Physics AUC, vol. 12 (part III), 84-103 (2002)

PHYSICS AUC

One-Casimir bi-Hamiltonian chains on Riemannian manifolds and related dispersionless bi-Hamiltonian field systems

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Abstract

One-Casimir bi-Hamiltonian theory of separable Stäckel systems on Riemannian manifolds is presented. A systematic passage to bi-Hamiltonian dispersionless field systems is constructed.

1 Preliminaries

The separation of variables is one of the most important methods of solving nonlinear ordinary differential equations of Hamiltonian type. It is known since 19*th* century, when Hamilton and Jacobi proved that given a set of appropriate coordinates, the so called separated coordinates, it is possible to solve a related Liouville integrable dynamic system by quadratures. Unfortunately in the 19*th* century and most of the 20*th* century, for a number of models of classical mechanics the separated variables were either guessed or found by some *ad hoc* methods. A fundamental progress in this field was made in 1985, when Sklyanin adopted the method of soliton systems, i.e. the Lax representation, to systematic construction of separated variables (see his review article [1]). In his approach, the appropriate Hamiltonians appear as coefficients of the spectral curve, i.e. the characteristic equation of the Lax matrix. Recently, a new constructive separability theory was presented, based on a bi-Hamiltonian property of integrable systems. In the frame of canonical coordinates the theory was developed in a series of papers [2]-[7] (see also the review article [8]), while the general case was considered in [9]-[13].

In this paper we briefly summarize the results of the theory in the special case of one-Casimir Poisson pencils on Riemannian manifolds, which is very important from the physical point of view. Moreover we show the relation of considered systems with a special class of dispersionless field systems in (1+1) dimension, which are also bi-Hamiltonian with infinite hierarchy of symmetries and conservation laws.

Let M be a differentiable manifold, TM and T^*M its tangent and cotangent bundle. At any point $u \in M$, the tangent and cotangent spaces are denoted by T_uM and T_u^*M, respectively. The pairing between them is given by the map $< \cdot, \cdot >: T_u^*M \times T_uM \to \mathbb{R}$. For each smooth function $F \in C^\infty(M)$, dF denotes the differential of F. M is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$, in general degenerate. The related Poisson tensor π is defined by $\{F,G\}_\pi(u) :=< dG, \pi \circ dF > (u) =< dG(u), \pi(u)dF(u) >$. So, at each point u, $\pi(u)$ is a linear map $\pi(u) : T_u^*M \to T_uM$ which is skew-symmetric and has vanishing Schouten bracket with itself, i.e. the related bracket fulfils the Jacobi identity. Any function $c \in C^\infty(M)$, such that $dc \in \ker \pi$, is called a Casimir of π. Let π_0,$\pi_1 : T^*M \to TM$ be two Poisson tensors on M. A vector field K is said to be bi-Hamiltonian with respect to π_0 and π_1 if there exist two smooth functions H,$F \in C^\infty(M)$ such that

$$K = \pi_0 \circ dH = \pi_1 \circ dF. \tag{1.1}$$

The Poisson tensors π_0 and π_1 are said to be compatible if the associated pencil $\pi_\xi = \pi_1 - \xi\pi_0$ is itself a Poisson tensor for any $\xi \in \mathbb{R}$.

In this paper we consider a particular Poisson manifold M of $\dim M = 2n+1$ equipped with a linear Poisson pencil π_ξ of maximal rank. Assuming that a Casimir of the pencil is a polynomial in ξ of an order n

$$h_\xi = h_0\xi^n + h_1\xi^{n-1} + ... + h_n \tag{1.2}$$

one gets a bi-Hamiltonian chain

$$\pi_\xi \circ dh_\xi = 0 \Longleftrightarrow \begin{array}{l} \pi_0 \circ dh_0 = 0 \\ \pi_0 \circ dh_1 = K_1 = \pi_1 \circ dh_0 \\ \pi_0 \circ dh_2 = K_2 = \pi_1 \circ dh_1 \\ \qquad\qquad \vdots \\ \pi_0 \circ dh_n = K_n = \pi_1 \circ dh_{n-1} \\ \qquad\qquad\quad 0 = \pi_1 \circ dh_n. \end{array} \tag{1.3}$$

where $\{h_i\}_{i=1}^n$ is a set of independent functions in involution with respect to both Poisson structures, so defines a Liouville integrable system on M.

In the following paper we restrict to a special case of quadratic in momenta constants of the motion, i.e. to the case of $M = T^*Q \times \mathbb{R}$, where Q is some Riemann space. So, let (Q, g) be a Riemann (pseudo-Riemann) manifold with covariant metric tensor g and local coordinates $q^1, ..., q^n$. Moreover, let $G := g^{-1}$ be a contravariant metric tensor satisfying $\sum_{j=1}^n g_{ij}G^{jk} = \delta_i^k$. The Levi-Civita connection components are defined by

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^n G^{il}(\partial_k g_{lj} + \partial_j g_{kl} - \partial_l g_{jk}), \qquad \partial_i \equiv \frac{\partial}{\partial q^i}. \tag{1.4}$$

The equations

$$q^i_{tt} + \Gamma^i_{jk}q^j_t q^k_t = G^{ik}\partial_k V(q), \qquad i = 1, ..., n, \qquad q_t \equiv \frac{dq}{dt} \tag{1.5}$$

describe the motion of a particle in the curved space with the metric g. Eqs.(1.5) can be obtained by varying the Lagrangian

$$\mathcal{L}(q, q_t) = \frac{1}{2} \sum_{i,j} g_{ij} q_t^i q_t^j - V(q). \tag{1.6}$$

One can pass in a standard way to the Hamiltonian description of dynamics, where the Hamiltonian function takes the form

$$H(q,p) = \sum_{i=1}^{n} q_t^i \frac{\partial \mathcal{L}}{\partial q_t^i} - \mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{n} G^{ij} p_i p_j + V(q), \qquad p_i := \frac{\partial \mathcal{L}}{\partial q_t^i} \tag{1.7}$$

and equations of motion are

$$\begin{gathered} \begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 \circ dH = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \partial p} \end{pmatrix} = X_H \\ \Updownarrow \\ q_t^i = \frac{\partial H}{\partial p_i}, \qquad p_{it} = -\frac{\partial H}{\partial q^i}. \end{gathered} \tag{1.8}$$

X_H denotes the Hamiltonian vector field and the whole dynamics takes place on the phase space $M = T^*G$ in local coordinates $(q^1, ..., q^n, p_1, ..., p_n)$.

Of a special importance is the geodesic motion $V(q) \equiv 0$, with Lagrangian equations

$$q_{tt}^i + \Gamma_{jk}^i q_t^j q_t^k = 0, \qquad i = 1, ..., n \tag{1.9}$$

and Hamiltonian representation

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 \circ dE = X_E, \qquad E = \frac{1}{2} \sum_{i,j=1}^{n} G^{ij} p_i p_j. \tag{1.10}$$

2 Stäckel manifolds and separable dynamic systems of Benenti type

2.1 Separable geodesics

The Stäckel space is the Riemann space with diagonal metric of such a form that the corresponding geodesic equations are separable. In 1893 Stäckel gave the first characterization of the Riemann (pseudo-Riemann) manifold (Q, g) on which the equations of geodesic motion can be solved by separation of variables. He proved that if in a system of orthogonal coordinates (λ, μ) there exists a non-singular matrix $\varphi = (\varphi_k^l(\lambda_k))$, called a *Stäckel matrix* such that the geodesic Hamiltonians E_r are of the form

$$E_r = \frac{1}{2} \sum_{i=1}^{n} (\varphi^{-1})_r^i \mu_i^2, \tag{2.1}$$

then E_r are functionally independent, pairwise commute with respect to the canonical Poisson bracket and the Hamilton-Jacobi equation associated to E_1 is separable.

Then, Eisenhart gave a coordinate-free representation for Stäckel geodesic motion introducing special family of *Killing tensors*. As it is known, a $(1,1)$-type tensor $B = (B^i_j)$ (or a $(2,0)$-type tensor $B = (B^{ij})$) is called a Killing tensor with respect to g if $\left\{ \sum (BG)^{ij} p_i p_j \, , \, E \right\}_{\theta_0} = 0$ (or $\left\{ \sum (B)^{ij} p_i p_j \, , \, E \right\}_{\theta_0} = 0$). He proved [14] that the geodesic Hamiltonians can be transformed into a Stäckel form (2.1) if the contravariant metric tensor $G = g^{-1}$ has $(n-1)$ commuting independent contravariant Killing tensors A_r of a second order such that

$$E_r = \frac{1}{2} \sum_{i,j} A_r^{ij} p_i p_j, \tag{2.2}$$

admitting a common system of closed eigenforms α_i

$$(A_r^* - v_r^i G)\alpha_i = 0, \;\; d\alpha_i = 0, \;\; i = 1, ..., n, \tag{2.3}$$

where v_r^i are eigenvalues of $(1,1)$ Killing tensor $K_r = A_r g$ ($K_r^* = g A_r^*$). In local coordinates q on Q we have

$$K_r = \sum_{i,j} (K_r)^i_j \frac{\partial}{\partial q^i} \otimes dq^j, \;\; K_r^* = \sum_{i,j} (K_r)^j_i dq^i \otimes \frac{\partial}{\partial q^j}, \tag{2.4}$$

$$A_r = \sum_{i,j} A_r^{ij} \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial q^j}, \qquad (K_r)^i_j = \sum_k A_r^{ik} g_{kj}.$$

Among all Stäckel systems a particular important subclass consists of these considered by Benenti [15]-[17] and constructed with the help of a so called *special conformal Killing tensor.*

Definition 1 *Let $L = (L^i_j)$ be a second order mixed type (i.e. $(1,1)$-) tensor on Q and let $\overline{L} : M \to \mathbb{R}$ be a function on M defined as $\overline{L} = \frac{1}{2} \sum_{i,j=1}^{n} (LG)^{i,j} p_i p_j$, where LG is a $(1,1)$ tensor with components $(LG)^{ij} = \sum_{i,j=1}^{n} L^i_k G^{kj}$. If*

$$\{\overline{L}, E\}_{\theta_0} = \alpha E, \qquad \text{where} \qquad \alpha = \sum_{i,j=1}^{n} G^{ij} \frac{\partial f}{\partial q^i} p_j, \qquad f = Tr(L), \tag{2.5}$$

then L is called a special conformal Killing tensor with the associated potential $f = Tr(L)$.

For the Riemannian manifold (Q, g, L), geodesic flow has n constants of motion of the form

$$E_r = \frac{1}{2} \sum_{i,j=1}^{n} A_r^{ij} p_i p_j = \frac{1}{2} \sum_{i,j=1}^{n} (K_r G)^{ij} p_i p_j, \qquad\qquad r = 1, ..., n, \tag{2.6}$$

$$(K_r G)^{ij} = \sum_{k=1}^{n} (K_r)^i_k G^{kj}$$

where A_r and K_r are Killing tensors of type $(2,0)$ and $(1,1)$, respectively. Moreover, as was shown by Benenti [15],[16], all the Killing tensors K_r with a common set of eigenvectors, are constructed from L by

$$K_{r+1} = \sum_{k=0}^{r} \rho_k L^{r-k}, \tag{2.7}$$

where ρ_r are coefficients of characteristic polynomial of L

$$\det(\xi I - L) = \xi^n + \rho_1 \xi^{n-1} + ... + \rho_n, \quad \rho_0 = 1. \tag{2.8}$$

¿From (2.7) we immediately find that

$$K_{r+1} - L \circ K_r = \rho_r I, \qquad -L \circ K_n = \rho_n I, \qquad K_1 = I. \tag{2.9}$$

Lemma 2 *¿From (2.7) and (2.9) it follows that appropriate Killing tensors K_r are given by the following 'cofactor' formula*

$$cof(\xi I - L) = \sum_{i=1}^{n-1} K_{n-i} \xi^i, \tag{2.10}$$

where $cof(A)$ stands for the matrix of cofactors, so that $cof(A)A = (\det A)I$. Notice that $K_1 = I$, hence $A_1 = G$ and $E_1 \equiv E$.

Proof.

$$\begin{aligned}(\xi I - L)(\sum_{i=1}^{n-1} K_{n-i}\xi^i) &= \sum_{i=1}^{n-1}(K_{n-i}\xi^{i+1} - L \circ K_{n-i}\xi^i) \\ &= -L \circ K_n + (K_n - L \circ K_{n-1})\xi + ... \\ &+ (K_2 - L \circ K_1)\xi^{n-1} + K_1 \xi^n \\ &= I(\rho_n + \rho_{n-1}\xi + ... + \rho_1 \xi^{n-1} + \xi^n) = I \det(\xi I - L).\end{aligned}$$

■

According to the above results, the functions E_r, satisfy

$$\{E_s, E_r\}_{\theta_0} = 0, \tag{2.11}$$

and thus constitute a system of n constants of motion in involution with respect to the Poisson structure θ_0. So, for a given metric tensor g, the existence of a special conformal Killing tensor L is a sufficient condition for the geodesic flow on M to be a Liouville integrable Hamiltonian system.

It turns out that with the tensor L we can (generically) associate a coordinate system on M in which the geodesic flows associated with all the functions E_r separate. Namely, let $(\lambda^1(q), ..., \lambda^n(q))$ be n distinct, functionally independent eigenvalues of L, i.e. solutions of the characteristic equation $\det(\xi I - L) = 0$. Solving these relations with respect to q we get the transformation $\lambda \to q$

$$q^i = \alpha_i(\lambda), \qquad i = 1, ..., n. \tag{2.12}$$

The remaining part of the transformation to the separation coordinates can be obtained as a canonical transformation reconstructed from the generating function $W(p, \lambda) = \sum_i p_i \alpha_i(\lambda)$ in the following way

$$\mu_i = \frac{\partial W(p, \lambda)}{\partial \lambda^i} \Longrightarrow p_i = \beta_i(\lambda, \mu) \qquad i = 1, ..., n. \tag{2.13}$$

In the (λ, μ) coordinates the tensor L is diagonal

$$L = \mathrm{diag}(\lambda^1, ..., \lambda^n) \equiv \Lambda, \tag{2.14}$$

while geodesic Hamiltonians have the following form [2]

$$E_r = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda^i} \frac{f_i(\lambda^i)}{\Delta_i} \mu_i^2, \qquad r = 1, ..., n, \tag{2.15}$$

where

$$\Delta_i = \prod_{k=1,...n,\, k \neq i} (\lambda^i - \lambda^k), \tag{2.16}$$

$\rho_r(\lambda)$ are symmetric polynomials (Viéte polynomials) defined by the relation

$$\det(\xi I - \Lambda) = (\xi - \lambda^1)(\xi - \lambda^2) ... (\xi - \lambda^n) = \sum_{r=0}^{n} \rho_r \xi^r \tag{2.17}$$

and f_i are arbitrary smooth functions of one real argument. From (2.15) it immediately follows that in (λ, μ) variables the contravariant metric tensor G and all the Killing tensors K_r are diagonal

$$G^{ij} = \frac{f_i(\lambda^i)}{\Delta_i} \delta^{ij}, \quad (K_r)^i_j = -\frac{\partial \rho_r}{\partial \lambda^i} \delta^i_j. \tag{2.18}$$

Remark 3 *When $f_i(\lambda^i)$ is a polynomial of order $\leq n$ the space is flat, if the order of f is equal $n+1$ the space is of constant curvature.*

What are the *separated coordinates* ? How to solve equations by quadratures? Let us consider a set of coordinates $\{\lambda^i, \mu_i\}_{i=1}^n$ on M canonical with respect to θ_0. On can try to linearize the system (1.10) through a canonical transformation $(\lambda, \mu) \to (b, a)$ in the form $b^j = \frac{\partial W}{\partial a_i}, \mu_i = \frac{\partial W}{\partial \lambda^i}$, where $W(\lambda, a)$ is a generating function that solves the related Hamilton-Jacobi (HJ) equations

$$E_r(\lambda, \frac{\partial W}{\partial \lambda}) = a_r, \qquad r = 0, ..., n. \tag{2.19}$$

In general, the HJ equations (2.19) are nonlinear partial differential equations that are very difficult to solve. In general it is a hopeless task!!! However, there are rare cases when one can find a solution of (2.19) in the separable form

$$W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda_i, a) \tag{2.20}$$

that turns the HJ equations into a set of decuple ordinary differential equations that can be solved by quadratures. Such (λ, μ) coordinates are called *separated coordinates.* In the (a, b) coordinates the flow d/dt_j associated with every Hamiltonian E_j is trivial

$$\frac{da_i}{dt_j} = 0, \quad \frac{db^i}{dt_j} = \delta_{ij} \;, \qquad i, j = 1, \ldots, n \tag{2.21}$$

and the implicit form of the trajectories $\lambda(t_j)$ is given by

$$b^i(\lambda, a) = \frac{\partial W}{\partial a_i} = \delta_{ij} t_j + const, \quad i, j = 1, ..., n. \tag{2.22}$$

Equivalently to (2.20), separated coordinates are defined by n relations of the form

$$\varphi_i(\lambda^i, \mu_i, E_1, ..., E_n) = 0, \qquad i = 1, ..., n \tag{2.23}$$

joining each pair (λ^i, μ_i) of conjugate coordinates and all Hamiltonians $E_i,\; i = 1, ..., n$. Fixing the values of Hamiltonians $E_j = const = a_j$ one obtains an explicit factorization of the Liouville tori given by the equations

$$\begin{gathered} \varphi_i(\lambda^i, \mu_i, a_1, ..., a_n) = 0, \qquad i = 1, ..., n \\ \Updownarrow \\ \varphi_i(\lambda^i, \frac{dW_i}{d\lambda^i}, a_1, ..., a_n) = 0, \qquad i = 1, ..., n \end{gathered} \tag{2.24}$$

i.e. a decuple system of ordinary differential equations. To ensure that in the (λ, μ) coordinates the geodesic Hamiltonians (2.15) are separable it is sufficient to observe that in these coordinates they actually have the Stäckel form

$$E_r = \frac{1}{2} \sum_{i=1}^{n} (\varphi^{-1})^i_r \mu_i^2, \qquad r = 1, ..., n, \tag{2.25}$$

with the related Stäckel matrix

$$\varphi = \begin{pmatrix} \frac{(\lambda^1)^{n-1}}{f_1(\lambda^1)} & \frac{(\lambda^1)^{n-2}}{f_1(\lambda^1)} & \cdots & \frac{\lambda^1}{f_1(\lambda^1)} & \frac{1}{f_1(\lambda^1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{(\lambda^n)^{n-1}}{f_n(\lambda^n)} & \frac{(\lambda^n)^{n-2}}{f_n(\lambda^n)} & \cdots & \frac{\lambda^n}{f_n(\lambda^n)} & \frac{1}{f_n(\lambda^n)} \end{pmatrix}. \tag{2.26}$$

Inverting eqs. (2.25) one obtains exactly n relations (2.23) in the form

$$E_1(\lambda^i)^{n-1} + E_2(\lambda^i)^{n-2} + \dots + E_n = \frac{1}{2} f_i(\lambda^i)\mu_i^2, \qquad i = 1, \dots, n. \tag{2.27}$$

For $f_i(\lambda^i) = f(\lambda^i)$ eqs. (2.27) can be represented by n different points $(\xi, \mu) = (\lambda^i, \mu_i), i = 1, \dots, n$ of some curve

$$E_1\xi^{n-1} + E_2\xi^{n-2} + \dots + E_n = \frac{1}{2} f(\xi)\mu^2 \tag{2.28}$$

called *separation curve.*

Let us now solve explicitly the Hamilton-Jacobi equations (2.19) and dynamic equations, with respect to the evolution parameter t_r, written in (λ, μ) coordinates:

$$\sum_i (\varphi^{-1})^i_r \left(\frac{\partial W}{\partial \lambda^i}\right)^2 = a_r, \qquad r = 1, \dots, n$$

$$\Downarrow$$

$$f(\lambda^i)\left(\frac{\partial W}{\partial \lambda^i}\right)^2 = a_1(\lambda^i)^{n-1} + a_2(\lambda^i)^{n-2} + \dots + a_n \equiv a_{\lambda^i}$$

$$\Downarrow \quad W = \sum_{i=1}^n W_i(\lambda_i, a)$$

$$\frac{1}{2} f(\lambda^i)\left(\frac{dW_i}{d\lambda^i}\right)^2 = a_1(\lambda^i)^{n-1} + a_2(\lambda^i)^{n-2} + \dots + a_n \equiv a_{\lambda^i}$$

$$\Downarrow$$

$$W_i(\lambda_i, a) = \int^{\lambda^i} \sqrt{\frac{a_\xi}{\frac{1}{2}f(\xi)}}\, d\xi$$

$$\Downarrow$$

$$W(\lambda, a) = \sum_{k=1}^n \int^{\lambda^k} \sqrt{\frac{a_\xi}{\frac{1}{2}f(\xi)}}\, d\xi$$

$$\Downarrow$$

$$b^i = \frac{\partial W}{\partial a_i} = \sum_{k=1}^n \int^{\lambda^k} \frac{\xi^{i-1}}{\sqrt{\frac{1}{2}f(\xi)a_\xi}}\, d\xi$$

$$\Downarrow \quad \sqrt{\frac{1}{2}f(\xi)a_\xi} \equiv \psi(\xi)$$

$$\sum_{k=1}^n \int^{\lambda^k} \frac{\xi^{i-1}}{\psi(\xi)}\, d\xi = t_r\delta_{ri} + const_i, \qquad i = 1, \dots, n, \tag{2.29}$$

where the eqs.(2.29) are implicit solutions called the *inverse Jacobi problem.*

2.2 Bi-Hamiltonian chains

The special conformal Killing tensor L can be lifted from Q to a $(1,1)$-type tensor on $M = T^*Q$ where it takes the form

$$N = \begin{pmatrix} L & 0_n \\ F & L^T \end{pmatrix}, \quad F_j^i = \frac{\partial}{\partial q^i}(Lp)_j - \frac{\partial}{\partial q^j}(p^T L)_i. \tag{2.30}$$

The lifted $(1,1)$ tensor N is called a *recursion operator*. An important property of N is that when it acts on the canonical Poisson tensor θ_0 it produces another Poisson tensor

$$\theta_1 = N \circ \theta_0 = \begin{pmatrix} 0 & L \\ -L^T & F \end{pmatrix}, \tag{2.31}$$

compatible with the canonical one (actually θ_0 is compatible with $N^k\theta_0$ for any integer k). In the (λ, μ) coordinates the recursion operator and the tensor θ_1 attain the form

$$N = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}. \tag{2.32}$$

It is now possible to show that the geodesic Hamiltonians E_r satisfy on $M = T^*Q$ the set of relations [19]

$$\begin{gathered} \theta_0 \circ dE_{r+1} = \theta_1 \circ dE_r + \rho_r \theta_0 \circ dE_1, \quad E_{n+1} = 0, \qquad r = 1, ..., n. \\ \Updownarrow \\ dE_{r+1} = N^* \circ dE_r + \rho_r dE_1, \qquad N^* = \theta_0^{-1} \circ \theta_1, \end{gathered} \tag{2.33}$$

which is called a *quasi-bi-Hamiltonian chain* [18]. On the extended phase space $M' = T^*Q \times \mathbb{R}$, the extended geodesic Hamiltonians

$$e_r = E_r + c\rho_r, \quad r = 1, ..., n, \quad e_0 = c, \tag{2.34}$$

where c is an additional coordinate, satisfy the following *bi-Hamiltonian chain* [19]

$$\begin{aligned} \pi_0 \circ de_0 &= 0 \\ \pi_0 \circ de_1 &= X_1 = \pi_1 \circ de_0 \\ \pi_0 \circ de_2 &= X_2 = \pi_1 \circ de_1 \\ &\vdots \\ \pi_0 \circ de_n &= X_n = \pi_1 \circ de_{n-1} \\ 0 &= \pi_1 \circ de_n \end{aligned} \tag{2.35}$$

with the Poisson operators π_0 and π_1

$$\pi_0 = \left(\begin{array}{c|c} \theta_0 & 0 \\ \hline 0 & 0 \end{array}\right), \quad \pi_1 = \left(\begin{array}{c|c} \theta_1 & \theta_0 \circ de_1 \\ \hline -(\theta_0 \circ de_1)^T & 0 \end{array}\right) \tag{2.36}$$

Both Poisson tensors π_0 and π_1 are compatible and degenerated. The Casimir of π_0 is e_0 and the Casimir of π_1 is e_n and they start and terminate the bi-Hamiltonian chain (2.35). The projections of π_0, π_1 onto a symplectic leaf of

π_0 ($c = const$) reconstructs our nondegenerate Poisson tensors θ_0, θ_1. If we introduce the Poisson pencil $\pi_\xi = \pi_1 - \xi\pi_0$, the chain (2.35) can be written in a compact form

$$\pi_\xi \circ de_\xi = 0, \quad e_\xi = \sum_{r=0}^{n} e_{n-r}\xi^r, \tag{2.37}$$

where e_ξ is a Casimir of the Poisson pencil π_ξ depending polynomial on ξ. The projection of (2.35) onto symplectic leaf of $\pi_0 : c = 0$ reconstructs the quasi-bi-Hamiltonian chain (2.33) in a compact form

$$\theta_\xi \circ d(E_\xi) + \rho_\xi \theta_0 \circ d(E_1) = 0, \;\; E_\xi = \sum_{r=0}^{n-1} E_{n-r}\xi^r, \;\; \rho_\xi = \det(\xi I - L) = \sum_{r=0}^{n} \rho_{n-r}\xi^r. \tag{2.38}$$

If we start with a bi-Hamiltonian chain (2.35) or a quasi-bi-Hamiltonian chain (2.33) written in the 'physical' (original) coordinates (q, p) then we can usually find the functions $q(\lambda)$ that constitute the first half of the transformation $(\lambda, \mu) \to (q, p)$ (and then complete it with the help of the generating function W - as in (2.13) - to the complete transformation $(\lambda, \mu) \to (q, p)$) simply by taking the functions $\rho_r(q)$ in the Hamiltonians (2.34) and solving the system of equations $\rho_r(q) = \rho_r(\lambda)$, $r = 1, \ldots, n$ with respect to q.

2.3 Separable potentials

What potentials can be added to geodesic Hamiltonians E_r without destroying their separability within the above schema? It turns out that there exists a sequence of generic separable potentials $V_r^{(k)}$, $k = \pm 1, \pm 2, \ldots$, which can be added to geodesic Hamiltonians E_r such that the new Hamiltonians

$$H_r(q, p) = E_r(q, p) + V_r^{(k)}(q), \quad r = 1, \ldots, n, \tag{2.39}$$

are still separable in the same coordinates (λ, μ). It means that H_r follow the quasi-bi-Hamiltonian chain (2.33)

$$dH_{r+1} = N^* \circ dH_r + \rho_r dH_1, \quad H_{n+1} = 0, \quad r = 1, \ldots, n, \tag{2.40}$$

while for potentials we have

$$dV_{r+1} = L^* \circ dV_r + \rho_r dV_1, \quad r = 1, \ldots, n, \tag{2.41}$$

$$\Updownarrow$$

$$dV_{r+1} = \sum_{k=0}^{r} \rho_k (L^*)^{r-k} \circ dV_1$$

$$\Updownarrow (2.7)$$

$$dV_{r+1} = K_{r+1}^* \circ dV_1 \Longrightarrow d(K_{r+1}^* \circ dV_1) = 0. \tag{2.42}$$

The relations (2.42) were derived for the first time by Benenti [15], [16].

Theorem 4 *The generic separable potentials $V_r^{(m)}$ are given by the following recursion relation [5], [11]*

$$V_r^{(k+1)} = V_{r+1}^{(k)} - V_r^{(1)} V_1^{(k)}, \quad V_r^{(1)} = \rho_r, \ \ k = 1, 2, ..., \tag{2.43}$$

and its inverse

$$V_r^{(-k-1)} = V_{r-1}^{(-k)} - V_r^{(-1)} V_n^{(-k)}, \quad V_r^{(-1)} = \rho_{r-1}/\rho_n, \quad k = 1, 2, \dots . \tag{2.44}$$

Proof. The proof is inductive. We show it for positive potentials. Assuming that potentials $V_r^{(m)}$ fulfil condition (2.41), we prove that potentials $V_r^{(m+1)}$ fulfil the same condition. The condition (2.41) is true for the first nontrivial potentials $V_r^{(1)} = \rho_r$, which are coefficients of characteristic polynomials of special conformal Killing tensor L [19]

$$d\rho_{r+1} = L^* \circ d\rho_r + \rho_r d\rho_1, \qquad r = 1, ..., n. \tag{2.45}$$

Then we have

$$\begin{aligned}
& L^* \circ dV_r^{(m+1)} + \rho_r dV_1^{(m+1)} \\
&= L^* \circ d(V_{r+1}^{(m)} - \rho_r V_1^{(m)}) - \rho_r d(V_2^{(m)} - \rho_1 V_1^{(m)}) \\
&= L^* \circ dV_{r+1}^{(m)} - \rho_r L^* \circ dV_1^{(m)} - V_1^{(m)}(L^* \circ d\rho_r + \rho_r d\rho_1) + \rho_r(dV_2^{(m)} - \rho_1 dV_1^{(m)}) \\
&= L^* \circ dV_{r+1}^{(m)} - \rho_r L^* \circ dV_1^{(m)} - V_1^{(m)} d\rho_{r+1} + \rho_r L^* \circ dV_1^{(m)} \\
&= L^* \circ dV_{r+1}^{(m)} - V_1^{(m)} d\rho_{r+1} \\
&= dV_{r+2}^{(m)} - \rho_{r+1} dV_1^{(m)} - V_1^{(m)} d\rho_{r+1} \\
&= d(V_{r+2}^{(m)} - \rho_{r+1} V_1^{(m)}) \\
&= dV_{r+1}^{(m+1)}.
\end{aligned}$$

■

Now, again the extended Hamiltonians $h_r : M \times \mathbb{R} \to \mathbb{R}$

$$h_r = H_r + c\rho_r \tag{2.46}$$

satisfy the bi-Hamiltonian chain (compare with (2.35))

$$\begin{aligned}
\pi_0 \circ dh_0 &= 0 \\
\pi_0 \circ dh_1 &= X_1 = \pi_1 \circ dh_0 \\
\pi_0 \circ dh_2 &= X_2 = \pi_1 \circ dh_1 \\
&\ \vdots \\
\pi_0 \circ dh_n &= X_n = \pi_1 \circ dh_{n-1} \\
0 &= \pi_1 \circ dh_n
\end{aligned} \tag{2.47}$$

with π_0 as in (2.36) and with

$$\pi_1 = . \left(\begin{array}{c|c} \theta_1 & \theta_0 \circ dh_1 \\ \hline -(\theta_0 \circ dh_1)^T & 0 \end{array} \right). \tag{2.48}$$

If we use the following notation

$$H_\xi(q,p) = E_\xi(q,p) + V_\xi(q), \qquad V_\xi = \sum_{j=0}^{n-1} V_{n-j}\xi^j,$$

then the recursion formulas (2.43) and (2.44) can be written in a compact form [20]

$$V_\xi^{(k+1)} = \lambda V_\xi^{(k)} - \det(\xi I - L)V_1^{(k)} \tag{2.49}$$

and

$$V_\xi^{(-k-1)} = \frac{1}{\xi}\left(V_\xi^{(-k)} - \frac{\det(\xi I - L)}{\det L}V_n^{(-k)}\right) \tag{2.50}$$

and our bi-Hamiltonian chain (2.47) is given by

$$\pi_\xi \circ dh_\xi(q,p,c) = 0, \quad h_\xi(q,p,c) = H_\xi(q,p) + c\rho_\xi, \tag{2.51}$$

while the corresponding quasi-bi-Hamiltonian (2.38) chain takes the form

$$\theta_\xi \circ dH_\xi(q,p) + \rho_\xi\theta_0 \circ dH_1(q,p) = 0. \tag{2.52}$$

In our (λ,μ) coordinates the full (i.e. with a non-zero potential part) Hamiltonians (2.46) of our bi-Hamiltonian chain (2.47) attain the form

$$h_r(\lambda,\mu,c) = -\sum_{i=1}^{n} \frac{\partial\rho_r}{\partial\lambda^i}\frac{\frac{1}{2}f_i(\lambda^i)\mu_i^2 + \gamma_i(\lambda^i)}{\Delta_i} + c\rho_r(\lambda) \quad , \quad r = 1,...,n, \tag{2.53}$$

Lemma 5 *Potentials* $V_r^{(k)}$ *and* $V_r^{(-k)}$ $k = 1,2,...$ *enter separation curve*

$$H_1\xi^{n-1} + H_2\xi^{n-2} + ... + H_n = \frac{1}{2}f(\xi)\mu^2 + \gamma(\xi) \tag{2.54}$$

as $\gamma(\xi) = \xi^{n+k-1}, \xi^{-k}$ *and hence* $\gamma_i(\lambda^i) = (\lambda^i)^{n+k-1}$ *and* $\gamma_i(\lambda^i) = (\lambda^i)^{-k}$*, respectively.*

Proof. Potentials $V_r^{(1)} = \rho_r$ are coefficients of characteristic equation of the special conformal Killing tensor L

$$\xi^n + \rho_1\xi^{n-1} + ... + \rho_n = 0. \tag{2.55}$$

Then, we define $V^{(k+1)}$ potentials as

$$\xi^{n+k} + V_1^{(k+1)}\xi^{n-1} + ... + V_n^{(k+1)} = 0. \tag{2.56}$$

The recursion formula (2.43) is reconstructed as follows. From (2.56) we have

$$\xi^{n+k+1} + V_1^{(k+1)}\xi^n + ... + V_n^{(k+1)}\xi = 0. \tag{2.57}$$

Substituting (2.55) we find

$$\xi^{n+k+1} + (V_2^{(k+1)} - \rho_1 V_1^{(k+1)})\xi^{n-1} + ... + (V_n^{(k+1)} - \rho_{n-1}V_1^{(k+1)})\xi - \rho_n V_1^{(n+1)} = 0$$

$$\Downarrow \tag{2.58}$$
$$V_r^{(k+2)} = V_{r+1}^{(k+1)} - \rho_r V_1^{(k+1)}.$$

For the inverse potentials we have from (2.55)

$$\frac{1}{\rho_n}\xi^{n-1} + \frac{\rho_1}{\rho_n}\xi^{n-2} + ... + \frac{\rho_{n-1}}{\rho_n} + \xi^{-1} = 0, \tag{2.59}$$

so $V_r^{(-1)} = \frac{\rho_{r-1}}{\rho_n}$. Then, we define $V^{(-k)}$ potentials as follows

$$V_1^{(-k)}\xi^{n-1} + ... + V_n^{(-k)} + \xi^{-k} = 0. \tag{2.60}$$

The recursion formula (2.44) is reconstructed as follows. From (2.60) we have

$$V_1^{(-k)}\xi^{n-2} + ... + V_n^{(-k)}\xi^{-1} + \xi^{-k-1} = 0. \tag{2.61}$$

Substituting (2.59) we have

$$V_1^{(-k)}\xi^{n-2} + ... + V_n^{(-k)}(-\frac{1}{\rho_n}\xi^{n-1} - ... - \frac{\rho_{n-1}}{\rho_n}) + \xi^{-k-1} = 0$$
$$\Downarrow \tag{2.62}$$
$$V_r^{(-k-1)} = V_{r-1}^{(-k)} - \frac{\rho_{r-1}}{\rho_n}V_n^{(-k)}.$$

■

Finally notice, that the inverse Jacobi problem (2.29) is modified via $\psi(\xi) = \sqrt{\frac{1}{2}f(\xi)(a_\xi + \gamma(\xi))}$.

2.4 Example: Henon-Heiles system

In last decade a considerable progress has been made in construction of new integrable finite dimensional dynamic systems showing bi-Hamiltonian property. The majority of them originate from stationary flows, restricted flows or nonlinearization of Lax equations of underlying soliton systems [21]-[33], [4]. Here we illustrate all previous considerations on the example of the Hamiltonian system with two degrees of freedom, i.e. the integrable case of the Henon-Heiles system related to the stationary flow of the $5th$ order Korteveg de Vries soliton equation [24]. The Newton equations of motion are the following

$$q^1_{tt} = -3(q^1)^2 - \frac{1}{2}(q^2)^2 + c, \qquad q^2_{tt} = -q^1q^2. \tag{2.63}$$

With conjugate coordinates $p_1 = q^1_t, p_2 = q^2_t$ the system (2.63) is a part of the bi-Hamiltonian chain [24]

$$\begin{aligned} \pi_0 \circ dh_0 &= 0 \\ \pi_0 \circ dh_1 &= X_1 = \pi_1 \circ dh_0 \\ \pi_0 \circ dh_2 &= X_2 = \pi_1 \circ dh_1 \\ 0 &= \pi_1 \circ dh_2 \end{aligned} \tag{2.64}$$

where

$$\begin{aligned} h_0 &= c, \\ h_1 &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + (q^1)^2 + \frac{1}{2}q^1(q^2)^2 - cq^1, \\ h_2 &= \frac{1}{2}q^2 p_1 p_2 - \frac{1}{2}q^1 p_2^2 + \frac{1}{16}(q^2)^4 + \frac{1}{4}(q^1)^2(q^2)^2 - \frac{1}{4}c(q^2)^2, \end{aligned} \tag{2.65}$$

the first Poisson structure is canonical

$$\pi_0 = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the second one takes the form

$$\pi_1 = \begin{pmatrix} 0 & 0 & q^1 & \frac{1}{2}q^2 & p_1 \\ 0 & 0 & \frac{1}{2}q^2 & 0 & p_2 \\ -q^1 & -\frac{1}{2}q^2 & 0 & \frac{1}{2}p_2 & -2q^1 - \frac{1}{2}(q^2)^2 - c \\ -\frac{1}{2}q^2 & 0 & -\frac{1}{2}p_2 & 0 & -q^1 q^2 \\ -p_1 & -p_2 & 2q^1 + \frac{1}{2}(q^2)^2 + c & q^1 q^2 & 0 \end{pmatrix}. \tag{2.66}$$

¿From Hamiltonian functions (2.50) and the second Poisson structure (2.66) one can reconstruct the contravariant metric tensor G, the special conformal Killing tensor L and Killing tensors A_1, A_2

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix}, \quad A_1 = G, \quad A_2 = \begin{pmatrix} 0 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & -q^1 \end{pmatrix}. \tag{2.67}$$

The first pair of separated coordinates are eigenvalues of L calculated from coefficients of characteristic polynomial of L, given also directly in (2.55)

$$\rho_1 = -\lambda^1 - \lambda^2 = -q^1, \qquad \rho_2 = \lambda^1\lambda^2 = -\frac{1}{4}(q^2)^2,$$

$$\Downarrow$$

$$q^1 = \lambda^1 + \lambda^2, \quad q^2 = 2\sqrt{-\lambda^1\lambda^2}.$$

The missing part of the transformation we reconstruct from the generating function $W(p,\lambda) = p_1(\lambda^1 + \lambda^2) + 2p_2\sqrt{-\lambda^1\lambda^2}$ (2.13)

$$p_1 = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \qquad p_2 = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right).$$

Expressing Hamiltonians (2.65) in new coordinates one finds

$$\frac{1}{2}f_i(\lambda^i)\mu_i^2 + g_i(\lambda^i) = \frac{1}{2}\lambda^i\mu_i^2 + (\lambda^i)^4, \qquad i = 1,2 \tag{2.68}$$

and hence the inverse Jacobi problem (2.29)

$$\sum_{k=1}^{2}\int^{\lambda^k}\frac{(\xi)^{i-1}}{\psi(\xi)}d\xi = t_r\delta_{ri} + const_i, \quad r,i = 1,2, \quad \psi(\xi) = \sqrt{\frac{1}{2}\xi(-\xi^4 + c\xi^2 + a_1\xi + a_2)}\,. \tag{2.69}$$

3 Bi-Hamiltonian dispersionless field systems in (1+1) dimension

There is a quite well developed theory of the passage between an integrable, infinite dimensional Hamiltonian system (soliton system) and its various constrained flows which are themselves completely integrable Hamiltonian systems. Actually, by using the Hamilton-Jacobi method with respect to two evolution parameters x and t, N-gap solutions and N-soliton solutions of a given PDE can be constructed directly from solutions of related ODEs (constrained flows) [34]-[38].

In the following section we are interested in, instead of soliton systems, the first order quasi-linear PDEs of the form

$$q^i_t = \sum_{j=1}^{n} w_{ij}(q) q^j_x, \qquad q_i = q_i(x,t), \;\; i = 1, ..., n, \tag{3.1}$$

called *hydrodynamic* or *dispersionless* systems. More precisely, we consider these systems among (3.1), which have bi-Hamiltonian structure, infinite hierarchy of symmetries and conservation laws.

Let again, as in the previous section, g_{ij} and G^{ij} be a covariant and contravariant metric tensors such that

$$\sum_k g_{ik} G^{kj} = \delta^j_i. \tag{3.2}$$

Coefficients of the Levi-Civita connection $\Gamma^i_{jk}(q)$ are given by (1.4), while the components of the tensor of Riemannian curvature $R^i_{jkl}(q)$ are

$$R^i_{jkl}(q) = \frac{\partial \Gamma^i_{jl}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^l} + \sum_p \left(\Gamma^i_{pk} \Gamma^p_{jl} - \Gamma^i_{pl} \Gamma^p_{jk} \right). \tag{3.3}$$

One can rise and lower indices using g and G metrics

$$\Gamma^{ij}_k(q) = \sum_s G^{is} \Gamma^j_{sk}(q), \qquad \Gamma^i_{jk}(q) = \sum_s g_{js} \Gamma^{si}_k,$$

$$R^{ij}_{kl}(q) = \sum_s G^{is} R^j_{skl}(q), \qquad R^i_{jkl}(q) = \sum_s g_{js} R^{si}_{kl}. \tag{3.4}$$

Definition 6 *[39] Two Riemannian or pseudo-Riemannian contravariant metrices $G^{ij}_1(q), G^{ij}_2(q)$ are called compatible if for any linear combination of these metrices*

$$G^{ij}(q) = G^{ij}_1(q) + \xi G^{ij}_2(q), \qquad \xi \in \mathbb{R}, \tag{3.5}$$

such that $\det(G^{ij}) \neq 0$, *the coefficients of the corresponding Levi-Civita connections and the components of the corresponding tensor of Riemannian curvature* *are related by the same linear formula:*

$$\Gamma^{ij}_{k}(q) = \Gamma_{1}{}^{ij}_{k}(q) + \xi \Gamma_{2}{}^{ij}_{k}(q), \tag{3.6a}$$

$$R^{ij}_{kl}(q) = R_{1}{}^{ij}_{kl}(q) + \xi R_{2}{}^{ij}_{kl}(q). \tag{3.6b}$$

Then we say that G_1 and G_2 form a pencil of metrices.

The simplest case is the one of two flat metrices where (3.6a) is fulfilled and $R^{ij}_{kl}(q) = 0$. Another important case is the one with constant curvature, then

$$\begin{gathered} R_{1}{}^{ij}_{kl} = K_1(\delta^i_l \delta^j_k - \delta^i_k \delta^j_l), \quad R_{2}{}^{ij}_{kl} = K_2(\delta^i_l \delta^j_k - \delta^i_k \delta^j_l) \\ \Downarrow \\ R^{ij}_{kl} = (K_1 + \xi K_2)(\delta^i_l \delta^j_k - \delta^i_k \delta^j_l), \end{gathered} \tag{3.7}$$

where K_1, K_2 are arbitrary real constants.

For the flat case a local Poisson structure of the hydrodynamic (dispersionless) type or Dubrovin-Novikov [40] structure is defined by a tensor

$$\theta^{ij}(q) = G^{ij}(q)\partial_x - \sum_k \Gamma^{ij}_k q^k_x. \tag{3.8}$$

For a constant curvature case, a nonlocal Poisson structure of hydrodynamic type, or Mokhov-Ferapontov [41] structure is defined by a tensor

$$\theta^{ij}(q) = G^{ij}(q)\partial_x - \sum_k \Gamma^{ij}_k q^k_x + K q^i_x \partial_x^{-1} q^j_x, \tag{3.9}$$

where $G^{ij}(q)$ is a contravariant metric of constant curvature K. That is, for arbitrary functionals $F[q]$ and $H[q]$ on the space of fields $q^i(x), i = 1, ..., n$ a bracket of the form

$$\{F, H\}_\theta = \int_\Omega \frac{\delta F}{\delta q} \theta(q) \frac{\delta H}{\delta q} dx \tag{3.10}$$

is a Poisson bracket.

Lemma 7 *Two Poisson tensors θ_1 and θ_2 are compatible if related metrices are compatible.*

Theorem 8 *[39] Any nonsingular pair of matrices is compatible if and only if there exist local coordinates $\lambda = (\lambda^1, ..., \lambda^n)$ such that*

$$G^{ij}_1(\lambda) = g^i(\lambda)\delta^{ij}, \qquad G^{ij}_2(\lambda) = \chi_i(\lambda^i) g^i(\lambda)\delta^{ij}, \tag{3.11}$$

where $\chi_i(\lambda^i)$ are arbitrary, different from zero, smooth functions.

¿From this theorem and the results of previous sections we find a direct passage from integrable Stäckel finite-dimensional systems to bi-Hamiltonian dispersionless field systems. Actually, having some Riemann manifold with a special conformal Killing tensor, i.e. a triple (Q, G, L), we have immediately a pair of compatible metrices

$$G_1 \equiv G, \qquad G_2 = LG, \tag{3.12}$$

as in λ coordinates they are of the special form of (3.11)

$$g^i(\lambda) = \frac{f_i(\lambda^i)}{\Delta_i}, \quad \chi_i(\lambda^i) = \lambda^i. \tag{3.13}$$

Here we illustrate the results of this section constructing appropriate bi-Hamiltonian dispersionless field hierarchy, related to the Henon-Heiles system considered in the previous section. From (2.67) we have

$$G_1 = G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_2 = \; LG = \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix}, \tag{3.14}$$

which, according to (2.18), (2.68) and the Remark 2, are compatible flat metrices. So, related Dubrovin-Novikov Poisson structures are

$$\begin{aligned} \theta_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_x, \\ \theta_1 &= \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} \frac{1}{2}q^1_x & 0 \\ \frac{1}{2}q^2_x & 0 \end{pmatrix} \\ &= \frac{1}{2}\partial_x \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & -\frac{1}{4}q^2_x \\ \frac{1}{4}q^2_x & 0 \end{pmatrix}. \end{aligned} \tag{3.15}$$

Bi-Hamiltonian chain is constructed starting with the Casimir $\gamma_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \delta \int_\Omega 2q^1 dx$ of the first Poisson structure θ_0. Applying the second Poisson structure θ_1 we get the first trivial flow

$$\begin{pmatrix} q^1 \\ q^2 \end{pmatrix}_{t_1} = \theta_1 \circ \gamma_0 = \sigma_1 = \begin{pmatrix} q^1_x \\ q^2_x \end{pmatrix}, \tag{3.16}$$

which is bi-Hamiltonian, as $\sigma_1 = \theta_0 \circ \gamma_1$, where

$$\gamma_1 = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = \delta \int_\Omega \left[\frac{1}{2}(q^1)^2 + \frac{1}{2}(q^2)^2 \right] dx. \tag{3.17}$$

The first nontrivial dispersionless flow we find acting with θ_1 on γ_1

$$\begin{pmatrix} q^1 \\ q^2 \end{pmatrix}_{t_2} = \theta_1 \circ \gamma_1 = \sigma_2 = \begin{pmatrix} \frac{3}{2}q^1 q^1_x + \frac{1}{2}q^2 q^2_x \\ \frac{1}{2}q^1 q^2_x + \frac{1}{2}q^2 q^1_x \end{pmatrix}, \tag{3.18}$$

which also is bi-Hamiltonian, as

$$\sigma_2 = \theta_0 \circ \gamma_2, \quad \gamma_2 = \begin{pmatrix} \frac{3}{4}(q^1)^2 + \frac{1}{4}(q^2)^2 \\ \frac{1}{2}q^1 q^2 \end{pmatrix} = \delta \int_\Omega \left[\frac{1}{4}(q^1)^3 + \frac{1}{4}q^1 (q^2)^2 \right] dx. \tag{3.19}$$

The infinite hierarchy of commuting bi-Hamiltonian vector fields σ_n one can construct with the help of the recursion operator

$$N = \theta_1 \circ \theta_0^{-1} = \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}q^1_x & 0 \\ \frac{1}{2}q^2_x & 0 \end{pmatrix} \partial_x^{-1}, \tag{3.20}$$

as $\sigma_{n+1} = N^n \circ \sigma_1$.

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