Boltzmann equation in de Sitter expanding universe

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I dedicate this paper to the memory of Professor Oliviu Gherman which was one of the pioneers in promoting and teaching statistical physics in his university and our country.

Abstract

It is shown that in the de Sitter expanding universe the Boltzmann equation can be rewritten in terms of conserved momentum which is different from the canonical one but has the advantage to be conserved. In this manner the Boltzmann equation takes simple forms that can be analytically solvable. A solution of the Boltzmann-Marle model in the comoving charts of the de Sitter expanding universe is derived for the first time.

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1 Introduction

The de Sitter spacetime is a hyperbolic manifold having the maximal symmetry given by the $SO(1,4)$ isometry group. These symmetries are generating conserved quantities on geodesics including three components of a conserved momentum that differ from those of the covariant one. Thus, the relativistic Boltzmann equation on the de Sitter spacetime can be rewritten in new variables by using the coordinates and the conserved momentum instead of the canonical covariant one. Then the Boltzmann equation becomes very simple and can be analytically solved at least in the case of the Boltzmann-Marle model. We studied this model in the static de Sitter and Anti-de Sitter charts showing how this model can be solved [1].

Another type of local charts of special interest in the de Sitter expanding universe are the so called co-moving ones in which we can introduce different types of coordinates among them the conformal (or Euclidean) coordinates and the FLRW ones are the most popular. Our aim is to study in this paper the Boltzmann equation in the co-moving charts by using the conserved momentum instead of the canonical one. Thus we obtain a simpler equation of the Boltzmann-Marle model in these charts whose solutions are derived and discussed here for the first time.

The paper is organized as follows. In the second section we present a rapid method of changing variables in the relativistic Boltzmann equation [2] that allows us to obtain simple equations when the momentum variables are conserved. The next section is devoted to the geodesic motion on de Sitter spacetimes that can be observed from natural frames as well as from orthogonal (non-holonomic) local ones. The fourth section presents our principal results concerning the structure of the distribution function of the Marle model. We show that its equilibrium part is of the Maxwell-Jüttner form having a temperature increasing to infinity when one approaches to the de Sitter event horizon. Moreover, it is remarkable that the corresponding macroscopic velocity complies with the Hubble law [3].

2 Changing canonical variables

A mesoscopic system on curved backgrounds, (M, g), constituted by identical particles of mass m, is successfully described by the relativistic Boltzmann equation [2],

$$\frac{\partial f_B(x,p)}{\partial x^\mu}p^\mu - \Gamma^i_{\alpha\beta}p^\alpha p^\beta \frac{\partial f_B(x,p)}{\partial p^i} = Q(f_B, f'_B)\,, \tag{1}$$

determining the scalar distribution function $f_B(x,p)$ which depends on the local coordinates x^μ $(\alpha,\mu,\nu,... = 0,1,2,3)$ and momentum components $p^\mu = mu^\mu = m\frac{dx^\mu}{ds}$ along geodesics. These satisfy the geodesic equation and the normalization condition $g_{\mu\nu}p^\mu p^\nu = m^2$ such that we remain only with three independent momentum independent variables, say p^i $(i,j,k,... = 1,2,3)$.

The first problem we need to discuss here is how could we to replace these variables with other three arbitrary variables. We start by choosing n vectors fields K^a, $a,b,... = 1,2,...n$ and introduce the new parameters $k^a = K^a_\mu p^\mu$. In this manner we obtain a system of $n+1$ equations (including the normalization condition) assumed to be complete such that we can solve the momentum components $p^\mu = p^\mu(x,k)$ in terms of the new variables k^a. The next step is to define the function $f(x,k) = f_B(x,p(x,k))$ observing that,

$$\frac{df}{ds} = \frac{\partial f}{\partial x^\mu}u^\mu + \frac{\partial f}{\partial k^a}\frac{\partial k^a}{\partial x^\mu}u^\mu\,. \tag{2}$$

On the other hand, we have

$$\frac{dk^a}{ds} = K^a_\mu\frac{du^\mu}{ds} + \frac{\partial K^a_\mu}{\partial x^\nu}u^\nu u^\mu = K^a_{\mu;\nu}u^\mu u^\nu\,, \tag{3}$$

since u satisfy the geodesic equation. Thus we obtain the new general equation

$$\frac{\partial f(x,p)}{\partial x^\mu}p^\mu(x,k) + \frac{\partial f(x,p)}{\partial k^a}K^a_{\mu;\nu}(x)p^\mu(x,k)p^\nu(x,k) = Q(f,f')\,, \tag{4}$$

depending on the variables (x,k). The simplest case is when K^a are Killing vector fields since then we have $K^a_{\mu;\nu}p^\mu p^\nu = 0$ remaining with the simpler equation

$$\frac{\partial f(x,k)}{\partial x^\mu}p^\mu(x,k) = Q(f,f')\,. \tag{5}$$

The Boltzmann equation can be rewritten at any time in the orthogonal non-holonomic local frames. In general, these are defined by the system of tetrad fields $e_{\hat\alpha} = e^\mu_{\hat\alpha}\partial_\mu$ and the corresponding co-frames given by the 1-forms $\omega^{\hat\alpha} = \omega^{\hat\alpha}_\mu dx^\mu$ labelled by the local indices $\hat\alpha, \hat\beta ... = 0,1,2,3$ of the Minkowskian metric $\eta = \text{diag}(1,-1,-1,-1)$ giving the line element as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{\hat\alpha\hat\beta}\omega^{\hat\alpha}\omega^{\hat\beta}$. Under such circumstances, we may identify $K^{\hat\alpha}_\mu = \omega^{\hat\alpha}_\mu$ finding that the new variables denoted now by $k^{\hat\alpha} \to p^{\hat\alpha} = \omega^{\hat\alpha}_\mu p^\mu$ that are just the local components of momentum that obey the normalization condition

$$\eta_{\hat\alpha\hat\beta}p^{\hat\alpha}p^{\hat\beta} = g_{\mu\nu}p^\mu p^\nu = m^2\,, \tag{6}$$

leaving us again with three independent variables, $p^{\hat\imath}$ $(\hat\imath, \hat\jmath, ... = 1,2,3)$. Then the duality property of the tetrad fields allows us to solve the covariant components of momentum as $p^\mu(x, p^{\hat\imath}) = e^\mu_{\hat\alpha}(x)p^{\hat\alpha}$. Furthermore, we calculate

$$K^{\hat\alpha}_{\mu;\nu} = \nabla_\nu \omega^{\hat\alpha}_\mu = \partial_\nu \omega^{\hat\alpha}_\mu - \Gamma^\sigma_{\nu\mu}\omega^{\hat\alpha}_\sigma = -\hat\Gamma^{\hat\alpha}_{\nu\hat\beta}\omega^{\hat\beta}_\mu\,, \tag{7}$$

where $\hat\Gamma^{\hat\alpha}_{\nu\hat\beta} = \omega^{\hat\gamma}_\nu \hat\Gamma^{\hat\alpha}_{\hat\gamma\hat\beta}$ are the components in local frames of the connection Γ_ν defined by the identity

$$\partial_\nu \omega^{\hat\alpha}_\mu - \Gamma^\sigma_{\nu\mu}\omega^{\hat\alpha}_\sigma + \hat\Gamma^{\hat\alpha}_{\nu\hat\beta}\omega^{\hat\beta}_\mu = 0\,. \tag{8}$$

In this manner we obtain simply the well-known form of the Boltzmann equation in local frames [2],

$$\frac{\partial f(x,p^{\hat\imath})}{\partial x^\mu} e^\mu_{\hat\alpha} p^{\hat\alpha} - \frac{\partial f(x,p^{\hat\imath})}{\partial p^{\hat\jmath}} \hat\Gamma^{\hat\jmath}_{\hat\gamma\hat\beta} p^{\hat\gamma}p^{\hat\beta} = Q(f,f')\,, \tag{9}$$

written in terms of coordinates and local components of momentum.

3 de Sitter geodesic motion

Let us consider the de Sitter spacetime (M,g) defined as the hyperboloid of radius $1/\omega$ [1] in the five-dimensional flat spacetime (M^5, η^5) of coordinates z^A (labeled by the indices $A, B, ... = 0,1,2,3,4$) and metric $\eta^5 = \text{diag}(1,-1,-1,-1,-1)$. The local charts $\{x\}$ can be introduced on (M,g) giving the set of functions $z^A(x)$ which solve the hyperboloid equation [4],

$$\eta^5_{AB} z^A(x) z^B(x) = -\frac{1}{\omega^2}\,. \tag{10}$$

[1] We denote by ω the Hubble de Sitter constant since H is reserved for the energy operator

The Euclidean chart $\{t_c, \vec{x}\}$ with the conformal time t_c and Cartesian spaces coordinates x^i is defined by

$$\begin{aligned}
z^0(x) &= -\frac{1}{2\omega^2 t_c}\left[1 - \omega^2({t_c}^2 - \vec{x}^2)\right] \\
z^i(x) &= -\frac{1}{\omega t_c} x^i \,, \\
z^4(x) &= -\frac{1}{2\omega^2 t_c}\left[1 + \omega^2({t_c}^2 - \vec{x}^2)\right]
\end{aligned} \tag{11}$$

This chart covers the expanding portion of M for $t_c \in (-\infty, 0)$ and $\vec{x} \in \mathbb{R}^3$ while the collapsing portion is covered by a similar chart with $t_c > 0$. Both these charts have the conformal flat line element,

$$ds^2 = \eta^5_{AB} dz^A(x) dz^B(x) = \frac{1}{\omega^2 {t_c}^2}\left(d{t_c}^2 - d\vec{x}^2\right) . \tag{12}$$

In what follows we restrict ourselves to the expanding portion where conformal time $t_c \in (-\infty, 0]$ is related to the *proper* (or 'physical') time

$$t = -\frac{1}{\omega}\ln(-\omega t_c) \in \mathbb{R}^+ \tag{13}$$

of the FLRW chart $\{t, \vec{x}\}$ of the metric $ds^2 = dt^2 - e^{2\omega t} d\vec{x}\cdot d\vec{x}$, or of the de Sitter-Pailevé one, $\{t, \vec{x}_s\}$, where $ds^2 = (1 - \omega^2 \vec{x}_s^2)dt^2 + 2\omega \vec{x}_s \cdot d\vec{x}_s dt - d\vec{x}_s \cdot d\vec{x}_s$, that depends on the *comoving* Cartesian coordinates,

$$x_s^i = -\frac{1}{\omega t_c} x^i = x^i e^{\omega t} \,, \quad |\vec{x}_s| < \frac{1}{\omega} \,, \tag{14}$$

that coincide to those of the static chart [4, 5]. In both these charts an obsever staying at rest in $\vec{x}_s = 0$ has a finite event horizon on the sphere of the radius $|\vec{x}_s| = \frac{1}{\omega}$.

The charts $\{t_c, \vec{x}\}$ and $\{t, \vec{x}\}$ have diagonal metrics which allow us to introduce orthogonal local frames by using a *common* diagonal gauge defined by the vector fields

$$e_{\hat{0}} = -\omega t_c \partial_{t_c} = \partial_t \,, \quad e_{\hat{i}} = -\omega t_c \partial_{x^i} = e^{-\omega t}\partial_{x^i} \,, \tag{15}$$

and the corresponding 1-forms

$$\omega^{\hat{0}} = -\frac{1}{\omega t_c} dt_c = dt \,, \quad \omega^{\hat{i}} = -\frac{1}{\omega t_c} dx^i = e^{\omega t} dx^i \,. \tag{16}$$

In the chart $\{t, \vec{x}_s\}$ the same gauge is no longer diagonal since

$$e_{\hat{0}} = \partial_t\,, \qquad e_{\hat{i}} = \omega x_s^i \partial_t + \partial_{x_s^i}\,, \tag{17}$$

$$\omega^{\hat{0}} = dt\,, \qquad \omega^{\hat{i}} = dx_s^i - \omega x_s^i dt\,. \tag{18}$$

The de Sitter spacetime is an homogeneous space of the group $SO(1,4)$ that leave invariant the metric η^5 of the embedding manifold M^5 and implicitly Eq. (10). Therefore, each transformation $\mathfrak{g} \in SO(1,4)$ defines the isometry $x \to x' = \phi_{\mathfrak{g}}(x)$ derived from the system of equations $z[\phi_{\mathfrak{g}}(x)] = \mathfrak{g}z(x)$. In the standard parametrization $\mathfrak{g} = \mathfrak{g}(\xi)$ of the group $SO(1,4)$ with real skew-symmetric parameters $\xi^{AB} = -\xi^{BA}$, each isometry can be expanded as $\phi_{\mathfrak{g}}(x) = x + \xi^{AB}k_{(AB)}(x) + ...$ where $k_{(AB)}$ are the basis Killing vectors (in this parametrization) of the de Sitter manifold. These can be related to the Killing vectors in the pseudo-Euclidean spacetime (M^5, η^5) that have the form

$$K_C^{(AB)} dz^C = z^A dz^B - z^B dz^A = k_\mu^{(AB)} dx^\mu \tag{19}$$

allowing us to extract the components $k_{(AB)\,\mu} = \eta^5_{AC}\eta^5_{BD}k_\mu^{(CD)}$. In the chart $\{t_c, \vec{x}\}$ the corresponding contravariant components have simpler forms [6],

$$k^0_{(0i)} = k^0_{(4i)} = \omega t_c x^i\,, \qquad k^j_{(0i)} = k^j_{(4i)} - \frac{1}{\omega}\,\delta^j_i = \omega x^i x^j - \delta^j_i \chi\,, \tag{20}$$

$$k^0_{(ij)} = 0\,, \quad k^l_{(ij)} = \delta^l_j x^i - \delta^l_i x^j\,; \qquad k^0_{(04)} = t_c\,, \quad k^i_{(04)} = x^i\,, \tag{21}$$

where

$$\chi = \frac{1}{2\omega}\left[1 - \omega^2(t_c^2 - \vec{x}^2)\right]\,. \tag{22}$$

Let us focus now on the geodesics of a particle of mass m in the conformal chart $\{t_c, \vec{x}\}$ assuming that this has the *conserved* momentum $\vec{P}$ whose components defined as $P^{\,i} = \omega(k_{(0i)\,\mu} - k_{(4i)\,\mu})p^\mu$ differ from those of the usual momentum that read [6]

$$p^0 = -\omega t\sqrt{m^2 + \omega^2 P^2 t_c^2}\,, \qquad p^i = (\omega t_c)^2 P^{\,i}\,, \tag{23}$$

(where we denote $P = |\vec{P}\,|$). Hereby we deduce the geodesic trajectory,

$$x^i(t_c) = x_0^i + \frac{P^{\,i}}{\omega P^2}\left(\sqrt{m^2 + P^2\omega^2 {t_{c0}}^2} - \sqrt{m^2 + P^2\omega^2 t_c^2}\right)\,, \tag{24}$$

of a massive particle of momentum $\vec{P}$ passing through the point $\vec{x}_0$ at time t_{c0}. Thus, as was expected, we find that the geodesics of a massive particle is completely determined by the initial conditions, $t_{c\,0}$, $\vec{x}_0$, and the conserved momentum $\vec{P}$. Particularly, when $\vec{P}=0$ then the mobile stays at rest in $\vec{x}=\vec{x}_0$ on a world line along the vector field $-\omega t_c \partial_{t_c}$.

In other respects, we observe that the geodesic equation can be brought in a simpler form if we introduce the new functions

$$\xi^i(t_c,\vec{x}) = x^i + \frac{P^i}{\omega P^2}\sqrt{m^2+P^2\omega^2 t_c^2}\,. \tag{25}$$

Indeed, then Eq. (24) can be written as $\vec{\xi}(t_c,\vec{x}) = \text{const.} = \vec{\xi}(t_{c\,0},\vec{x}_0)$ where the integration constants are determined by the initial conditions. Another advantage of this notation is that the conserved energy,

$$E = \omega k_{(04)\,\mu}p^\mu = \omega\,\vec{x}(t_c)\cdot\vec{P} + \sqrt{m^2+P^2\omega^2 t_c^2}\,, \tag{26}$$

can be written now as

$$E = \omega\,\vec{\xi}(t_c,\vec{x})\cdot\vec{P}\,. \tag{27}$$

In the orthogonal local frames defined by the diagonal tetrad fields (15) the components of the momentum,

$$p^{\hat{0}} = \sqrt{m^2+\omega^2 P^2 t_c^2} = \omega\vec{P}\cdot(\vec{\xi}-\vec{x})\,, \qquad p^{\hat{i}} = -\omega t_c P^{\,i} \quad (\hat{i}=i)\,, \tag{28}$$

can also be expressed in terms of coordinates and above introduced conserved quantities.

4 Distribution function of the Marle model

In what follows we study the Merle model on de Sitter sapcetimes assuming that the distribution function $f(t,\vec{x},\vec{P})$ depends on coordinates and conserved momentum, $\vec{P}$, while the Boltzmann equation of this model takes the simpler form

$$p^0\partial_{t_c}f + p^i\partial_i f = -\frac{m}{\tau}\left(f-f^{(\mathrm{eq})}\right), \quad \tau>0\,, \tag{29}$$

where the momentum components are given by Eqs. (23). Our principal goal is to discuss the properties of the analytical solutions of this equation.

Let us observe first that the equilibrium distribution $f^{(\mathrm{eq})}$ must be a solution of the homogeneous equation $p^0\partial_{t_c}f^{(\mathrm{eq})} + p^i\partial_i f^{(\mathrm{eq})} = 0$ even though, in general, this restriction is no mandatory. Then we obtain our main result that is given by the following theorem.

Theorem 1 *The general solution of the Boltzmann equation (29), when $f^{(\mathrm{eq})}$ satisfies the homogeneous version of this equation, reads*

$$f(t_c,\vec{x},\vec{P})=F(\vec{\xi},\vec{P})+H(\vec{\xi},\vec{P})\,(-\omega t_c)^{\frac{1}{\omega\tau}}\left(1+\sqrt{1+\frac{\omega^2P^2t_c^2}{m^2}}\right)^{-\frac{1}{\omega\tau}}\,,\tag{30}$$

where $F=f^{(\mathrm{eq})}$ and H are arbitrary functions of $\vec{\xi}$ and $\vec{P}$.

Proof: By using suitable algebraic codes on computer we find that the solution of the homogeneous equation is the arbitrary function $f^{(\mathrm{eq})}=F$ that depends only on the conserved quantities $\vec{\xi}$ and $\vec{P}$. Then we derive the solution $f-f^{(\mathrm{eq})}$ of the complete equation (29) that is determined up to an arbitrary function $H\equiv H(\vec{\xi},\vec{P})$. ∎

Furthermore, based on the above result, we can find the form of $f^{(eq)}$.

Corollary 1 *The restriction $f^{(\mathrm{eq})}=F(\vec{\xi},\vec{P})$ fixes $f^{(\mathrm{eq})}$ up to two free parameters: the total number of particles and the the temperature at the origin ($\vec{x}=0$).*

Proof: In the Marle model for the collision term, $f^{(\mathrm{eq})}$ is given by the Maxwell-Jüttner distribution:

$$f^{(\mathrm{eq})}=\frac{Z}{(2\pi)^3}\exp\left(\beta_E\mu_E-\beta_E U_{\hat{\alpha}}p^{\hat{\alpha}}\right),\tag{31}$$

where Z denotes the degrees of freedom and $U_{\hat{\alpha}}$ are the components of the macroscopic velocity in the local frames defined by Eqs. (15) and (16). The quantities β_E and $U_{\hat{\alpha}}$ are point-wise functions that have to be determined assuming that $\beta_E U_{\hat{\alpha}}p^{\hat{\alpha}}$ depends only on $\vec{\xi}$ and $\vec{P}$. According to Eqs. (28) we can write

$$\beta_E U_{\hat{\alpha}}p^{\hat{\alpha}}=\omega\beta_E\left[U_{\hat{0}}\vec{P}\cdot\vec{\xi}-\vec{P}\cdot(U_{\hat{0}}\vec{x}-t_c\vec{U})\right]\tag{32}$$

where we denoted $\vec{U}=(U^{\hat{1}},U^{\hat{2}},U^{\hat{3}})=(-U_{\hat{1}},-U_{\hat{2}},-U_{\hat{3}})$. In order for $f^{(\mathrm{eq})}$ to depend only on $\vec{\xi}$ and $\vec{P}$, we must take $\beta_E U_{\hat{0}}=$ const. while the second term above must vanish. This constrains the macroscopic velocity $U=(U^{\hat{0}},\vec{U})$ to satisfy $\vec{U}=U^{\hat{0}}\frac{\vec{x}}{t_c}$ which combined with the normalization condition $1=(U^{\hat{0}})^2-\vec{U}\cdot\vec{U}$ leads to two possible solutions between which we chose

$$U=-\frac{1}{\sqrt{1-\frac{\vec{x}^2}{t_c^2}}}\left(1,\frac{\vec{x}}{t_c}\right),\tag{33}$$

on the expanding portion of M. Moreover, we can write

$$\beta_E = \beta_0 \sqrt{1 - \frac{\vec{x}^2}{t_c^2}}\,, \tag{34}$$

assuming that β_0 is a positively defined arbitrary constant. Thus, $f^{(\mathrm{eq})}$ can be put in the definitive form,

$$f^{(\mathrm{eq})} = \frac{Z}{(2\pi)^3} \exp\left(\beta_E \mu_E - \beta_0 \omega \vec{P}\cdot\vec{\xi}\right), \tag{35}$$

that depends on the arbitrary constants Z and β_0. ∎

It can be seen that β_0 corresponds to the inverse temperature at the origin of the coordinate system and it represents a free parameter. Moreover, the product $\beta_E \mu_E$ should also be constant, since any dependence on $\vec{\xi}$ or $\vec{P}$ implies also a dependence on $p^{\hat{\alpha}}$, which is not allowed by the form of the Maxwell-Jüttner distribution (31). The connection between $\beta_E \mu_E$ and the particle number density will be investigated in section 5.

The Euclidean chart used so far offered us some technical advantages but shadowing the physical meaning because of the unusual nature of the conformal time. Therefore, the physical meaning of the above obtained results can be better understood in the charts $\{t, \vec{x}\}$ or $\{t, \vec{x}_s\}$ with the proper time (13). The distribution function in the FLRW chart has the form,

$$f(t,\vec{x},\vec{P}) = f^{(eq)}(\vec{\xi},\vec{P}) + H(\vec{\xi},\vec{P})\, e^{-\frac{t}{\tau}} \left(1 + \sqrt{1 + \frac{P^2}{m^2} e^{-2\omega t}}\right)^{-\frac{1}{\omega\tau}}, \tag{36}$$

pointing out the term $\exp(-\frac{t}{\tau})$ which is independent on the Hubble de Sitter parameter ω. In the limit of $t \to \infty$ (when $t_c \to 0$) this produces the relaxation

$$\lim_{t\to\infty} f(t,\vec{x},\vec{P}) = f^{(eq)}(\vec{\xi}_0,\vec{P})\,, \tag{37}$$

where

$$\xi_0^i = \lim_{t_c\to 0} \xi^i = x^i + \frac{mP^i}{\omega P^2}\,. \tag{38}$$

The conclusion is that the system tends to equilibrium in the limit of $t \to \infty$.

The inverse temperature (34) corresponding to $f^{(\mathrm{eq})}$ can be written in FLRW or de Sitter-Painlevé coordinates as,

$$\beta_E = \beta_0 \sqrt{1 - \omega^2 \vec{x}^2 e^{2\omega t}} = \beta_0 \sqrt{1 - \omega^2 \vec{x}_s^2}\,, \tag{39}$$

giving the temperature distribution observed by an observer staying at rest in $\vec{x} = \vec{x}_s = 0$. This observes that when one approaches to the event horizon, $|\vec{x}_s| \to \frac{1}{\omega}$, the temperature increases to infinity since then $\beta_E \to 0$.

The local components of the four-velocity (33) can be expressed in the local frames of the chart $\{t, \vec{x}\}$ as

$$U = \frac{1}{\sqrt{1 - \omega^2 \vec{x}^2 e^{2\omega t}}} \left(1, \omega \vec{x} e^{\omega t}\right), \tag{40}$$

allowing us to define the three-velocity

$$\vec{V} = \omega \vec{x} e^{\omega t} = \omega \vec{x}_s \tag{41}$$

which recovers just the Hubble law [3] (for details see also Ref. [7]). This suggests that the distribution function (30) is a suitable candidate to describe the current state of our Universe.

Finally, we consider the problem of the flat limit, when $\omega \to 0$, which is more sensitive since the solution of the Merle model on the Minkowski spacetime is somewhat trivial. Nevertheless, we observe that in this limit we have

$$\beta_E \to \beta_0 , \quad \omega \vec{\xi} \cdot \vec{P} \to \sqrt{m^2 + P^2} = E_0 , \tag{42}$$

while the last term of Eq. (36) vanishes. This means that in the limit $\omega \to 0$ the function $f \to f^{(eq)}$ becomes proportional with the traditional form $\exp(-\beta_0 E_0)$ we meet in special relativity.

5 Conclusion

We derived solved here the Boltzmann-Marle model in the co-moving frames of the de Sitter expanding universe. The obtained distributions have relatively simple forms such that the compatibility conditions and the momenta giving physical quantities can be solved analytically. Thus we may have a framework for studying transport phenomena in the rarefied gases or plasmas in our expanding universe.

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