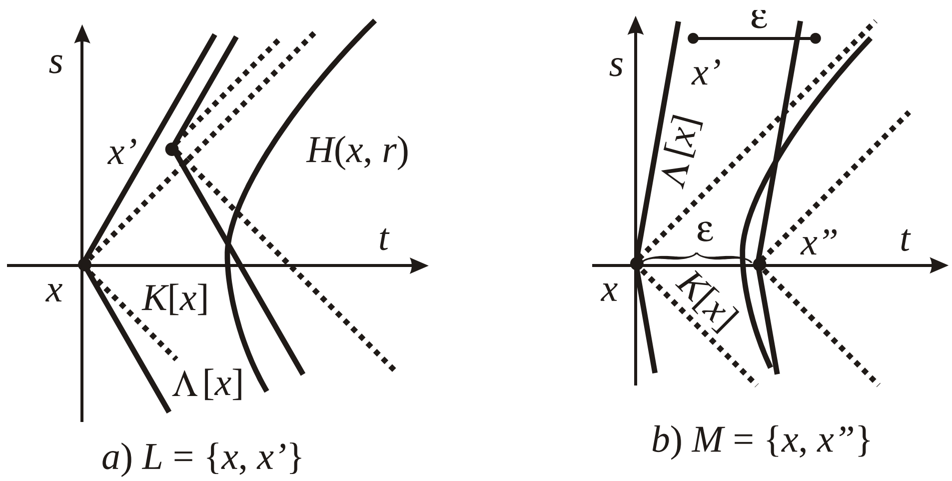

Figure 1: $\Lambda \supset K_\chi$ disturbs discreteness

For brevity we note $x = (0, 0)$, $x' = (t', s')$ *and* $x" = (\varepsilon, 0)$, *where* $\varepsilon > 0$ *is chosen such that* $0 < t' + \varepsilon < s'$ *too. Finally, let us search the sets* $L = \{x, x'\}$ *and* $M = \{x, x"\}$ *for discreteness (see Fig.1.):*

Because $L \cap \overset{0}{\Lambda}[x] = \{x'\} \notin \chi(x)$, *it follows that* $x \notin \partial_\Lambda(L)$. *This means that* L *is not discrete, contrarily to* $[d_1]$.

In the other case, $M \cap \overset{0}{\Lambda}[x] = \{x"\} \in \chi(x)$ *shows that* $x \in \partial_\Lambda(M)$. *Since also* $x" \in \partial_\Lambda(M)$, *we conclude that* M *is discrete. However, because*

$$x' + x" = (t' + \varepsilon, s') \notin K[x],$$

it follows that $\Lambda[x"] \notin \chi(x)$. *Consequently, according to* (1), *for the second part of* $[d_4]$ *we obtain*

$$\overset{0}{\Lambda}[x] \cap \left(\{x\} \cup \Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \right) = \Lambda[x"] \notin \chi(x).$$

This means that $x \notin \partial_\Lambda \left(\{x\} \cup \Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \right)$, *i.e.* $[d_4]$ *fails again.*

This remark is useful in solving the converse problem, namely to recover the horistology from a given operator of discreteness. We start this study *by selecting several properties from the above Proposition 7 as axioms of an operator of discreteness:*

Definition 9 *Let S be a non-void set. A function $\partial : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ is said to be an* ***operator of discreteness*** *if it satisfies the conditions:*

$[\partial_1]$ *card* $M \in \mathbb{N} \Longrightarrow \partial(M) = M$;

$[\partial_2]$ $L \subseteq M \Longrightarrow L \cap \partial(M) \subseteq \partial(L)$;

$[\partial_3]$ $\partial(M) \cap \partial(L) \subseteq \partial(M \cup L)$.

In addition, if Λ is a (s.s.) order on S, such that the equivalence

$[\partial_4]$ $x \in \partial(M) \Longleftrightarrow x \in M \cap \partial\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right)$

holds for all $M \in \mathcal{P}(S)$, then we say that Λ is ***compatible*** *with ∂.*

The triplet (S, ∂, Λ), where $[\partial_1] - [\partial_4]$ hold, is called *discreteness space.* As before, we say that the points $x \in \partial(M)$ are *detachable* from M, $\partial(M)$ is the *discrete part* of M, and $M \in \mathcal{P}(S)$ is a *discrete set* if $\partial(M) = M$.

Remark 10 *We will see later that the discreteness spaces always exist inside a horistological framework; more exactly, we will show that each operator of discreteness defines a horistology. Until then, it is useful to mention a couple of aspects concerning the axioms $[\partial_1] - [\partial_4]$:*

(i) According to Proposition 7, (S, χ) is a discreteness space, and $\Lambda = K$ is compatible with ∂_K. Consequently, real examples of horistological spaces (e.g. event spaces) show that the axioms $[\partial_1] - [\partial_4]$ are not contradictory.

(ii) If $\Lambda = \delta$ is compatible with an operator of discreteness ∂, then every set is discrete. This behavior is specific to the "small" orders, in the sense of cardinality. More exactly, if card $\overset{0}{\Lambda} \in \mathbb{N}$, then both $M \cap \overset{0}{\Lambda}[x]$ and $\Lambda[M \cap \overset{0}{\Lambda}[x]]$ are finite for arbitrary M, hence $\left(\{x\} \cup \Lambda[M \cap \overset{0}{\Lambda}[x]]\right)$ is discrete by virtue of $[\partial_1]$. Consequently, for each $x \in M$ we have

$$x \in M \cap \partial\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right).$$

By $[\partial_4]$, we obtain $x \in \partial(M)$, i.e. M consists of detachable points only.

(iii) If card $S \in \mathbb{N}$, then $\partial(M) = M$ holds for all $M \in \mathcal{P}(S)$, hence the operator ∂ reduces to the identity of $\mathcal{P}(S)$. In this case, large orders, e.g. $\Lambda = S \times S$, can be compatible with ∂.

Properties similar to $[d_5] - [d_9]$ can be directly deduced from $[\partial_1] - [\partial_4]$, as in the following:

Lemma 11 *If the order* Λ *is compatible with* $\partial : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$, *then*

$[\partial_5]$ $x \in \partial(\{x\} \cup M) \Longleftrightarrow x \in \partial(\{x\} \cup \Lambda[M])$ *for all* $x \in S$ *and* $M \subseteq \overset{0}{\Lambda}[x]$.

Proof. We have to show that $[\partial_5]$ follows from $[\partial_4]$ alone. In fact, $[\partial_5]$ refers to points $x \in S$ and sets $M \in \mathcal{P}(S)$, for which $M \subseteq \overset{0}{\Lambda}[x]$. In this case we have $M \cap \overset{0}{\Lambda}[x] = M$, so that $[\partial_4]$ implies $[\partial_5]$. ■

Now, let us analyze how a pair (∂, Λ) defines a horistology:

Theorem 12 *If* (S, ∂, Λ) *is a discreteness space, then the function*

$$\chi_{(\partial,\Lambda)} : S \to \mathcal{P}(\mathcal{P}(S)),$$

of values

$$\chi_{(\partial,\Lambda)}(x) = \{P \subseteq \overset{0}{\Lambda}[x] : x \in x \in \partial(\{x\} \cup P)\}, \tag{2}$$

is a horistology on S. *In addition, the proper order of* $\chi_{(\partial,\Lambda)}$ *is*

$$K_{\chi_{(\partial,\Lambda)}} = \Lambda. \tag{3}$$

Proof. We have to show that $\chi_{(\partial,\Lambda)}$ fulfils the conditions $[h_1]$, $[h_2]$, $[h_3]$, and $[h_4]$. In fact, according to (2), we have $P \subseteq \overset{0}{\Lambda}[x]$, hence $x \notin P$ for all $P \in \chi_{(\partial,\Lambda)}(x)$, as in $[h_1]$.

Condition $[h_2]$ is based on $[\partial_2]$. Primarily, if $P \in \chi_{(\partial,\Lambda)}(x)$ and $Q \subseteq P$, then $Q \subseteq \overset{0}{\Lambda}[x]$ because $P \subseteq \overset{0}{\Lambda}[x]$. To obtain $Q \in \chi_{(\partial,\Lambda)}(x)$ via (2), let us take $M = \{x\} \cup P$ and $L = \{x\} \cup Q$ in $[\partial_2]$. From $x \in L \cap \partial(M)$, we obtain

$$x \in \partial(L) = \partial(\{x\} \cup Q).$$

To prove $[h_3]$, let $P, Q \in \chi_{(\partial,\Lambda)}(x)$, i.e. according to (2), we have:

$$P \subseteq \overset{0}{\Lambda}[x],\ Q \subseteq \overset{0}{\Lambda}[x],\ x \in \partial(\{x\} \cup P) \text{ and } x \in \partial(\{x\} \cup Q).$$

It is easy to see that $P \cup Q \subseteq \overset{0}{\Lambda}[x]$, and according to $[\partial_3]$,

$$x \in \partial(\{x\} \cup P) \cap \partial(\{x\} \cup Q) \subseteq \partial(\{x\} \cup P \cup Q).$$

To conclude, $P \cup Q \in \chi_{(\partial,\Lambda)}(x)$ in accordance with (2).

Before proving $[h_4]$, we have to prove the equality (3). As usually, we may reduce this equality to a double inclusion. According to (2), the $\chi_{(\partial,\Lambda)}$–perspectives of x are parts of $\overset{0}{\Lambda}[x]$. Because this property holds for arbitrary x in S, it follows that $K_{\chi_{(\partial,\Lambda)}} \subseteq \Lambda$. The opposite inclusion is a consequence of $[\partial_1]$. More exactly, for each $(x,y) \in \overset{0}{\Lambda}$, we obviously have $card\ \{x,y\} = 2$. If we note $P = \{y\}$, then the conditions in (2) are satisfied in the sense that $P \subseteq \overset{0}{\Lambda}[x]$ and

$$x \in \partial(\{x\} \cup P) \equiv \partial(\{x,y\}) \overset{[\partial_1]}{=} \{x,y\}.$$

Consequently, $\{y\} \in \chi_{(\partial,\Lambda)}(x)$, hence $(x,y) \in \overset{0}{K}_{\chi_{(\partial,\Lambda)}}$, which proves (3).

The equality (3) is tacitly used in the proof of $[h_4]$. More exactly, based on Lemma 5, with $K_{\chi_{(\partial,\Lambda)}} = \Lambda$, we may prove $[h_4]^*$ instead of $[h_4]$.

Part "$\Longrightarrow$" in $[h_4]^*$ is based on Lemma 11 and s.s. transitivity. In fact,

$$x \in \partial(\{x\} \cup P) \Longrightarrow x \in \partial(\{x\} \cup \Lambda[P])$$

is a consequence of $P \subseteq \overset{0}{\Lambda}[x]$ and $[\partial_5]$. In addition, $\Lambda[P] \subseteq \overset{0}{\Lambda}[x]$ follows from s.s. transitivity, hence $\Lambda[P] \in \chi_{(\partial,\Lambda)}(x)$.

Part "$\Longleftarrow$" is based on the reflexivity of Λ, which leads to $P \subseteq \Lambda[P]$, so that we may use the fact that $[\partial_2]$ implies $[h_2]$. ■

Corollary 13 *Let (S,χ) be a horistological space, and let $K = K_\chi$ be the order attached to χ. If ∂ is the operator of discreteness in (S,χ), relative to K, then K is compatible with ∂, and*

$$\chi_{(\partial,K)} = \chi. \tag{4}$$

Proof. As mentioned in Remark 10 (i), the compatibility of K with ∂ follows from the property $[d_4]$ in Proposition 7.

To prove (4), let us remind that by virtue of Theorem 12, $\chi_{(\partial,K)}$ really is a horistology, and its proper order is K. Because the equality (4) refers to functions on S, we have to prove that a set equality

$$\chi_{(\partial,K)}(x) = \chi(x)$$

holds at each $x \in S$. As usually, we will prove two inclusions:

Part "$\subseteq$": According to the rule (2) of deriving a horistology from ∂, if $P \in \chi_{(\partial,K)}(x)$, then $P \subseteq \overset{0}{K}[x]$, and $x \in \partial(\{x\} \cup P)$. Because ∂ derives from χ, the condition $x \in \partial(\{x\} \cup P)$ means

$$(\{x\} \cup P) \cap \overset{0}{K}[x] \in \chi(x).$$

Obviously, $(\{x\} \cup P) \cap \overset{0}{K}[x] = P$, hence $\chi_{(\partial,K)}(x) \subseteq \chi(x)$.

Part "$\supseteq$": If $P \in \chi(x)$, then $P \subseteq \overset{0}{K}[x]$, hence $(\{x\} \cup P) \cap \overset{0}{K}[x] = P$. Consequently, $x \in \partial(\{x\} \cup P)$, so that using (2), we obtain $P \in \chi_{(\partial,K)}(x)$. This proves the inclusion $\chi_{(\partial,K)}(x) \supseteq \chi(x)$. ■

Corollary 14 *Let (S, ∂, Λ) be a discreteness space, and let $\chi_{(\partial,\Lambda)}$ be the attached horistology via Theorem 12. If ∂_Λ is the operator of discreteness in the horistological space $(S, \chi_{(\partial,\Lambda)})$, then*

$$\partial_\Lambda = \partial. \tag{5}$$

Proof. ∂_Λ and ∂ are functions on $\mathcal{P}(S)$, hence (5) means $\partial_\Lambda(M) = \partial(M)$ at each $M \in \mathcal{P}(S)$. The proof has two parts again:

Part "$\subseteq$": Let $x \in \partial_\Lambda(M)$ be arbitrary. According to Theorem 12, the proper order of $\chi_{(\partial,\Lambda)}$ is Λ, hence by Definition 6, we have $x \in M$ and $P \overset{not.}{=} M \cap \overset{0}{\Lambda}[x] \in \chi_{(\partial,\Lambda)}(x)$. Taking into account the formula (2), which describes the perspectives of x in the horistology $\chi_{(\partial,\Lambda)}$, we see that $P \subseteq \overset{0}{\Lambda}[x]$ and $x \in \partial(\{x\} \cup P)$. Now, we may use Lemma 11: Replacing M by P in the left hand side of $[\partial_5]$, we obtain

$$x \in \partial\left(\{x\} \cup \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right).$$

Finally, using the implication "$\Longleftarrow$" in $[\partial_4]$, we conclude that $x \in \partial(M)$.

Part "$\supseteq$" is based on the opposite implications in $[\partial_4]$ and $[\partial_5]$. ■

Remark 15 *The two corollaries from above are customarily interpreted in the sense that the perspectives and the discrete sets are equivalent methods for defining a horistology. Other equivalent ways to introduce horistologies can be found in [BT] and [PM]. All these results reveal a deep similarity between topology, viewed as theory of continuity, and horistology, which turns out to be very adequate for the study of discreteness.*

Besides their structural importance, the discrete sets allow a simple form to the discreteness of a function. In this respect we remind (see $[BT]$, etc.):

Definition 16 *Let* $\left(S_1, \chi_1\right)$ *and* $\left(S_2, \chi_2\right)$ *be horistological spaces. We say that function* $f : S_1 \to S_2$ *is discrete at a point* $x \in S_1$ *if*

$$f(\chi_1(x)) \subseteq \chi_2(f(x)).$$

If this property holds at each $x \in S_1$, then we consider that f is *discrete on the space* S_1. We reduce the discreteness on a subset $A \subset S_1$ to the previous one, where A is considered a horistological subspace (in the sense of $[BT_1]$, etc.).

Lemma 17 *Let* $\left(S_1, \chi_1\right)$ *and* $\left(S_2, \chi_2\right)$ *be horistological spaces. If the function* $f : S_1 \to S_2$ *is discrete at* $x \in S_1$*, then*

$$f\left(\overset{0}{K}_{\chi_1}[x]\right) \subseteq \overset{0}{K}_{\chi_2}[f(x)].$$

In addition, if f *is discrete on the whole* S_1*, then*

$$\Lambda \overset{not.}{=} f_{II}(K_{\chi_1}) \overset{def.}{=}$$

$$\left\{(X,Y) \in S_2 \times S_2 : \exists (x,y) \in K_{\chi_1} \text{ such that } X = f(x) \text{ and } Y = f(y)\right\}$$

is a (s.s.) order on S_2*, and* $\Lambda \subseteq K_{\chi_2}$*.*

Proof. Because χ_1 satisfies $[h_2]$, we have

$$\overset{0}{K}_{\chi_1}[x] = \{y \in S_1 : \{y\} \in \chi_1(x)\},$$

so that

$$f\left(\overset{0}{K}_{\chi_1}[x]\right) = \{z \in S_2 : \exists y \in S_1 \text{ such that } \{y\} \in \chi_1(x) \text{ and } z = f(y)\}.$$

Using the discreteness of f again, we obtain $\{f(y)\} \in \chi_2(f(x))$, hence

$$z \overset{not.}{=} f(y) \in \overset{0}{K}_{\chi_2}[f(x)].$$

To prove the second assertion, we first remark the reflexivity of Λ, then we examine condition (0). If, for example, $(X,Y)\in\Lambda$ and $(Y,Z)\in\overset{0}{\Lambda}$, then there exist $x,y,z\in S_1$ such that $X=f(x)$, $Y=f(y)$, $Z=f(z)$, $(x,y)\in K_{\chi_1}$ and $(y,z)\in\overset{0}{K}_{\chi_1}$. Because K_{χ_1} is s.s. transitive, we deduce that $(x,z)\in\overset{0}{K}_{\chi_1}$, hence $(X,Z)\in\overset{0}{\Lambda}$. The case $(X,Y)\in\overset{0}{\Lambda}$ and $(Y,Z)\in\Lambda$ is similar.

Finally, let us take an arbitrary $(X,Y)\in\overset{0}{\Lambda}$. The corresponding x and y satisfy $(x,y)\in\overset{0}{K}_{\chi_1}$, hence $\{y\}\in\chi_1(x)$. Because f is discrete at x, we have $\{Y\}\in\chi_2(X)$, i.e. $(X,Y)\in\overset{0}{K}_{\chi_2}$. ■

Theorem 18 *Let (S_1,χ_1) and (S_2,χ_2) be horistological spaces, and let the function $f:S_1\to S_2$ be $1:1$ and discrete on S_1. If $x\in M$ is detachable from the set $M\in\mathcal{P}(S_1)$, then $f(x)$ is Λ-detachable from $f(M)$ in S_2, where*

$$\Lambda=f_{II}(K_{\chi_1}).$$

Proof. By hypothesis, $M\cap\overset{0}{K}_{\chi_1}[x]\in\chi_1(x)$, where $x\in M$. The discreteness of f at x leads to $f\left(M\cap\overset{0}{K}_{\chi_1}[x]\right)\in\chi_2(f(x))$. Because f is $1:1$, we have

$$f\left(M\cap\overset{0}{K}_{\chi_1}[x]\right)=f(M)\cap f\left(\overset{0}{K}_{\chi_1}[x]\right).$$

Based on Lemma 17, we may easily see that

$$f\left(\overset{0}{K}_{\chi_1}[x]\right)=\overset{0}{\Lambda}[f(x)].$$

Consequently, $f(M)\cap\overset{0}{\Lambda}[f(x)]\in\chi_2(f(x))$, i.e.$f(x)$, which obviously is a member of $f(M)$, is Λ-detachable from $f(M)$. ■

The property of f in this theorem justifies the following:

Definition 19 *Let $(S_1,\partial_1,\Lambda_1)$ and $(S_2,\partial_2,\Lambda_2)$ be discreteness spaces. If the function $f:S_1\to S_2$ satisfies the condition*

$$f(\partial_1(M))\subseteq\partial_2(f(M))\tag{6}$$

for all $M \subseteq S_1$, then f is said to be a ***detachability preserving*** *function.*

Alternatively, if the implication

$$M = \partial_1(M) \Longrightarrow f(M) = \partial_2(f(M)) \tag{7}$$

is valid at each $M \subseteq S_1$, then f is named ***discreteness preserving*** *function.*

In the conditions of Theorem 18, f preserves discreteness too, namely:

Corollary 20 *Let (S_1, χ_1) and (S_2, χ_2) be horistological spaces, and let the function $f : S_1 \to S_2$ be $1:1$ and discrete on S_1. If ∂_1 and ∂_2 denote the discreteness operators on (S_1, χ_1) and $(S_2, \chi_2|_\Lambda)$, where $\Lambda = f_{II}(K_{\chi_1})$, then function f is discreteness preserving.*

Proof. The hypothesis $M = \partial_1(M)$ means that each $x \in M$ is detachable from M. According to the previous theorem, respectively (6), it follows that each $f(x)$ is Λ-detachable from $f(M)$. In other words, $f(M)$ is ∂_2-discrete, i.e. the conclusion in (7) is correct. ■

The discreteness preserving functions are specific to spaces of events. In particular, the following property refers to $\mathbb{R}^4$, but it is valid in much more general cases where Zeeman's Theorem holds (see [BT], [BT_2], etc.).

Corollary 21 *The causal automorphisms of the Minkowskian space-time $\mathbb{R}^4$ preserve the discreteness of the sets relative to the intrinsic horistology.*

Proof. According to[ZEC], the group of causal automorphisms consists of Lorentz transformations, translations and dilations, which are all discrete functions. In addition, because $f_{II}(K_\chi) = K_\chi$ holds for each causal automorphism f, we may use Theorem 18 to conclude that these functions do preserve the discreteness. ■

Theorem 22 *Let (S_1, χ_1) and (S_2, χ_2) be horistological spaces, and let ∂_1 and ∂_2 be the corresponding discreteness operators. If $f : S_1 \to S_2$ is a strictly monotonic function relative to the orders K_{χ_1} and K_{χ_2}, which preserves detachability, then f is discrete on S_1.*

Proof. Condition (6) holds by hypothesis, i.e. the implication

$$x \in \partial_1(M) \Longrightarrow f(x) \in \partial_2(f(M))$$

is valid at every $x \in M$ and $M \in \mathcal{P}(S_1)$. According to Definition 16, we have to show that

$$\forall x \in S_1 \text{ and } \forall P \in \chi_1(x) \Longrightarrow f(P) \in \chi_2(f(x)).$$

Because $P \subseteq \overset{0}{K}_{\chi_1}[x]$ holds for all $P \in \chi_1(x)$, it follows that x is detachable from $M = P \cup \{x\}$. In other words, $x \in \partial_1(M)$, hence by hypothesis,

$$f(x) \in \partial_2(f(M)).$$

If we remind Definition 6, we may transform this membership into

$$f(M) \cap \overset{0}{K}_{\chi_2}[f(x)] \in \chi_2(f(x)).$$

The proof is complete if we remark that $f(M) \cap \overset{0}{K}_{\chi_2}[f(x)] = f(P)$. In fact, because $f(M) = f(P) \cup \{f(x)\}$, the problem reduces to the inclusion

$$f(P) \subseteq \overset{0}{K}_{\chi_2}[f(x)].$$

Routinely, if $Y \in f(P)$, then there exists $y \in P$ such that $Y = f(y)$. Using the monotony of f, from $(x, y) \in \overset{0}{K}_{\chi_1}$ we deduce $(f(x), f(y)) \in \overset{0}{K}_{\chi_2}$, hence $Y \in \overset{0}{K}_{\chi_2}[f(x)]$. ■

We may combine Theorems 18 and 22 in the following:

Corollary 23 *Let (S_1, χ_1) and (S_2, χ_2) be horistological spaces, and let the function $f : S_1 \to S_2$ be $1:1$ and strictly monotonic relative to the orders K_{χ_1} and K_{χ_2}. If in addition $f_{II}(K_{\chi_1}) = K_{\chi_2}$, then*

$$(f \text{ is discrete on } S_1) \Longrightarrow (f \text{ preserves detachability}).$$

The proof is immediate.

Conclusion 24 *The above properties of the discrete sets, together with those of the discrete functions, may enforce the idea that the horistologies really represent structures of discreteness. The analogy with topological families of sets (e.g. open) completes the duality continuity – discreteness. The most significant interpretations are expected in domains where the horistologies represent intrinsic structures, e.g. relativity (as in Corollary 21 from above, etc.), concave programming (see $[B - C]$, etc.), discrete system theory (see $[B - P_1]$, etc.), L^p duality theory for $p < 1$ (see $[CB]$), etc.*

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