Aczél's Inequality, Superadditivity, Horistology

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Abstract. We show how the fundamental inequality (Cauchy / Aczél) for two vectors in a complex indefinite inner product space depends on the nature of the linear span of these vectors. The superadditivity is referred to restrained norms and metrics, from which some qualitative structures dual to topologies, and called *horistologies,* are refined.

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Introduction. The frame of this note will be the inner product spaces in the sense of [BJ]. If such a space, say $(E, (.,.))$, is semi-definite, then the Cauchy's inequality holds for all $x, y \in E$, namely

$$|(x,y)|^2 \leq (x,x)(y,y). \tag{1}$$

Nowadays it is well known that, in this case, $(.,.)$ generates a semi-norm, a semi-metric, and a (uniform) topology, in a standard, classical manner, i.e. $(.,.)$ determines the geometry as well as the analysis on E. The problem "what happens in indefinite spaces" is still unclassified. Simple examples show that the contrary of (1), i.e.

$$|(x,y)|^2 > (x,x)(y,y) \tag{2}$$

is possible, so the first step is to specify conditions that allow us to choose between (1) and (2).

When referred to finite dimensional spaces, relation (2) is known as $Aczél's\, inequality$ (e.g. see [MDS]), and takes physical meaning in relativity (see [NLG], etc.), but it is also mentioned in function spaces like the L^p with $p < 1$ (see the Holder's inequalities in [H-S], etc.). It is a matter of course that (2) leads to superadditivity, and this is not suitable to topological structures, but the actual practice is to ignore it, and to operate with less natural topologies on the indefinite inner product spaces; this is the case of the "complexified" Minkowskian space-time in relativity, and the "transformation" of a decomposable nondegenerate inner product space into a (pre) Hilbert space via the fundamental symmetry $J = P^+ - P^-$. Contrary to this classical tendency, we claim that the superadditive norms and metrics generate qualitative structures comparable to topologies, but of dual nature (called *horistologies* in [BT2], [PM], etc.). Till now these structures turned out to be useful in relativity (compare to [ZEC]), duality theory (see [CDB]), and concave programming (see [B-C]).

1. **Theorem 1.** *If* $(E, (.,.))$ *is a (generally complex) inner product space, and* $x, y \in E$*, then (2) holds iff* $F = Lin\{x, y\}$ *is indefinite.*

Proof. If F is semi-definite, that is contrary to indefinite, then (1) holds, and this is opposite to (2). In the other case, if F is indefinite, it is easy to see that x and y are linearly independent, and the trinomial $T_{x,y} : \mathbb{C} \to \mathbb{R}$, of the form

$$T_{x,y}(\lambda) = (x + \lambda y, x + \lambda y)$$

takes both strictly positive and strictly negative values. In particular, if $(x, y) = 0$, then $T_{x,y}(\lambda) = (x, x) + |\lambda|^2 (y, y)$, hence $T_{x,y}$ will change the sign iff $(x, x)(y, y) < 0$. Consequently (2) holds with $|(x, y)| = 0$.

In the remaining case when $(x, y) \neq 0$, let us note $(x, y) = re^{i\varphi}$, $r > 0$, and $y^* = e^{i\varphi} y$. It is easy to see that $(x, y^*) = (y^*, x) = r$, and $Lin\{x, y) = Lin\{x, y^*\}$. On the other hand $G = Lin_{\mathbb{R}}\{x, y^*\}$, endowed with $(.,.)\,|_{G\times G}$, is a real indefinite inner product space. Consequently the trinomial $T_{x,y^*}\,|_{\mathbb{R}}$ takes the form

$$T_{x,y^*}\,|_{\mathbb{R}}\,(\lambda) = (x, x) + 2\lambda r + \lambda^2 (y, y),$$

which shows that it changes the sign iff $\Delta = r^2 - (x, x)(y, y) > 0$. ■

Remark. a) Inequality (2) is useful only if both (x, x) and (y, y) take either positive or negative values, otherwise it is trivial. If we note $\|x\|_+ = \sqrt{(x, x)}$ in the case $(x, x) > 0$, and $\|x\|_- = \sqrt{-(x, x)}$ if $(x, x) < 0$, then (2) leads to

$$|(x, y)| > \|x\|_\pm \|y\|_\pm \quad . \tag{3}$$

If we intend to use this inequality in the same way as (1), i.e. to construct a norm, we first have to accept *restrained* norms since $\|.\|_\pm$ cannot be defined on the whole space. Based on relativistic interpretations like *causality* and *proper time* (see [NLG], etc.), the order cones appear to be the most appropriate domains of the restrained norms. Considering orders restricts us to work with **real** linear spaces.

b) The construction of the usual norms (based on scalar products) involves two inequalities of the same sense, namely

$$\mathrm{Re}(x, y) \le |(x, y)| \overset{(1)}{\le} \|x\| \cdot \|y\| \ .$$

If we want to use (2) with a similar purpose, we have to consider only **real** inner product spaces again. Therefore the investigation of its consequences will be restricted to the frame of the real linear spaces, even if (2) generally holds for the complex ones. An immediate result in this sense is *superadditivity* (briefly *Sa*):

Theorem 2. *Let $(E,(.,.))$ be a real indefinite nondegenerate inner product space. If $e \in E$ is a unit positive element, $S \subseteq e^{\perp}$ is a negative linear subspace of E, and $F = Lin(\{e\}, S)$ then:*

1. *F is a Pontrjagin Π_1 space;*
2. *$K_{e,S} = \{(x,y) = (\lambda e + u, \mu e + v) \in F \times F : \mu - \lambda > \sqrt{-(v-u, v-u)}\} \cup \delta$ is an order relation on F;*
3. *$\|.\|_+ : K_{e,S}[0] \to \mathbb{R}_+$, expressed by $\|x\|_+ = \sqrt{(x,x)}$, is superadditive, i.e.*

$$\|x+y\|_+ > \|x\|_+ + \|y\|_+ \quad . \tag{4}$$

The proof is direct; it makes use of inequality (1)on S.

Structural consequences, in terms of [BT1], [BT2], [PM], etc.:

a) The functional $\|.\|_+$ is a *superadditive norm* in the sense that the conditions

[San1] $\|x\|_+ = 0$ iff $x = 0$, and

[San2] $\|\lambda x\|_+ = \lambda \|x\|_+$ whenever $\lambda \in \mathbb{R}_+$ and $x \in K_{e,S}$

hold besides (4)≡[San3].

b) The functional $d : K_{e,S} \to \mathbb{R}_+$, generated by $\|.\|_+$ via the usual formula $d(x,y) = \|y - x\|_+$, is *superadditive* too, i.e.

$$d(x,y) > d(x,z) + d(z,y) \ for\, all \ (x,z),(z,y) \in K_{e,S} \ . \tag{5}$$

Considering d as a *Sa metric* means that, in addition to (5), we have $d(x,y) = 0$ if and only if $x = y$.

c) The Sa metric d induces a (uniform) *horistology* on $(E,(.,.))$. In fact, using the so called *hyperbolic perspectives* of $x \in E$, and of *radius* $r > 0$, defined by

$$H(x,r) = \{y \in E : d(x,y) > r\},$$

we may attach a set of *perspectives*, say $\chi_d(x)$, to each $x \in E$, by taking

$$\chi_d(x) = \{V \subset E : \exists r > 0 \, s.t. \, V \subseteq H(x,r)\}.$$

More exactly, this means that function $\chi_d : E \to \mathcal{P}(\mathcal{P}(E))$, satisfies the conditions:

[h1] $x \notin V$ at any $x \in E$ and for all $V \in \chi_d(x)$;

[h2] $V \in \chi_d(x)$ and $U \subseteq V$ imply $U \in \chi_d(x)$;

[h3] $U, V \in \chi_d(x) \Longrightarrow U \cup V \in \chi_d(x)$;

[h4] $\forall\ V \in \chi_d(x)$, $\exists U \in \chi_d(x)$ s.t. $\forall y \in V$ and $W \in \chi_d(y)$, we have $W \subseteq U$.

A theory similar in many respects to the point-set topology can be developed by following [BT2], [PM], etc. This similarity is based on a duality of the basic notions:

filters of neighborhoods - ideals of perspectives, continuity - discreteness, convergence - emergence, etc.

Example. Let $E = \mathbb{R}^4$ be the relativistic universe of events, endowed with the inner product

$$(e_1, e_2) = c^2 t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 \,,$$

where $e_1 = (t_1, x_1, y_1, z_1)$ and $e_2 = (t_2, x_2, y_2, z_2)$. Obviously, $(\mathbb{R}^4, (.,.))$ is a Pontrjagin space of index 1. Each positive event e corresponds to an inertial observer $\omega = \{\lambda e : \lambda \in \mathbb{R}\}$, and $S = e^\perp$ represents the set of simultaneous events relative to ω. Independently of e, $K_{e,S} = K$ is the *causal relation,* and the Sa metric generated by $(.,.)$ measures the *proper time.* The corresponding horistology is involved in the analysis of some qualitative properties like the discreteness of a function (e.g. a Lorentz transformation), the emergence of a sequence of events, etc.

Other simple examples bring forward Sa norms independent of inner products, Sa metrics in nonlinear spaces, as well as nonuniform horistologies.

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