Admitted Events and Discreteness

Maria PREDOI and Trandafir BALAN*

Abstract

The purpose of this paper is to define and study the *admitted events* in the framework of the *horistological worlds* (defined in [BT]). The most important properties are connected to discrete functions and discrete sets (studied in [B-P]). The *operator of admittance*, which attaches to each set the ensemble of its admitted events, turns out to allow the reconstruction of the initial horistological structure.

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1 Introduction

In the present work we first define the admitted events, then we study the operator of admittance with the aim of clarifying its relations with the basic horistological structures. The terminology is inspired from the relativist physics, e.g. we speak of events, perspectives, causality, etc., but for the sake of rigor, we will formulate the notions and the properties in the most general framework of horistological structures. However, the discourse may change its practical significance if referred to other particular objects, for example sentences in a deductive scheme, states of a dynamic system, etc.

In this introductory part we recall some notions and results from [BT] and [B-P] as they are used later. The second section is dedicated to the study of the admitted events and the operator of admittance in horistological spaces. In the final section we identify the axioms of an abstract operator of admittance, and we analyze the problem of reconstructing a horistology from such an operator.

*e-mail: mpred@central.ucv.ro, ttbalan@yahoo.com

We will use the similar topic from topology as a model for our study: It is well-known that, except neighborhoods, we may define a topological structure by many other tools, e.g. convergent nets, open/closed sets, and operators of interior/adherence (see [KJ], [PG], etc.). By the investigation of the same problem in horistology we aim to reinforce the idea that the horistologies - as structures of discreteness, and the topologies - as structures of continuum, are dual in many respects.

The binary relations play an important role in the sequel. In particular,

$$\delta = \{(x, x) : x \in S\}$$

represents the *equality* on S. If K is a binary relation on S, then $\overset{0}{K} = K \setminus \delta$ denotes the so-called *strict* K. The *section* (*cut*, *cone*, etc.) of K at a point x is defined by

$$K[x] = \{y \in S : (x, y) \in K\}.$$

By extension, the *section* of K at a set M is

$$K[M] = \cup\{K[x] : x \in M\}.$$

We use the term *order* in its strict sense, which is more suitable to *causality*. More exactly, we say that $K \subseteq S \times S$ is an *order* on S if it is *reflexive*, i.e. $\delta \subseteq K$, and *sensu stricto* (briefly *s.s.*) *transitive*, which means

$$K \circ \overset{0}{K} \subseteq \overset{0}{K} \text{ and } \overset{0}{K} \circ K \subseteq \overset{0}{K}.$$

Obviously, if K is s.s. transitive, then it is *antisymmetric*, i.e. $K \cap K^{-1} = \delta$.

Definition 1.1 *Let S be an arbitrary non-void set. We say that the function*

$$\chi : S \to \mathcal{P}(\mathcal{P}(S))$$

is a ***horistology*** *on S iff it satisfies the conditions:*

$[h_1]$ $x \notin P$ *for all* $x \in S$ *and* $P \in \chi(x)$;

$[h_2]$ $P \in \chi(x), Q \subseteq P \Longrightarrow Q \in \chi(x)$;

$[h_3]$ $P, Q \in \chi(x), \Longrightarrow P \cup Q \in \chi(x)$;

$[h_4]$ $\forall P \in \chi(x), \exists H \in \chi(x)$ *such that* $[y \in P$ *and* $Q \in \chi(y)] \Longrightarrow [Q \subseteq H]$.

We say that the pair (S, χ) is a *horistological space*. The conditions $[h_2]$ and $[h_3]$ show that $\chi(x)$ forms an ideal at each $x \in S$. The elements of $\chi(x)$ are called *perspectives* of x, respectively x is considered to be a *premise* of each $P \in \chi(x)$.

It is easy to see that:

Proposition 1.2 *If (S, χ) is a horistological space, then*

$$K_\chi = \{(x, y) : \exists P \in \chi(x) \text{ such that } y \in P\} \cup \delta$$

is an order on S.

This order is called *proper order* of χ, or χ-*causality*, etc.

Proposition 1.3 *Let $\Lambda \subseteq S \times S$ be an order on S. If $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ is a horistology on S, then the restriction $\chi|_\Lambda : S \to \mathcal{P}(\mathcal{P}(S))$, of values*

$$\chi|_\Lambda(x) = \{P \cap \Lambda[x] : P \in \chi(x)\}$$

is a horistology too. In addition, the proper order of $\chi|_\Lambda$ is $K_{\chi|_\Lambda} = \Lambda \cap K_\chi$.

Lemma 1.4 *If the function $\chi : S \to \mathcal{P}(\mathcal{P}(S))$ satisfies the condition $[h_2]$, then $[h_4]$ is equivalent to:*

$[h_4^*]$ $P \in \chi(x) \Longleftrightarrow K_\chi[P] \in \chi(x)$.

Definition 1.5 *Let Λ be an order on the horistological space (S, χ), such that $\Lambda \subseteq K_\chi$, and let M be a subset of S. We say that a point $x \in M$ is Λ-**detachable** from M if*

$$M \cap \overset{0}{\Lambda}[x] \in \chi(x).$$

*The set of all Λ-detachable points of M is called Λ-**discrete part** of M; it is noted $\partial_\Lambda(M)$. If each point of M is Λ-detachable, i.e. $\partial_\Lambda(M) = M$, then M is considered Λ-**discrete**. The function*

$$\partial_\Lambda : \mathcal{P}(S) \longrightarrow \mathcal{P}(S),$$

*which extracts the Λ-discrete part $\partial_\Lambda(M)$ of each subset $M \in \mathcal{P}(S)$, is called **operator of Λ-discreteness**.*

In the case $\Lambda = K_\chi$, we may omit mentioning Λ, and simply speak of detachability, discreteness, etc. Alternatively, we may interpret the Λ-discreteness as discreteness relative to the horistology $\chi|_\Lambda$.

Definition 1.6 *Let (S, χ) be a horistological space. The function*

$$p : \mathcal{P}(S) \longrightarrow \mathcal{P}(S),$$

of values

$$p(M) = \{x \in S : M \in \chi(x)\},$$

*is called **premise operator** on S.*

Definition 1.7 *Let $\left(S_1, \chi_{_1}\right)$ and $\left(S_2, \chi_{_2}\right)$ be horistological spaces. We consider that function $f : S_1 \longrightarrow S_2$ is **discrete at a point** $x \in S_1$ if*

$$f(\chi_{_1}(x) \subseteq \chi_{_2}(f(x)).$$

*If this property holds at each $x \in S_1$, then f is said to be **discrete on the space** S_1.*

Lemma 1.8 *Let $\left(S_1, \chi_{_1}\right)$ and $\left(S_2, \chi_{_2}\right)$ be horistological spaces. If function $f : S_1 \longrightarrow S_2$ is discrete at $x \in S_1$, then*

$$f\left(\overset{0}{K}_{\chi_1}[x]\right) \subseteq \overset{0}{K}_{\chi_2}[f(x)]\,.$$

In addition, if f is discrete on the whole S_1, then

$$\Lambda \overset{not.}{=} f_{II}\left(K_{\chi_1}\right) \overset{def.}{=}$$

$$= \Big\{(X, Y) \in S_2 \times S_2 : \exists (x, y) \in K_{\chi_1} \text{ such that } X = f(x) \text{ and } Y = f(y)\Big\}$$

is a (s.s.) order on S_2. In addition, $\Lambda \subseteq K_{\chi_2}$.

More details about detachability and discreteness can be found in our recent paper [B-P].

2 Admitted events in horistology

Definition 2.1 *Let Λ be an order on the horistological space (S,χ), such that $\Lambda \subseteq K_\chi$, and let M be a subset of S. We say that a point $x \in S$ is a* Λ***-admitted event*** *of M, (or admitted relative to Λ, etc.) if*

$$M \cap \overset{0}{\Lambda}[x] \in \chi(x).$$

The set of all admitted events of M relative to Λ is called Λ***-admittance*** *of M, and we note it $\mathcal{A}_\Lambda(M)$. The function $\mathcal{A}_\Lambda : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$, of values*

$$\mathcal{A}_\Lambda(M) = \{x \in S : M \cap \overset{0}{\Lambda}[x] \in \chi(x)\},$$

is called ***operator of*** Λ***-admittance****.*

If $\Lambda = K_\chi$, and no confusion is possible, i.e. the admittance is completely determined by χ, then we don't mention Λ any more, and we simply note the admittance by $\mathcal{A}$. In particular, every Λ-admittance is an admittance relative to the horistology $\chi|_\Lambda$.

The following proposition shows immediate relations between the operators ∂_Λ and $\mathcal{A}_\Lambda$, respectively between Λ-discreteness and Λ-admittance.

Proposition 2.2 *If (S,χ) is a horistological space, and Λ is an order on S such that $\Lambda \subseteq K_\chi$, then:*
(a) $\partial_\Lambda(M) = M \cap \mathcal{A}_\Lambda(M)$ for all $M \in \mathcal{P}(S)$;
(b) M is Λ-discrete $\Longleftrightarrow M \subseteq \mathcal{A}_\Lambda(M)$;
(c) $\mathcal{A}_\Lambda(M) = \{x \in S : x \in \partial_\Lambda(\{x\} \cup M)\}$.

Proof. (a) follows from the Definitions 1.5. of ∂_Λ and 2.1. of $\mathcal{A}_\Lambda$.

(b) According to Definition 1.5., M is Λ-discrete iff $M = \partial_\Lambda(M)$. From (a) it follows that M is Λ-discrete iff $M = M \cap \mathcal{A}_\Lambda(M)$, hence $M \subseteq \mathcal{A}_\Lambda(M)$.

(c) can be directly deduced from the Definitions 1.5. and 2.1. if we remark that $(\{x\} \cup M) \cap \overset{0}{\Lambda}[x] = M \cap \overset{0}{\Lambda}[x]$. ∎

Remark 2.3 (i) *It is easy to see that $\mathcal{A}_\Lambda(\varnothing) = S$, $\partial_\Lambda(\varnothing) = \varnothing$, and independently of Λ we have*

$$\partial_\Lambda(S) = \mathcal{A}_\Lambda(S) = \{x \in S : \overset{0}{\Lambda}[x] \in \chi(x)\}.$$

(ii) The extreme case when $\Lambda = \delta$ is still acceptable, but $\overset{0}{\delta}[x] = \varnothing$ leads to a trivial situation when all subsets of S are δ-discrete. More exactly, we have

$$\partial_\delta(M) = M \cap \mathcal{A}_\delta(M) = M \cap S = M \text{ and}$$

$$\mathcal{A}_\delta(M) = \{x \in S : \varnothing \in \chi(x)\} = S$$

for all $M \in \mathcal{P}(S)$. In particular,

$$\partial_\delta(S) = \mathcal{A}_\delta(S) = \{x \in S : \varnothing \in \chi(x)\} = S.$$

In [B-P] we have shown that the discreteness of a function allows simple description in terms of discrete sets. In essence, if $f : S_1 \longrightarrow S_2$ is 1:1 and discrete on S_1, then $f$preserves detachability and set discreteness. By analogy, let us now analyze the connection between discrete functions and admittance.

Theorem 2.4 *Let $\left(S_1, \chi_{_1}\right)$ and $\left(S_2, \chi_{_2}\right)$ be horistological spaces, and let the function $f : S_1 \longrightarrow S_2$ be 1:1 and discrete on S_1. If $x \in S_1$ is an admitted event of the set $M \in \mathcal{P}(S_1)$, then $f(x)$ is an admitted event of $f(M)$ in S_2, relative to the order $\Lambda = f_{II}(K_{\chi_1})$.*

Proof. By hypothesis, $M \cap \overset{0}{K}_{\chi_1}[x] \in \chi_{_1}(x)$. By Definition 1.7, the discreteness of f at x leads to

$$f\left(M \cap \overset{0}{K}_{\chi_1}[x]\right) \in \chi_{_2}(f(x)).$$

Because f is 1:1, we have

$$f\left(M \cap \overset{0}{K}_{\chi_1}[x]\right) = f(M) \cap f\left(\overset{0}{K}_{\chi_1}[x]\right).$$

Based on Lemma 1.8, we may easily see that

$$f\left(\overset{0}{K}_{\chi_1}[x]\right) = \overset{0}{\Lambda}\left[f(x)\right].$$

Consequently,

$$f(M) \cap \overset{0}{\Lambda}\left[f(x)\right] \in \chi_{_2}(f(x)),$$

i.e. $f(x)$ is an admitted event of $f(M)$ in S_2. ■

Corollary 2.5 *As before, let* $(S_1, \chi_{_1})$ *and* $(S_2, \chi_{_2})$ *be horistological spaces, and let the function* $f : S_1 \longrightarrow S_2$ *be 1:1 and discrete on the whole* S_1. *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *denote the admittance operators on* $(S_1, \chi_{_1})$ *and* $(S_2, \chi_{_2}|_\Lambda)$, *where* $\Lambda = f_{II}(K_{\chi_{_1}})$, *then* f *preserves the admittance, i.e. the inclusion*

$$f\left(\mathcal{A}_1(M)\right) \subseteq \mathcal{A}_2\left(f(M)\right)$$

holds for all $M \in \mathcal{P}(S_1)$.

This assertion is an immediate consequence of Theorem 2.4. Particularly, if M is a discrete set in S_1, then $f(M)$ is discrete in S_2, i.e. f preserves the discreteness of the sets too. In fact, according to Proposition 2.2(b), M is a discrete set iff $M \subseteq \mathcal{A}_1(M)$. Because $f(M) \subseteq f\left(\mathcal{A}_1(M)\right)$, we obtain

$$f(M) \subseteq \mathcal{A}_2\left(f(M)\right),$$

which shows that $f(M)$ is a discrete set in the space $(S_2, \chi_{_2}|_\Lambda)$.

Theorem 2.6 *Let* $(S_1, \chi_{_1})$ *and* $(S_2, \chi_{_2})$ *be horistological spaces, and let* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *be the corresponding admittance operators. If function* $f : S_1 \longrightarrow S_2$ *is strictly monotonic relative to* $K_{\chi_{_1}}$ *and* K_{χ_2}, *and admittance preserving, then* f *is discrete on* S_1.

Proof. According to Definition 1.7, we have to show that

$$\forall x \in S_1 \text{ and } \forall P \in \chi_{_1}(x) \Longrightarrow f(P) \in \chi_{_2}(f(x)).$$

Because $P \subseteq \overset{0}{K}_{\chi_{_1}}[x]$ holds for all $P \in \chi_{_1}(x)$, it follows that

$$P \cap \overset{0}{K}_{\chi_{_1}}[x] = P \in \chi_{_1}(x),$$

i.e. $x \in \mathcal{A}_1(P)$. By hypothesis, $f(x) \in \mathcal{A}_2\left(f(P)\right)$, hence

$$f(P) \cap \overset{0}{K}_{\chi_2}[f(x)] \in \chi_{_2}(f(x)).$$

But $f(P) \subseteq \overset{0}{K}_{\chi_2}[f(x)]$: Routinely, if $Y \in f(P)$, then there exists $y \in P$ such that $Y = f(y)$; using the monotony of f, from $(x, y) \in K_{\chi_{_1}}$, we deduce that $(f(x), f(y)) \in K_{\chi_2}$, hence $Y \in \overset{0}{K}_{\chi_2}[f(x)]$. Based on this inclusion, we may conclude that $f(P) \in \chi_{_2}(f(x))$. ∎

We may combine Corollary 2.5 and Theorem 2.6 in the following:

Corollary 2.7 *Let $\left(S_1,\chi_{_1}\right)$ and $\left(S_2,\chi_{_2}\right)$ be horistological spaces, and let the function $f: S_1 \longrightarrow S_2$ be 1:1 and strictly monotonic relative to the orders $K_{\chi_{_1}}$ and K_{χ_2}. If in addition $f_{II}(K_{\chi_{_1}}) = K_{\chi_2}$, then f is discrete on S_1 if and only if it is admittance preserving.*

The proof is immediate.

Relative to the notion of admittance we mention:

Theorem 2.8 *Let (S,χ) is a horistological space, and let Λ be an order on S, such that $\Lambda \subseteq K_\chi$. The operator $\mathcal{A}_\Lambda$ has the properties:*

[a_1] $cardM \in \mathbb{N} \Longrightarrow \mathcal{A}_\Lambda(M) = S$;

[a_2] $L \subseteq M \Longrightarrow \mathcal{A}_\Lambda(M) \subseteq \mathcal{A}_\Lambda(L)$;

[a_3] $\mathcal{A}_\Lambda(M \cup L) = \mathcal{A}_\Lambda(M) \cap \mathcal{A}_\Lambda(M)$ *for all* $M, L \in \mathcal{P}(S)$;

[a_4] $\mathcal{A}_\Lambda(M) = \left\{x \in S : x \in \mathcal{A}_\Lambda\left(\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right)\right\}$ *for all* $M \in \mathcal{P}(S)$;

[a_5] $\mathcal{A}_\Lambda(M) \cap \left\{x \in S : M \subseteq \overset{0}{\Lambda}[x]\right\} = \mathcal{A}_\Lambda\left(\Lambda[M]\right) \cap \left\{x \in S : M \subseteq \overset{0}{\Lambda}[x]\right\}$,

which means that the equivalence

$$x \in \mathcal{A}_\Lambda(M) \Longleftrightarrow x \in \mathcal{A}_\Lambda\left(\Lambda[M]\right)$$

holds for all $x \in S$ and $M \subseteq \overset{0}{\Lambda}[x]$;

[a_6] $\mathcal{A}_\Lambda(M) = \left\{x \in S : x \in \mathcal{A}_\Lambda\left(M \cap \overset{0}{\Lambda}[x]\right)\right\}$ *for all* $M \in \mathcal{P}(S)$;

[a_7] $\Pi \subseteq \Lambda \Longrightarrow \mathcal{A}_\Lambda(M) \subseteq \mathcal{A}_\Pi(M)$ *for all* $M \in \mathcal{P}(S)$;

[a_8] $M \subseteq \mathcal{A}_\Lambda(M)$ *and* $\Pi \subseteq \Lambda \Longrightarrow M \subseteq \mathcal{A}_\Pi(M)$;

[a_9] $M \subseteq \mathcal{A}_\Lambda(M)$ *and* $L \subseteq M \Longrightarrow L \subseteq \mathcal{A}_\Lambda(L)$;

[a_{10}] $\partial_\Lambda\left(\mathcal{A}_\Lambda(M)\right) \subseteq \mathcal{A}_\Lambda(\mathcal{A}_\Lambda(M)) \subseteq \mathcal{A}_\Lambda(\partial_\Lambda(M))$ *for all* $M \in \mathcal{P}(S)$.

Proof.

[a_1] Property [d_1] from Proposition 3 in [B-P] says that

$$cardM \in \mathbb{N} \Longrightarrow \partial_\Lambda(M) = M.$$

From the above Proposition 2.2(a) it follows that $M \cap \mathcal{A}_\Lambda(M) = M$, hence

$$M \subseteq \mathcal{A}_\Lambda(M). \qquad (1)$$

On the other hand, if $x \in \complement M$, then for $M_x \overset{not.}{=} M \cup \{x\}$ we have $card M_x \in \mathbb{N}$ too, hence $\partial_\Lambda(M_x) = M_x$. Consequently $x \in \partial_\Lambda(M_x)$, i.e. $M_x \cap \overset{0}{\Lambda}[x] \in \chi(x)$. Because $M_x \cap \overset{0}{\Lambda}[x] = M \cap \overset{0}{\Lambda}[x]$, we may conclude that $x \in \mathcal{A}_\Lambda(M)$, hence

$$\complement M \subseteq \mathcal{A}_\Lambda(M). \tag{2}$$

From (1) and (2) we immediately obtain $\mathcal{A}_\Lambda(M) = S$.

[a$_2$] Obviously, $L \subseteq M$ implies $L \cap \overset{0}{\Lambda}[x] \subseteq M \cap \overset{0}{\Lambda}[x]$ for all $x \in S$. If we take $x \in \mathcal{A}_\Lambda(M)$, which means $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, then using [h$_2$] we deduce $L \cap \overset{0}{\Lambda}[x] \in \chi(x)$. Consequently, $x \in \mathcal{A}_\Lambda(L)$, hence $\mathcal{A}_\Lambda(M) \subseteq \mathcal{A}_\Lambda(L)$.

[a$_3$] From $M \subseteq M \cup L$, $L \subseteq M \cup L$, and [a$_2$] we deduce that

$$\mathcal{A}_\Lambda(M) \supseteq \mathcal{A}_\Lambda(M \cup L) \text{ and } \mathcal{A}_\Lambda(L) \supseteq \mathcal{A}_\Lambda(M \cup L),$$

hence

$$\mathcal{A}_\Lambda(M) \cap \mathcal{A}_\Lambda(L) \supseteq \mathcal{A}_\Lambda(M \cup L). \tag{3}$$

Conversely, if $x \in \mathcal{A}_\Lambda(M) \cap \mathcal{A}_\Lambda(L)$, then $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, and $L \cap \overset{0}{\Lambda}[x] \in \chi(x)$. According to [h$_3$], it follows that

$$(M \cup L) \cap \overset{0}{\Lambda}[x] \in \chi(x),$$

i.e. $x \in \mathcal{A}_\Lambda(M \cup L)$. Therefore,

$$\mathcal{A}_\Lambda(M) \cap \mathcal{A}_\Lambda(L) \subseteq \mathcal{A}_\Lambda(M \cup L). \tag{4}$$

The equality [a$_3$] is an immediate consequence of (3) and (4).

[a$_4$] is a characterization of the Λ-admittance in terms of Λ-sections. First off, let us remark that the s.s. transitivity of Λ leads to the equality

$$\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right] \cap \overset{0}{\Lambda}[x] = \Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]. \tag{5}$$

Now, if we start with $x \in \mathcal{A}_\Lambda(M)$, i.e. $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, then by $[h_4^*]$ and $[h_2]$ we obtain $\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right] \in \chi(x)$. Equivalently, via (5), we have

$$\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right] \cap \overset{0}{\Lambda}[x] \in \chi(x).$$

Consequently, according to Definition 2.1, we deduce that

$$x \in \mathcal{A}_\Lambda \left(\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \right),$$

hence

$$\mathcal{A}_\Lambda(M) \subseteq \left\{ x \in S : x \in \mathcal{A}_\Lambda \left(\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \right) \right\}. \tag{6}$$

Conversely, let us take $x \in S$ with the property $x \in \mathcal{A}_\Lambda \left(\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \right)$. By the definition of $\mathcal{A}_\Lambda$ it follows that

$$\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \cap \overset{0}{\Lambda}[x] \in \chi(x),$$

and by (5) $\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \in \chi(x)$. Using $[h_2]$ we obtain $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, hence $x \in \mathcal{A}_\Lambda(M)$. So we conclude that the opposite of (6) holds, i.e.

$$\left\{ x \in S : x \in \mathcal{A}_\Lambda \left(\Lambda \left[M \cap \overset{0}{\Lambda}[x] \right] \right) \right\} \subseteq \mathcal{A}_\Lambda(M). \tag{7}$$

Finally, the equality [a$_4$] is a consequence of (6) and (7).

[a$_5$] Let us take $x \in \mathcal{A}_\Lambda (\Lambda [M])$. By the definition of $\mathcal{A}_\Lambda$ this means that $\Lambda[M] \cap \overset{0}{\Lambda}[x] \in \chi(x)$. Because Λ is reflexive, it follows that $M \subseteq \Lambda[M]$, hence

$$M \cap \overset{0}{\Lambda}[x] \subseteq \Lambda[M] \cap \overset{0}{\Lambda}[x].$$

Using [h$_2$], we deduce that also $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, i.e. $x \in \mathcal{A}_\Lambda(M)$. This means that the inclusion

$$\mathcal{A}_\Lambda (\Lambda [M]) \subseteq \mathcal{A}_\Lambda(M) \tag{8}$$

holds for all $M \in \mathcal{P}(S)$.

To complete the proof of [a$_5$], we have to show that

$$\mathcal{A}_\Lambda(M) \cap \left\{ x \in S : M \subseteq \overset{0}{\Lambda}[x] \right\} \subseteq \mathcal{A}_\Lambda (\Lambda[M]). \tag{9}$$

With this aim, let us take $x \in \mathcal{A}_\Lambda(M)$ such that $M \subseteq \overset{0}{\Lambda}[x]$. In this case, from $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, and $M \subseteq \overset{0}{\Lambda}[x]$ we obtain $M \in \chi(x)$. According to $[h_4^*]$ and $[h_2]$, $\Lambda[M] \in \chi(x)$ too. Because $\Lambda[M] \cap \overset{0}{\Lambda}[x] \subseteq \Lambda[M]$, by [h$_2$] we obtain $\Lambda[M] \cap \overset{0}{\Lambda}[x] \in \chi(x)$, hence $x \in \mathcal{A}_\Lambda(\Lambda[M])$. Property [a$_5$] follows from (8) and (9).

[a$_6$] is a shorter characterization of the Λ-admittance, which follows from the very definition of $\mathcal{A}_\Lambda$. In fact, $x \in \mathcal{A}_\Lambda(M)$ means that $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, hence $\left(M \cap \overset{0}{\Lambda}[x]\right) \cap \overset{0}{\Lambda}[x] \in \chi(x)$. Using Definition 2.1 again, we obtain $x \in \mathcal{A}_\Lambda\left(M \cap \overset{0}{\Lambda}[x]\right)$, so that

$$\mathcal{A}_\Lambda(M) \subseteq \left\{x \in S : x \in \mathcal{A}_\Lambda\left(M \cap \overset{0}{\Lambda}[x]\right)\right\}. \tag{10}$$

Conversely, let us take $x \in S$ with the property $x \in \mathcal{A}_\Lambda\left(M \cap \overset{0}{\Lambda}[x]\right)$. The definition of $\mathcal{A}_\Lambda$ shows that $\left(M \cap \overset{0}{\Lambda}[x]\right) \cap \overset{0}{\Lambda}[x] \in \chi(x)$, which reduces to $M \cap \overset{0}{\Lambda}[x] \in \chi(x)$, hence $x \in \mathcal{A}_\Lambda(M)$. Therefore,

$$\left\{x \in S : x \in \mathcal{A}_\Lambda\left(M \cap \overset{0}{\Lambda}[x]\right)\right\} \subseteq \mathcal{A}_\Lambda(M). \tag{11}$$

The equality [a$_6$] follows from (10) and (11).

[a$_7$] shows that smaller orders lead to more admitted events. This fact is based on [h$_2$] and the obvious inclusion $M \cap \overset{0}{\Pi}[x] \subseteq M \cap \overset{0}{\Lambda}[x]$.

[a$_8$] can be considered a reformulation of [a$_7$] in terms of discreteness. In fact, if M is Λ-discrete and $\Pi \subseteq \Lambda$, then M is Π-discrete. Using the connection between discreteness and admittance from Proposition 2.2(b), the problem reduces to

$$(\mathcal{A}_\Lambda(M) \subseteq \mathcal{A}_\Pi(M) \text{ and } M \subseteq \mathcal{A}_\Lambda(M)) \Longrightarrow (M \subseteq \mathcal{A}_\Pi(M)).$$

[a$_9$] follows from [a$_2$], and shows that the subsets of a Λ-discrete set are Λ-discrete too. By [a$_2$], $L \subseteq M$ implies $\mathcal{A}_\Lambda(M) \subseteq \mathcal{A}_\Lambda(L)$, and by Proposition 2.2(b) we have $L \subseteq M \subseteq \mathcal{A}_\Lambda(M) \subseteq \mathcal{A}_\Lambda(L)$, hence $L \subseteq \mathcal{A}_\Lambda(L)$.

[a$_{10}$] According to Proposition 2.2(a), $\partial_\Lambda(P) \subseteq \mathcal{A}_\Lambda(P)$ holds for arbitrary $P \in \mathcal{P}(S)$. Putting $P = \mathcal{A}_\Lambda(M)$ for arbitrary $M \in \mathcal{P}(S)$, we obtain

$$\partial_\Lambda \left(\mathcal{A}_\Lambda(M)\right) \subseteq \mathcal{A}_\Lambda \left(\mathcal{A}_\Lambda(M)\right). \tag{12}$$

On the other hand, the action of [a$_2$] on the inclusion $\partial_\Lambda(M) \subseteq \mathcal{A}_\Lambda(M)$, where M is arbitrary in $\mathcal{P}(S)$, leads to

$$\mathcal{A}_\Lambda \left(\mathcal{A}_\Lambda(M)\right) \subseteq \mathcal{A}_\Lambda \left(\partial_\Lambda(M)\right). \tag{13}$$

Property [a$_{10}$] is a simple juxtaposition of the inclusions (12) and (13). ■

Remark 2.9 *Property* $[a_1]$ *and Proposition 2.2(b) establish that the finite sets are* Λ*-discrete for all* $\Lambda \subseteq K_\chi$*. Simple examples (as those from [B-P], Remark 1) reveal the importance of the restriction* $\Lambda \subseteq K_\chi$*.*

Obviously, the properties $[a_5]$ *and* $[a_6]$ *represent equivalent characterization of the* Λ*-admittance in horistological spaces. However, there is a "small" difference:* $[a_5]$ *refers to sets* $M \in \mathcal{P}(S)$ *for which* $M \subseteq \overset{0}{\Lambda}[x]$*, while* $[a_6]$ *is non-trivial in the contrary case, when* $M \cap \overset{0}{\Lambda}[x] \neq M$*.*

Sometimes it is important to know relations between the conditions $[a_4]$*,* $[a_5]$ *and* $[a_6]$*, as for example when we analyze the problem of defining a horistology by an operator of admittance. In this respect we notice the implications:* $[a_4] \Longrightarrow [a_5]$*,* $[a_4] \Longrightarrow [a_6]$*, and* $([a_5]\&[a_6]) \Longrightarrow [a_4]$*. This remark explains why* $[a_4]$ *is a preferred axiom of an abstract operator of admittance (see the next section).*

In the case $\Lambda = K_\chi$, the operator of admittance is closely related to the premise operator:

Theorem 2.10 *Let* $\mathcal{A}$ *be the operator of admittance, and let* p *be the premise operator in a horistological space* (S, χ)*. The following statements represent equivalent properties:*

(a) $M \in \chi(x)$*;*
(b) $x \in p(M)$*;*
(c) $x \in \mathcal{A}(\mathcal{M})$ *and* $M \subseteq \overset{0}{K}[x]$*;*
(d) $x \in p(K[M])$*;*
(e) $x \in \mathcal{A}(K[M])$ *and* $M \subseteq \overset{0}{K}[x]$*.*

The proof is routine and will be omitted.

3 Structures of admittance

It is already known that except perspectives, a horistology is well defined by other means, e.g. a premise operator [BT], discrete sets [B-P], emergent nets [PM], etc. Now, to enlarge this list, we will show how the horistology can be recovered by an abstract operator of admittance. To start, we have to select several properties from the above Theorem 2.4 as axioms of such an operator:

Definition 3.1 *Let S be an arbitrary nonvoid set. A function*

$$\mathcal{A} : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$$

is named ***operator of admittance*** *if it satisfies the conditions:*
$[A_1]$ $cardM \in \mathbb{N} \Longrightarrow \mathcal{A}(M) = S$;
$[A_2]$ $L \subseteq M \Longrightarrow \mathcal{A}(M) \subseteq \mathcal{A}(L)$;
$[A_3]$ $\mathcal{A}(M \cup L) \supseteq \mathcal{A}(M) \cap \mathcal{A}(L)$ *for all* $M, L \in \mathcal{P}(S)$.
In addition, if Λ is a s.s. order on S, such that
$[A_4]$ $x \in \mathcal{A}(M) \Longleftrightarrow x \in \mathcal{A}\left(\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right)$ *for all* $M \in \mathcal{P}(S)$,
then we say that Λ is ***compatible*** *with $\mathcal{A}$.*

The triplet $(S, \mathcal{A}, \Lambda)$, for which the conditions $[A_1]$ - $[A_4]$ are valid, is called *admittance space*. As before, $\mathcal{A}(M)$ is called *admittance* of M, and the points of $\mathcal{A}(M)$ are named *admitted events* of M. In this context, a set $M \in \mathcal{P}(S)$ is considered *discrete* iff $M \subseteq \mathcal{A}(M)$.

Remark 3.2 *Before approaching the construction of horistologies by means of admittance operators, it is useful to discuss a couple of aspects concerning the axioms $[A_1]$ - $[A_4]$:*

(i) According to Theorem 2.4, every horistological space (S, χ) is an admittance space, and the proper order K_χ is compatible with $\mathcal{A}$. Consequently, the concrete examples of horistological spaces, e.g. the event spaces, show that the axioms $[A_1]$ - $[A_4]$ are not contradictory.

(ii) If the equality $\Lambda = \delta$ is compatible with an operator of admittance $\mathcal{A}$, then every set is discrete. This behavior is specific to all orders, which are "small" in the sense of cardinality. More exactly, if $card\overset{0}{\Lambda} \in \mathbb{N}$, then both $M \cap \overset{0}{\Lambda}[x]$ and $\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]$ are finite, hence $\mathcal{A}\left(\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right) = S$

by virtue of $[A_1]$. *In this case, according to* $[A_4]$, *we obtain* $\mathcal{A}(M) = S$ *for all* $M \in \mathcal{P}(S)$. *Therefore* $M \subseteq \mathcal{A}(M)$, *i.e. every set* $M \in \mathcal{P}(S)$ *is discrete.*

(iii) If $cardS \in \mathbb{N}$, *then* $\mathcal{A}(M) = S$ *for all* $M \in \mathcal{P}(S)$, *hence* $\mathcal{A}$ *reduces to a constant function. In this case each order is compatible with* $\mathcal{A}$.

Properties similar to $[A_5]$ - $[A_{10}]$ in Theorem 2.8 can be directly deduced from $[A_1]$ - $[A_4]$, as in the following:

Lemma 3.3 *If the order* Λ *is compatible with* $\mathcal{A}$, *then:*

$[A_5]$ $x \in \mathcal{A}(M) \Longleftrightarrow x \in \mathcal{A}(\Lambda[M])$ *for all* $x \in S$ *and* $M \subseteq \overset{0}{\Lambda}[x]$;

$[A_6]$ $x \in \mathcal{A}(M) \Longleftrightarrow x \in \mathcal{A}\left(M \cap \overset{0}{\Lambda}[x]\right)$ *for all* $M \in \mathcal{P}(S)$.

Proof. We have to show that $[A_5]$ and $[A_6]$ follow from $[A_4]$ alone. In fact, $[A_5]$ refers to points $x \in S$ and sets $M \in \mathcal{P}(S)$ for which $M \subseteq \overset{0}{\Lambda}[x]$. In this case we have $M \cap \overset{0}{\Lambda}[x] = M$, so that $[A_4]$ implies $[A_5]$. Further on, because $M \cap \overset{0}{\Lambda}[x] \subseteq \overset{0}{\Lambda}[x]$, from $[A_5]$ we may deduce that

$$x \in \mathcal{A}\left(M \cap \overset{0}{\Lambda}[x]\right) \Longleftrightarrow x \in \mathcal{A}\left(\Lambda\left[M \cap \overset{0}{\Lambda}[x]\right]\right),$$

which proves that $[A_4]$ implies $[A_6]$. ■

In the sequel we will investigate how a pair $(\mathcal{A}, \Lambda)$, for which the axioms $[A_1]$ - $[A_4]$ hold, define a horistology.

Theorem 3.4 *If* $(S, \mathcal{A}, \Lambda)$ *is an admittance space, then the function*

$$\chi_{(\mathcal{A},\ \Lambda)} : S \longrightarrow \mathcal{P}(\mathcal{P}(S)),$$

of values

$$\chi_{(\mathcal{A},\ \Lambda)}(x) = \{P \subseteq \overset{0}{\Lambda}[x] : x \in \mathcal{A}(P)\}, \tag{14}$$

is a horistology on S. *In addition, the proper order of* $\chi_{(\mathcal{A},\ \Lambda)}$ *is*

$$K_{\chi_{(\mathcal{A},\ \Lambda)}} = \Lambda. \tag{15}$$

Proof. We have to show that $\chi_{(\mathcal{A},\,\Lambda)}$ fulfils the conditions $[h_1]-[h_4]$. To prove $[h_1]$, we may remark that in (14) we have $P \subseteq \overset{0}{\Lambda}[x]$, hence $x \notin P$ holds for arbitrary $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$.

Condition $[h_2]$ is based on $[A_2]$. Primarily, if $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$ and $Q \subseteq P$, then $Q \subseteq \overset{0}{\Lambda}[x]$ too. The other condition follows from $[A_2]$, which states that $Q \subseteq P$ implies $\mathcal{A}(Q) \supseteq \mathcal{A}(P)$. Consequently, $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$.

To prove $[h_3]$, let us chose arbitrary $P, Q \in \chi_{(\mathcal{A},\,\Lambda)}(x)$. According to (14), this means $P \subseteq \overset{0}{\Lambda}[x]$, $Q \subseteq \overset{0}{\Lambda}[x]$, $x \in \mathcal{A}(P)$, and $x \in \mathcal{A}(Q)$. It is easy to see that $P \cup Q \subseteq \overset{0}{\Lambda}[x]$, and $x \in \mathcal{A}(P \cup Q) \overset{[A_3]}{=} \mathcal{A}(P) \cap \mathcal{A}(Q)$. To conclude, $P \cup Q \in \chi_{(\mathcal{A},\,\Lambda)}(x)$, in accordance to (14).

Before proving $[h_4]$, we prefer to establish the equality (15). As usually, we reduce it to a double inclusion: $K_{\chi_{(\mathcal{A},\,\Lambda)}} \subseteq \Lambda$ is a direct consequence of the definitions of $\chi_{(\mathcal{A},\,\Lambda)}$ and $K_{\chi_{(\mathcal{A},\,\Lambda)}}$. More exactly, if $(x,y) \in K_{\chi_{(\mathcal{A},\,\Lambda)}}$, $x \neq y$, then there is some $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$ such that $y \in P$. From $P \subseteq \overset{0}{\Lambda}[x]$ and $y \in P$ we deduce $y \in \overset{0}{\Lambda}[x]$, which means $(x,y) \in \overset{0}{\Lambda}$. The opposite inclusion follows from $[A_1]$. In fact, if $(x,y) \in \overset{0}{\Lambda}$, or equivalently $y \in \overset{0}{\Lambda}[x]$, then for $P = \{y\}$ we obviously have $P \subseteq \overset{0}{\Lambda}[x]$ and $x \in \mathcal{A}(P) \overset{[A_1]}{=} S$. Therefore $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$ and $y \in P$, hence $(x,y) \in K_{\chi_{(\mathcal{A},\,\Lambda)}}$, and $x \neq y$.

The equality (15) is tacitly present in the proof of $[h_4]$. More exactly, based on Lemma 1.4 with $K_{\chi_{(\mathcal{A},\,\Lambda)}} = \Lambda$, we may prove $[h_4^*]$ instead of $[h_4]$, i.e.

$$P \in \chi_{(\mathcal{A},\,\Lambda)}(x) \Longleftrightarrow \Lambda[P] \in \chi_{(\mathcal{A},\,\Lambda)}(x).$$

Part "$\Longrightarrow$" follows from Lemma 3.3 and the strict transitivity of Λ. In fact, if $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$, then $P \subseteq \overset{0}{\Lambda}[x]$ and $x \in \mathcal{A}(P)$. According to $[A_5]$, we obtain $x \in \mathcal{A}(\Lambda[P])$, which, together with $\Lambda[P] \subseteq \overset{0}{\Lambda}[x]$ show that $\Lambda[P] \in \chi_{(\mathcal{A},\,\Lambda)}(x)$. Part "$\Longleftarrow$" is based on $[A_1]$ and the reflexivity of Λ. In more details, if $\Lambda[P] \in \chi_{(\mathcal{A},\,\Lambda)}(x)$, then $\Lambda[P] \subseteq \overset{0}{\Lambda}[x]$, and $x \in \mathcal{A}(\Lambda[P])$. The reflexivity of Λ gives $P \subseteq \Lambda[P]$, so that $P \subseteq \overset{0}{\Lambda}[x]$. On the other hand,

$$P \subseteq \Lambda[P] \overset{[A_2]}{\Longrightarrow} \mathcal{A}(P) \supseteq \mathcal{A}(\Lambda[P]),$$

hence $x \in \mathcal{A}(P)$. Finally, $P \in \chi_{(\mathcal{A},\,\Lambda)}(x)$. ■

Corollary 3.5 *Let (S, χ) be a horistological space. If we briefly note by K the proper order K_χ of χ, and by $\mathcal{A}$ the admittance operator $\mathcal{A}_K$ in (S, χ), then K is compatible with $\mathcal{A}$, and*

$$\chi_{(\mathcal{A},\ K)} = \chi. \tag{16}$$

Proof. As mentioned in Remark 3.2(i), the compatibility of K with $\mathcal{A}$ is a consequence of the property [a_4] in Theorem 2.8. To prove (16), we first remind that $\chi_{(\mathcal{A},\ K)}$ really is a horistology (Theorem 3.4), and its proper order is K. The equality (16), which refers to functions on S, means that the set equality

$$\chi_{(\mathcal{A},\ K)}(x) = \chi(x)$$

holds at each $x \in S$. As usually, we have to prove two inclusions:

"$\subseteq$" : According to the rule (14) of deriving a horistology from $\mathcal{A}$, if $P \in \chi_{(\mathcal{A},\ K)}(x)$, then $P \subseteq \overset{0}{K}[x]$, and $x \in \mathcal{A}(P)$. Because $\mathcal{A}$ derives from χ, the condition $x \in \mathcal{A}(P)$ takes the form $P \cap \overset{0}{K}[x] \in \chi(x)$. Furthermore, because $P \cap \overset{0}{K}[x] = P$, we have $P \in \chi(x)$, which proves that

$$\chi_{(\mathcal{A},\ K)}(x) \subseteq \chi(x).$$

"$\supseteq$" : If $P \in \chi(x)$, then $P \subseteq \overset{0}{K}[x]$ follows from the definition of $K_\chi = K$. Therefore $P \cap \overset{0}{K}[x] = P \in \chi(x)$, hence $x \in \mathcal{A}(P)$. Finally, using (14), we obtain $P \in \chi_{(\mathcal{A},\ K)}(x)$, and the proof of the relation

$$\chi_{(\mathcal{A},\ K)}(x) \supseteq \chi(x)$$

is accomplished. ■

Corollary 3.6 *Let $(S, \mathcal{A}, \Lambda)$ be an admittance space, and let $\chi_{(\mathcal{A},\ \Lambda)}$ be the attached horistology via Theorem 3.4. If $\mathcal{A}_\Lambda$ is the operator of admittance in the horistological space $(S, \chi_{(\mathcal{A},\ \Lambda)})$, then*

$$\mathcal{A}_\Lambda = \mathcal{A}. \tag{17}$$

Proof. $\mathcal{A}_\Lambda$ and $\mathcal{A}$ are functions on $\mathcal{P}(S)$, hence (17) means $\mathcal{A}_\Lambda(M) = \mathcal{A}(M)$ at each $M \in \mathcal{P}(S)$. The proof has two parts again:

Part "$\subseteq$": Let us take an arbitrary $x \in \mathcal{A}_\Lambda(M)$. According to Theorem 3.4, the proper order of $\chi_{(\mathcal{A},\,\Lambda)}$ is Λ, hence $M \cap \overset{0}{\Lambda}[x] \in \chi_{(\mathcal{A},\,\Lambda)}(x)$ holds by Definition 2.1. Taking into account formula (14), which describes the perspectives of x in the horistology $\chi_{(\mathcal{A},\,\Lambda)}$, we see that $x \in \mathcal{A}\left(M \cap \overset{0}{\Lambda}[x]\right)$. Now, according to $[A_6]$ in Lemma 3.3, we have $x \in \mathcal{A}(M)$. Consequently, $\mathcal{A}_\Lambda(M) \subseteq \mathcal{A}(M)$.

Part "$\supseteq$": Conversely, if $x \in \mathcal{A}(M)$, then $x \in \mathcal{A}\left(M \cap \overset{0}{\Lambda}[x]\right)$ follows by $[A_6]$. Together with $M \cap \overset{0}{\Lambda}[x] \subseteq \overset{0}{\Lambda}[x]$, it leads to $M \cap \overset{0}{\Lambda}[x] \in \chi_{(\mathcal{A},\,\Lambda)}(x)$, hence $x \in \mathcal{A}_\Lambda(M)$. This proves the inclusion $\mathcal{A}_\Lambda(M) \supseteq \mathcal{A}(M)$. ■

Remark 3.7 *The operators of discreteness and admittance are very closed each other. The distinction is made by the membership of a point to the considered set, which is asked for detachable points only. A similar situation holds in topology relative to adherent and accumulation points.*

The parallel between topology and horistology reveals new features of the continuum - discreteness duality from a structural point of view. At least in principle, a deeper knowledge of these mathematical structures shall contribute to a better understanding of duality as a universal phenomenon.

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