

# Conserved charges in gravitational theories: contribution from scalar fields

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## Abstract

In order to illustrate a recently derived covariant formalism for computing asymptotic symmetries and asymptotically conserved superpotentials in gauge theories, the case of gravity with minimally coupled scalar fields is considered and the matter contribution to the asymptotically conserved superpotentials is computed.

## 1 Introduction

Noether currents associated to gauge symmetries are problematic because they are given on-shell by the divergence of arbitrary superpotentials. It can however be shown [1, 2] that reducibility parameters, i.e., parameters of gauge symmetries that leave the solutions of the field equations invariant, are related to conserved superpotentials. Furthermore, if one defines suitable equivalence classes of reducibility parameters and of conserved superpotential, this relation becomes one-to-one and onto, and the algebra associated with the classes of reducibility parameters can be represented in a covariant way on the classes of the superpotentials [3].

In full semi-simple Yang-Mills theory or in full general relativity, there are in fact no non trivial reducibility parameters and thus also no non trivial

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conserved superpotentials. However, non trivial results are obtained if one considers the linearized theory around a given background solution. This observation serves as a starting point for developing a systematic and covariant theory of asymptotic symmetries and conservation laws for generic gauge theories [4, 5, 3].

Concerning the choice of the boundary and fall-off conditions, two approaches are possible.

The first is to define from the outset what one means by a field that is “asymptotically background” by specifying the boundary and the fall-off conditions on these fields. The formalism developed in [3] then allows to define and compute asymptotic reducibility parameters and the associated asymptotically conserved  $n - 2$  forms.

An alternative approach is to first compute the exact reducibility parameters and the associated conserved  $n - 2$  forms of the linearized theory around the background. For a given boundary, their asymptotic behaviour can then be determined. In this case, the formalism of [3] is used not only to define asymptotic reducibility parameters and the associated asymptotically conserved  $n - 2$  forms, but also to specify boundary conditions on the left hand sides of the field equations. These latter conditions then serve to define fields that are “asymptotically background”: their fall-off is such that, when inserted in the left hand sides of the field equations, the boundary conditions are satisfied.

The way superpotentials are associated to reducibility parameters is controlled by the linearized theory and is independent of the choice for the boundary and the fall-off conditions. In this short note, we apply the corresponding part of the formalism of [3] in the context of matter coupled gravity to derive the matter contribution to the (asymptotically) conserved  $n - 2$  forms, and thus also to the charges. Even though various types of matter fields can be dealt with in a straightforward way, we consider here for simplicity only minimally coupled scalar fields.

## 2 Contribution of scalars to superpotentials

As in section 6.3 of [3], we suppose that the spacetime dimension is  $n \geq 3$  and that the Lagrangian is given by

$$L = \frac{1}{16\pi} \sqrt{-g} (R - 2\Lambda) + L_{\text{matter}}. \quad (1)$$

In the present case, the matter Lagrangian is explicitly given by

$$L_{\text{matter}} = -\frac{\sqrt{-g}}{8\pi}[g^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi_i + V(\phi)]. \quad (2)$$

The metric is decomposed as  $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ , where  $\bar{g}_{\mu\nu}$  denotes the background metric, which together with its inverse is used to lower and raise the indices in the linearized theory, while  $h_{\mu\nu}$  denotes the metric perturbations. Similarly, the scalar fields are decomposed as  $\phi^i = \bar{\phi}^i + \varphi^i$ . The exact reducibility parameters of the linearized theory are given by the Killing vectors  $\xi^\mu$  of the background metric,  $L_\xi\bar{g}_{\mu\nu} = 0$ , that satisfy in addition  $L_\xi\bar{\phi}^i = 0$ , while the asymptotic reducibility parameters are the vectors  $\xi^\mu$  that satisfy the asymptotic counterparts (see [3]) of these two conditions.

The linearized equations of motion are

$$\mathcal{H}^{\mu\nu} + \frac{\sqrt{-\bar{g}}}{2} T_{\text{lin}}^{\mu\nu} = 0, \quad (3)$$

where  $\mathcal{H}^{\mu\nu}[h; \bar{g}]$  is the linear part of  $\delta[(1/16\pi)\sqrt{-g}(R - 2\Lambda)]/\delta g_{\mu\nu}$ , (see e.g. (6.13) of [3]) for the explicit expression) while  $\sqrt{-\bar{g}} T_{\text{lin}}^{\mu\nu}/2$  is the linear part of  $\sqrt{-g}T^{\mu\nu}/2 = \delta L_{\text{matter}}/\delta g_{\mu\nu}$ . Explicitly,

$$\frac{\sqrt{-\bar{g}}}{2} T_{\text{lin}}^{\mu\nu} = \frac{\sqrt{-\bar{g}}}{8\pi}[\partial^\mu\varphi^i\partial^\nu\bar{\phi}^i + \partial^\mu\bar{\phi}^i\partial^\nu\varphi^i + \dots], \quad (4)$$

where the dots denote terms that do not involve derivatives of  $h_{\mu\nu}$  or  $\varphi^i$ . Because the gauge transformations  $L_\xi\bar{\phi}^i$  do not involve derivatives of the gauge parameters, the only contribution from the scalar fields to the  $n - 1$  form  $(d^{n-1}x)_\mu s_\xi^\mu$  (associated to the (asymptotic) reducibility parameters  $\xi^\mu$  and used to determine the asymptotically conserved  $n - 2$  form according to (1.13) of [3]) is given by

$$(d^{n-1}x)_\mu\sqrt{-\bar{g}}T_{\text{lin}}^{\mu\rho}\xi_\rho, \quad (5)$$

The contribution from the scalar fields to the (asymptotically) conserved  $n - 2$  form  $\tilde{k}_\xi^{[\nu\mu]}(d^{n-2}x)_{\nu\mu}$  is in turn given by

$$\begin{aligned} (d^{n-2}x)_{\nu\mu}[\varphi^i\frac{\partial}{\partial\varphi_\nu^i}\sqrt{-\bar{g}}T_{\text{lin}}^{\mu\rho}\xi_\rho - (\mu \leftrightarrow \nu)] &= \\ &= (d^{n-2}x)_{\nu\mu}\frac{\sqrt{-\bar{g}}}{4\pi}\varphi_i[\partial^\mu\bar{\phi}^i\xi^\nu - \partial^\nu\bar{\phi}^i\xi^\mu]. \end{aligned} \quad (6)$$

In the three dimensional case with coordinates  $t, r, \theta$ , and boundary the circle at  $r \rightarrow \infty$ , the matter contribution to the charges  $Q_\xi$  is

$$\lim_{r \rightarrow \infty} \int_0^{2\pi} d\theta \frac{\sqrt{-\bar{g}}}{4\pi} \varphi_i [\partial^r \bar{\phi}^i \xi^t - \partial^t \bar{\phi}^i \xi^r]. \quad (7)$$

The contribution of the scalar fields to these charges has recently turned out to be relevant [6] in the context of black hole solutions of 2 + 1 gravity with a scalar field.

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