DISCRETE INSTABILITY

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Abstract We analyze a variant of instability, in which instead of non-continuity of the function *initial state – evolution*, we propose the dual of continuity, known as *discreteness*. The aim is to refine the stability theory and to gain new instruments to investigate concrete cases.

Key words Instability; discreteness; super-additivity; horistology

1. Introduction. Even if restricted to the Lyapunov's sense, the notion of stability has plenty of meanings. For example, the standard form of the stability of a linear system (see [Ionescu, 1985], etc) reduces to the boundedness of the evolutions, while for nonlinear smooth systems (see [Belea, 1985], etc.) it is expressed by the continuity of the map *initial state – evolution*. Many variants have been produced by altering the space of functions that stand for evolutions, or by stressing on particular components of the evolution. The common feature of all these variants, which was therefore assumed in the most general system theory, is that of continuity of a particular function. More exactly (according to [Mesarovic et al. 1975], [Belea 1985], etc.), the *evolution* of a dynamical system is defined as a function $x:T\longrightarrow\mathcal{X}$, where $T\subseteq\mathbb{R}$ is the *time set*, and $\mathcal{X}$ is the set of *states*. Most frequently, we have $T = [t_0 ,\longrightarrow)$ for some $t_0 \in\mathbb{R}$, where t_0 is referred as *initial moment*. Correspondingly, $x_0 = x(t_0)$ is called *initial state*. The further change of states is described by an internal rule of state transition, λ: $\mathcal{X}$x$K \longrightarrow\mathcal{X}$, where K is the usual order on T (i.e. induced from $\mathbb{R}$), and $y = \lambda(x, t_1, t_2)$ means that state x at the moment t_1 is transformed into state y at the later moment t_2. Such a dynamical system is considered with *time evolution*, and it is shortly noted as a triplet $(\mathcal{X}, T, \lambda)$.

Generalizing the case of smooth systems, where the evolutions are solutions of some differential equations, it is always assumed that the evolutions are uniquely determined by the initial states via the internal rule of state transition. This correspondence, noted

$$\Psi: \mathcal{X}\longrightarrow\mathcal{X}^T$$

is known as *initial state – evolution* function, and the notation $x = \Psi(x_0)$ shows that the evolution x starts with the initial value x_0. Consequently, we get another way to specify a dynamical system, namely $(\mathcal{X}, T, \Psi)$.

2. Stability. The notion of stability involves particular structures on $\mathcal{X}$, e.g. those of a normed linear space. The norm is used to produce topologies on $\mathcal{X}$ and $\mathcal{X}^T$, which are involved in the condition of continuity. To simplify the formalism of stability, using linearity, we may always reduce the problem to the (stationary, i.e. constant) null solution θ, where $\theta(t) = 0$ for all $t \geq t_0$. More explicitly, the null evolution $\theta \in\mathcal{X}^T$ is said to be *stable* iff Ψ is continuous at $0 = \theta(t_0) \in\mathcal{X}$, i.e.

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{\|x_0\|<\delta}\ \forall_{t\geq t_0}\ \Rightarrow \|\Psi(x_0)(t)\| < \varepsilon .$$

As far as we know, the notion of *instability* was reduced to NON-continuity, and most frequently to the exact negation of the condition from above. Our aim in this paper is to conceive instability by a condition dual but not opposite to continuity. In our opinion, the dual of continuity is *discreteness* (see [Bălan 1992, Part II], [Bălan et al. 2002], etc.).

3. Discreteness. Similarly to the continuous functions, which are the morphisms of the topological structures, the *discrete* functions are conceived as the morphisms of the *horistological* spaces, introduced by [Bălan 1992] as qualitative structures of the super-additivity. In other words, discreteness is dual to continuity since the horistologies are dual to the topologies, as we can see from the very starting definition, where the filters of neighborhoods are replaced by ideals of *perspectives*.

We remember that a *horistology* on $\mathcal{X}$ is a function

$$\chi: \mathcal{X}\longrightarrow\mathscr{P}(\mathscr{P}(\mathcal{X})),$$

which attaches a family of perspectives $\chi(e)\subset\mathscr{P}(\mathcal{X})$ to each $e\in\mathcal{X}$, such that the following conditions (easily seen to be dual to those that occur in the definition of a topology) are fulfilled:

[h$_1$] $\forall_{e\in\mathcal{X}} \forall_{P\in\chi(e)} \Rightarrow e \notin P$;

[h$_2$] $P, Q \in \chi(e) \Rightarrow P \cup Q \in \chi(e)$;

[h$_3$] $P \in \chi(e), Q \subseteq P \Rightarrow Q \in \chi(e)$;

[h$_4$] $\forall_{P\in\chi(e)} \exists_{W\in\chi(e)} \forall_{v\in P} \forall_{Q\in\chi(v)} \Rightarrow Q \subseteq W$.

The pair $(\mathcal{X}, \chi)$ is called *horistological space*.

It is significant to recall that each horistological space is endowed with a *proper order*, defined by

$$\Pi_\chi = \{(e,v) \in \mathcal{X}^2 : \{v\} \in \chi(e)\} \cup \Delta,$$

where $\Delta = \{(e, e): e \in \mathcal{X}\}$ is called *diagonal* of $\mathcal{X}^2$, and represents the equality on $\mathcal{X}$.

If $(\mathcal{X}, \chi)$ and $(\mathcal{Y}, \xi)$ are two horistological spaces, then function $f: \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be *discrete* at $e \in \mathcal{X}$ iff $f(P) \in \xi(f(e))$ whenever $P \in \chi(e)$. If f is discrete at each $e \in \mathcal{D} \subseteq \mathcal{X}$, we say that f is discrete on $\mathcal{D}$. In this case f is monotonic on $\mathcal{D}$ relative to the proper orders of the horistologies χ and ξ.

It is remarkable that differently from the continuous functions, for which the counter-images of the $f(x)$-neighborhoods are asked to be x-neighborhoods, the discrete functions directly carry e-perspectives into $f(e)$-perspectives (compare to *boundedness*, *Darboux property*, etc.).

4. Concrete horistologies. For the sake of explicitness, we recall that *sub-additivity* refers to the usual triangle's rule of a metric. By duality, $\rho: K \longrightarrow \mathbb{R}_+$, where K is an order relation on $\mathcal{X}$, is named *super-additive* (briefly Sa) *metric* on $\mathcal{X}$, iff the following conditions hold:

[M$_1$] $\rho(x, y) = 0$ iff $x = y$, and

[M$_2$] $\rho(x, y) \geq \rho(x, z) + \rho(z, y)$ at all $(x, z), (z, y) \in K$.

Obviously, Sa metrics cannot be defined on the whole $\mathcal{X}^2$; the orders are their most natural domains.

In [Bălan 1992, part I] we find enough examples to conclude that all sub-additive norms/metrics have dual super-additive norms/metrics. Therefore topological and horistological structures on the same space generally can be coupled in pairs. In fact, super-additivity leads to horistology in a way similar to the construction of a metric topology. More exactly, if K is an order on $\mathcal{X}$, and $\rho: K \longrightarrow \mathbb{R}_+$ is a super-additive metric, then we may take $P \subset \mathcal{X}$ to be a perspective of $e \in \mathcal{X}$ iff there is some $\varepsilon > 0$ such that $P \subseteq H(e, \varepsilon)$, where

$$H(e, \varepsilon) = \{v \in \mathcal{X}: \rho(e, v) > \varepsilon\}$$

is a *hyperbolic perspective* of e, of *radius* ε.

It is significant to mention that $K = \Pi_\chi$ whenever χ is generated by the S.a. metric $\rho: K \longrightarrow \mathbb{R}_+$.

Because the study of the practical problems usually involves measurements, the metric horistologies will be of primary interest in the context of (in)stability too. The following examples of super-additive metrics, and corresponding horistologies, which are mentioned in the previous works too, seem to be particularly useful in the study of (in)stability:

(i) $\mathcal{X} = \mathbb{R}$, K is the natural order on $\mathbb{R}$, and

$$\rho(x, y) = y - x$$

at any $(x, y) \in K$. Consequently, $P \subset \mathcal{X}$ is a perspective of $x \in \mathbb{R}$ iff $P \subseteq [y, +\infty)$ for some $y > x$.

(ii) $\mathcal{X} = \mathbb{R}^2$, K is the product order, i.e.

$$K = \{((x, y), (u, v)): x < u, y < v\} \cup \Delta,$$

and ρ is the *hyperbolic* metric of values

$$\rho((x, y), (u, v)) = (u - x)(v - y).$$

(iii) $\mathcal{X} = \mathbb{R}^2$, K is the product order, and

$$\delta((x, y), (u, v)) = \min\{u - x, v - y\}.$$

(iv) $\mathcal{X} = \mathbb{R}^n$, where $n \in \mathbb{N} \setminus \{0, 1, 2\}$, K is the product order, and the super-additive metrics are similar to ρ and δ from the cases (*ii*) and (*iii*).

(v) $\mathcal{X} = \mathbb{R}^T$, Λ is the functional product order,

$$\Lambda = \{(x, y) \in \mathcal{X}^2 : \exists_{\eta>0} \forall_{t\in T} \Rightarrow x(t) + \eta < y(t)\} \cup \Delta,$$

and

$$\sigma(x, y) = \inf\{y(t) - x(t): t \in T\}.$$

It is easy to identify the usual (i.e. sub-additive) metrics, which are dual to the examples from above. In addition, the dual is not uniquely determined. For example, the Euclidean metric on $\mathbb{R}^n$, where $n \in \mathbb{N}^*$, can be conceived as dual to several super-additive metrics, e.g.

$$\mu((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \left(\sum_{k=1}^{n} \sqrt{y_k - x_k}\right)^2,$$

or

$$\nu((t, x_1, \ldots, x_n), (\tau, y_1, \ldots, y_n)) = \left[c^2(\tau - t)^2 - \sum_{k=1}^{n}(y_k - x_k)^2\right]^{1/2},$$

where $c > 0$, and the other elements take physical significance in the special relativity (e.g. see [Bălan 2001], etc.).

5. Instability. The null evolution $\theta \in \mathcal{X}^T$ of a dynamical system $(\mathcal{X}, T, \Psi)$ is said to be *discretely instable* (briefly d.i.) iff Ψ is discrete at the initial state $0 \in \mathcal{X}$. Because discreteness involves particular horistologies, say χ on $\mathcal{X}$ and ξ on $\mathcal{X}^T$, we can be more specific and mention it as $\chi - \xi$ d.i.

Because no relationship between topologies and horistologies, including duality, is a priori imposed on individual spaces, stability and discrete instability are in principle independent properties of the dynamical systems. In particular, we may expect them to hold in particular cases simultaneously.

6. The mathematical pendulum is generally agreed as a standard example in the dichotomy *stability – instability*, because the normal pendulum is stable, but the reversed one is not. We will show that the reversed pendulum is a very natural example of d.i. system too.

Let us remember that the free pendulum of length l, in the gravitational field of acceleration g, evolves in accordance to the equation

$$\varphi'' - \omega^2 \sin \varphi = 0,$$

where $\omega^2 = g/l$, and φ is the angle between the rod and the vertical through the fixed end of the rod, directed upwards (the value of the mass m, carried at the other end of the rod, does not influence on the evolution). This equation is non-linear and difficult to solve, but we may change the unknown function, and introduce

$$\zeta(\varphi) = \varphi'(t).$$

By integrating the resulting equation

$$\zeta' \zeta - \omega^2 \sin \varphi = 0$$

we obtain the relation

$$\zeta^2(\varphi) = -2\omega^2 \cos \varphi + C.$$

In order for us to simplify the further analysis, let us consider the *non-struck* pendulum, for which there is no impulse ($\varphi' = 0$) at the initial state $\varphi = \varphi_0$. Then we have

$$\zeta^2(\varphi) = 2\omega^2 (\cos \varphi_0 - \cos \varphi),$$

where φ_0 stands for the constant of integration. To keep up the evolution in the real range (i.e. to avoid complex quantities) we have to restrict $\varphi \in [\varphi_0, 2\pi - \varphi_0]$, so that

$$\cos \varphi \le \cos \varphi_0 .$$

It is easy to see that $\varphi_1(t) = 0$ and $\varphi_2(t) = \pi$ are stationary evolutions of this pendulum, which correspond to the initial states 0, respectively π.

Now, to speak in dynamic system language, we shell take $\mathcal{X} = [0, 2\pi] \subset \mathbb{R}$, K and ρ as in the above example (*i*), $T = \mathbb{R}_+$, i.e. $t_0 = 0$, and finally Λ and σ from example (*v*) on $\mathcal{X}^T \cap \mathbf{C}^2(T)$. According to the very definition, φ_1 is a d.i. evolution relative to the horistologies generated by ρ and σ. On the other hand, φ_2 hasn't this property, but it is stable relative to the usual topological structures on $\mathcal{X}$ and $\mathcal{X}^T$ (e.g. see [Barbu 1985], etc.).

7. Time invariant linear systems. It is well known (see [Ionescu 1985], [Barbu 1985], etc.) that a linear system of the form

$$x' = Ax + Bu ,$$

where A and B are constant matrices, and u is the input, has the solution (state evolution)

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau .$$

A remarkable theorem states that such a system is (internally) stable iff the proper values of A all have negative real parts, and those with null real parts are simple. Taking into account the form of the fundamental matrix, stability is immediately reduced to the property of boundedness for some exponential functions. Consequently, a single proper value with strictly positive real part makes the system non-stable, while d.i. holds iff all proper values of A are real and strictly positive. More exactly, because x is a vector function, the product horistological structure of $\mathcal{X}^T$ (similar to (*iv*)) asks each component of x to be discrete in the sense of (*v*). This is obviously possible for increasing exponentials only. To conclude, in this case, d.i. is much stronger than non-stability.

8. Concomitance of stability and d.i. is possible in spite of their opposite nature. For example, let us consider the second order linear differential equation

$$x'' \cosh t + 2x' \sinh t = 0,$$

where $t \ge t_0 = 0$, and $x \in \mathcal{X} = \mathbf{C}^2_{\mathbb{R}}(T)$. If we note

$$x' = y,$$

then the equation becomes

$$y' + 2y \tanh t = 0,$$

which can be integrated, and we deduce

$$y(t) = C_1 \cosh^{-2} t,$$

so that finally we obtain the general solution

$$x(t) = C_1 \tanh t + C_2.$$

Obviously, $C_2 = x(0)$, and $C_1 = x'(0) = y(0)$, and the null evolution corresponds to null initial conditions

$$x(0) = x'(0) = 0.$$

According to the property of the function tanh of being bounded, the double inequality

$$C_2 \le x(t) \le C_1 + C_2$$

holds at any $t \ge 0$. Using it we can easily show that the null solution is concomitantly stable and discretely instable. In particular, the discrete instability refers to the horistology in the example (*iii*), at the initial conditions

$$(x(0), x'(0)) \in \mathbb{R}^2 .$$

9. Symmetric d.i. It is easy to see that to each Sa metric

$$\rho: K \longrightarrow \mathbb{R}_+ ,$$

there corresponds a *symmetric* Sa metric

$$\rho^* : K^{-1} \longrightarrow \mathbb{R}_+ ,$$

which takes the values $\rho^*(x, y) = \rho(y, x)$. Alternatively, if we reformulate condition [M_2], we may speak of a symmetric Sa metric from the very beginning, but we prefer to distinguish two *symmetric* horistologies, χ and χ^*, generated by ρ and respectively ρ^*. The existence of a symmetric horistology is essential for a horistology to be uniform / metric (see [Bălan 1992*]).

Because the metrical horistologies always appear in symmetric pairs, the property of discreteness, and in particular d.i., can be studied in pairs of horistologies. For example, the study of the pendulum can be similarly developed for $\varphi \in [-2\pi, 0]$, and the result is the same. Reversing time in linear systems may affect the physical meaning, but it is possible in principle, and the resulting properties are similar to the increasing time.

Besides symmetric pairs, the examples from above show that we obtain plenty of variants of d.i. by altering the Sa metrics, hence the horistologies.

10. Preserving d.i. Changing the horistology naturally affects the discrete instability, but it is possible to make changes that preserve d.i., based on *comparable* horistologies. A horistology χ^- is said to be *smaller* than χ on $\mathcal{X}$ iff $\chi^-(e) \subseteq \chi(e)$ at any $e \in \mathcal{X}$; in this case we note $\chi^- \subseteq \chi$. Similarly we define a *greater* horistology, which is noted $\chi^+ \supseteq \chi$.

Let ($\mathcal{X}$, T, Ψ) be a $\chi - \xi$ discretely instable dynamical system (its null evolution is discretely instable relative to the horistologies χ on $\mathcal{X}$ and ξ on $\mathcal{X}^T$). If $\chi^- \subseteq \chi$, and $\xi^+ \supseteq \xi$, then the system ($\mathcal{X}$, T, Ψ) will be $\chi^- - \xi^+$ discretely instable too.

The proof of this property reduces to the more general fact that such a change of horistologies preserves the discreteness of Ψ at 0.

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