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Gravity, hyperbolic billiards and Lorentzian Kac-Moody algebras

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Abstract

Lorentzian Kac-Moody algebras control the asymptotic dynamics of gravitational theories in the vicinity of a cosmological singularity. The list of the Lorentzian Kac-Moody algebras that have already been encountered in this context is explicitly given.

1 Introduction

Motivated by the classic work of Belinskii, Khalatnikov and Lifshitz [1], it was recently found that the dynamics of the Einstein-dilaton-p-form system in any number of spacetime dimensions can be described, near a (past or future) spacelike singularity, as a billiard motion in a region of hyperbolic space [2, 3]. The dimension of the billiard, which is called "Einstein billiard", as well as its precise shape, depend on the theory at hand. For four-dimensional vacuum gravity, one recovers the billiards described in [4, 5].

The billiard picture emerges once one realizes that the spatial points effectively decouple as one approaches the singularity, and that the non trivial asymptotic dynamics of the system essentially reduces to the evolution of the local scale factors (controlling how distances along independent directions scale) and of the dilaton(s). The emergence of hyperbolic geometry is

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related to the fact that the De Witt "supermetric" [6] in the space $\mathcal{M}$ of the scale factors and the dilaton(s) has Lorentzian signature. Spatial derivatives and p-form terms in the Hamiltonian define exponential potentials in $\mathcal{M}$, which become sharp wall potentials in the asymptotic limit. The resulting available region is the "billiard table". The motion of the scale factors and the dilaton(s) is a free motion interrupted by collisions with the billiard walls. A detailed derivation of the billiards is given in [7].

It was further realized in [3] that the billiards associated (i) with the low energy bosonic sectors of 11-dimensional supergravity, or of types II_A and II_B supergravities in 10-dimensions, or with 10-dimensional type I supergravities (ii) with or (iii) without a vector multiplet, could be identified with the fundamental Weyl chamber of the Kac-Moody algebra E_{10} (case (i)), BE_{10} (case (ii)) or DE_{10} (case (iii)), which are all of indefinite type. This is because the linear forms α_i that determine the (dominant) walls define a matrix through

$$A_{ij} = 2\,\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} \tag{1}$$

which can be identified with the (generalized) Cartan matrix of the Kac-Moody algebra at hand, the linear forms themselves being identified with the simple roots. These striking symmetry properties of the billiards relevant to the asymptotic dynamics hold also for pure gravity in any number of spacetime dimensions $D = d + 1$, for which the billiard turns out to be the fundamental Weyl chamber of AE_d [8]. The billiard dynamics is chaotic when the Kac-Moody algebra is hyperbolic [8], which is the case for E_{10}, BE_{10}, DE_{10} and AE_d with $d \leq 9$. Vacuum gravity is no longer chaotic in spacetime dimensions ≥ 11, as observed previously in [9].

Many other models have been investigated since then, including pure $D = 4$ supergravity theories, with similar conclusions [10, 11]: in each case, the billiard has remarkable symmetry properties, i.e., is the fundamental Weyl chamber of a Lorentzian Kac-Moody algebra. The purpose of this review is to provide the explicit list of all the Lorentzian Kac-Moody that have already been uncovered in this context. We refer to the original references [3, 8, 10, 11] for the detailed derivation, which includes the explicit identification of the dominant wall forms α_i and the computation of the generalized Cartan matrix through the use of (1).

2 A few definitions

Kac-Moody algebras are defined in [12]. A Kac-Moody algebra is Lorentzian if its Cartan matrix is non degenerate and symmetrizable, and such that the invariant metric in the Cartan subalgebra has Lorentzian signature $-, +, \cdots, +$. It is hyperbolic if, in addition, its Dynkin diagram is such that if one removes any node from it, one gets the Dynkin diagram of an affine or finite Kac-Moody algebra. Hyperbolic Kac-Moody algebras exist only in ranks ≤ 10.

Given a finite-dimensional simple Lie algebra $\mathcal{G}$, there is a standard way to derive from it a Lorentzian Kac-Moody algebra $\mathcal{G}^{\wedge\wedge}$, called its "double extension", "overextension" or "canonical Lorentzian extension" [13, 12, 14]. On first adds the affine root that turns $\mathcal{G}$ into its untwisted affine extension $\mathcal{G}^{\wedge}$; one then adds a further root attached to the affine root by a single link. There are other ways to make Lorentzian Kac-Moody algebras out of a finite-dimensional simple Lie algebra, in which the intermediate affine step involves a twist [11].

It turns out that for each finite-dimensional simple Lie algebra $\mathcal{G}$, a model exists that exhibits its overextension $\mathcal{G}^{\wedge\wedge}$ [10]. These models have in fact been studied previously in [15, 16] with different purposes. Some twisted extensions appear in pure $D = 4$ supergravities [11]. They are listed below.

3 Models leading to $A_n^{\wedge\wedge}$

In each case, we give the Lagrangian of the theory (in the maximum dimension = "endpoint of the oxidation sequence" [16]) and the generalized Cartan matrix of the underlying Lorentzian Kac-Moody algebra.

The model leading to $A_n^{\wedge\wedge}$ is pure gravity in spacetime dimensions $D = n + 3$,

$$\mathcal{L}_D = R \star 1, \quad D = n + 3 \tag{2}$$

The computation of the relevant walls and of the Cartan matrix has been done in [8], where it was indeed found to be just the generalized Cartan matrix of the overextension $A_n^{\wedge\wedge}$ of the Lie algebra A_n. [$A_n^{\wedge\wedge}$ is also denoted

$AE_{(n+2)}$]. For $D > 4$, the generalized Cartan matrix reads

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

For $D = 4$, one has

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

The algebra is hyperbolic for $n < 8$. The extension $AE_9 \equiv A_7^{\wedge\wedge}$ is the last hyperbolic algebra in the family; $AE_{10} \equiv A_8^{\wedge\wedge}$ is not hyperbolic.

4 Models leading to $B_n^{\wedge\wedge}$

The model leading to $B_n^{\wedge\wedge}$ is formulated in $D = n + 2$ spacetime dimensions and contains the metric, a dilaton, a 2-form, B, and a 1-form, A. The Lagrangian reads [16]

$$\mathcal{L}_D = R \star 1 - \star d\phi \wedge d\phi - \frac{1}{2} e^{a\sqrt{2}\phi} \star G \wedge G - \frac{1}{2} e^{a\frac{\sqrt{2}}{2}\phi} \star F \wedge F, \qquad (3)$$

where $a^2 = 8/n$ and

$$G = dB + \frac{1}{2} A \wedge dA, \qquad F = dA. \qquad (4)$$

A straightforward computation yields the dominant walls and the following generalized Cartan matrix of the overextension $B_n^{\wedge\wedge}$ [10]

$$\left(\begin{array}{ccccccccc} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -2 & 2 \end{array}\right)$$

The overextension $B_n^{\wedge\wedge}$ is hyperbolic for $n \leq 8$. For $n = 8$, one recovers the Kac-Moody billiard of BE_{10} given in [3], which controls the asymptotic dynamics of the low-energy bosonic sectors of the heterotic and type I superstrings. The hyperbolic character of BE_{10} is another way to see that these models are chaotic.

5 Models leading to $C_n^{\wedge\wedge}$

There is no known formulation in $D > 4$ spacetime dimensions. The Lagrangian of the $C_n^{\wedge\wedge}$ theories is given in $D = 4$ dimensions by [16]

$$\begin{aligned} \mathcal{L}_4 &= R \star 1 - \star d\vec{\phi} \wedge d\vec{\phi} - \frac{1}{2} \sum_{\alpha} e^{2\vec{\sigma}_\alpha . \vec{\phi}} \star (d\chi^\alpha + \cdots) \wedge (d\chi^\alpha + \cdots) \\ &\quad - \frac{1}{2} \sum_{a=1}^{n-1} e^{\vec{e}_a . \vec{\phi}\sqrt{2}} \star dA^a_{(1)} \wedge dA^a_{(1)} \end{aligned} \tag{5}$$

where the ellipsis complete the "curvatures" of the χ's [16]. The $(n-1)$ dilatons $\vec{\phi} = (\phi^1, ..., \phi^{n-1})$ are associated with the Cartan subalgebra of $Sp(2n-2, R)$ and the $\frac{1}{2}n(n-1)$ axions χ^α are associated with the positive roots of $Sp(2n-2, R)$. The fields $A^a_{(1)}$ are one-forms. The $\vec{\sigma}_\alpha$ are the positive roots of $Sp(2n-2, R)$; these can be written in terms of an orthonormalized basis of $(n-1)$ vectors in Euclidean space ($\vec{e}_a . \vec{e}_b = \delta_{ab}$) $\vec{e}_a$ ($a = 1, \cdots, n-1$) as

$$\vec{\sigma}_\alpha = \{\sqrt{2}\vec{e}_a, \frac{1}{\sqrt{2}}(\vec{e}_a \pm \vec{e}_b),\ a > b\}. \tag{6}$$

The normalization is such that the long roots have squared length equal to two. The notation $\vec{\sigma}_\alpha.\vec{\phi}$ means

$$\vec{\sigma}_\alpha.\vec{\phi} \equiv \sum_{a=1}^{n-1} \sigma_\alpha^a \phi^a.$$

The simple roots are

$$\sqrt{2}\vec{e}_1 \quad \text{and} \quad \frac{1}{\sqrt{2}}(\vec{e}_{a+1} - \vec{e}_a),\ a = 1, ..., n-2. \tag{7}$$

The walls of the billiard are easily determined and the Cartan matrix is found to be [10]

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2 \end{pmatrix},$$

which is recognized to be the generalized Cartan matrix of the overextension $C_n^{\wedge\wedge}$. It is hyperbolic for $n \leq 4$.

6 Models leading to $D_n^{\wedge\wedge}$

For $D_n^{\wedge\wedge}$, the Lagrangian (in $D = n + 2$ dimensions) is [16]

$$\mathcal{L}_D = R \star 1 - \star d\phi \wedge d\phi - \frac{1}{2} e^{a\sqrt{2}\phi} \star dB \wedge dB \tag{8}$$

where B is a 2-form and $a^2 = 8/n$.

The resulting Cartan matrix is given by [10]

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}$$

It is indeed the generalized Cartan matrix of $D_n^{\wedge\wedge} = DE_{n+2}$ which is known to be hyperbolic for $n \leq 8$.

For $n = 8$, one gets the last hyperbolic algebra in this family, namely $DE_{10} \equiv D_8^{\wedge\wedge}$, [3]. For $n = 24$, which is the case relevant to the bosonic string, one gets $D_{24}^{\wedge\wedge}$.

7 Models associated with exceptional groups

7.1 $G_2^{\wedge\wedge}$ [8]

The model is the Einstein-Maxwell system in $D = 5$ with an extra FFA term:

$$\mathcal{L}_5 = R \star 1 - \frac{1}{2} \star F \wedge F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A, \qquad F = dA. \tag{9}$$

The Cartan matrix reads

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

which is the generalized Cartan matrix of $G_2^{\wedge\wedge}$. This algebra is hyperbolic.

7.2 $F_4^{\wedge\wedge}$ [10]

The model is a $D = 6$ dimensional theory containing the metric, a dilaton (ϕ), an axion (χ), two one-forms ($A^{\pm}$), a two-form (B) and a self-dual 3-form field strength (G) [16]. The Lagrangian is given by

$$\begin{aligned}\mathcal{L}_6 = \quad & R\star 1 - \star d\phi\wedge d\phi - \frac{1}{2}e^{2\phi}\star d\chi\wedge d\chi - \frac{1}{2}e^{-2\phi}\star H\wedge H & (10)\\ & -\frac{1}{2}\star G\wedge G - \frac{1}{2}e^{\phi}\star F^{+}\wedge F^{+} - \frac{1}{2}e^{-\phi}\star F^{-}\wedge F^{-} & (11)\\ & -\frac{1}{\sqrt{2}}\chi\, H\wedge G - \frac{1}{2}A^{+}\wedge F^{+}\wedge H - \frac{1}{2}A^{+}\wedge F^{-}\wedge G. & (12)\end{aligned}$$

The field strengths are given in terms of potentials as:

$$F^{+} = dA^{+} + \frac{1}{\sqrt{2}}\chi\, dA^{-} \tag{13}$$

$$F^{-} = dA^{-} \tag{14}$$

$$H = dB + \frac{1}{2}A^{-}\wedge dA^{-} \tag{15}$$

$$G = dC - \frac{1}{\sqrt{2}}\chi\, H - \frac{1}{2}A^{+}\wedge dA^{-}. \tag{16}$$

The Cartan matrix of the dominant walls is given by [10]

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0 & 0\\ 0 & 0 & -1 & 2 & -1 & 0\\ 0 & 0 & 0 & -2 & 2 & -1\\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

This is the generalized Cartan matrix of the overextension $F_4^{\wedge\wedge}$, which is hyperbolic.

7.3 $E_6^{\wedge\wedge}$ [10]

The $E_6^{\wedge\wedge}$-theory can be obtained as a $D = 8$ truncation of maximal supergravity in which the 3-form potential is retained. It includes the metric, a dilaton and an axion,χ, together with the 3-form, C [16]. The 8-dimensional Lagrangian reads

$$\mathcal{L}_8 = R\star 1 - \star d\phi\wedge d\phi - \frac{1}{2}e^{2\sqrt{2}\phi}\star d\chi\wedge d\chi - \frac{1}{2}e^{-\sqrt{2}\phi}\star G\wedge G + \chi\, G\wedge G, \tag{17}$$

where $G = dC$.

The Cartan matrix is given by [10]

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

It is the generalized Cartan matrix of the hyperbolic algebra $E_6^{\wedge\wedge}$.

7.4 $E_7^{\wedge\wedge}$ [10]

This model is obtained as a consistent (albeit non supersymmetric) truncation of $D = 9$ maximal supergravity to the theory whose bosonic sector comprises the metric, a dilaton, a 1-form, A, and a 3-form potential C [16]. The Lagrangian reads

$$\begin{aligned} \mathcal{L}_9 = \quad & R \star 1 - \star d\phi \wedge d\phi - \frac{1}{2} e^{\frac{2\sqrt{2}}{\sqrt{7}}\phi} \star dC \wedge dC \\ & -\frac{1}{2} e^{-\frac{4\sqrt{2}}{\sqrt{7}}\phi} \star dA \wedge dA - \frac{1}{2} dC \wedge dC \wedge A. \end{aligned} \tag{18}$$

The Cartan matrix is [10]

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

It is the overextension $E_7^{\wedge\wedge}$, which is hyperbolic.

7.5 $E_8^{\wedge\wedge}$ [3]

This is the $D = 11$-dimensional supergravity whose bosonic sector is given by

$$\mathcal{L}_{11} = R \star 1 - \frac{1}{2} \star dC \wedge dC - \frac{1}{6} dC \wedge dC \wedge C \tag{19}$$

C is a 3-form.

The Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

As pointed out in [3] this is the Cartan matrix of the overextension $E_8^{\wedge\wedge}$, better known as E_{10}. As shown in that paper, it is also the Cartan matrix relevant to type IIA supergravity in ten dimensions (dimensional reduction) as well as type IIB. The algebra is hyperbolic.

8 Twisted extensions

The previous models exhaust all overextensions $\mathcal{G}^{\wedge\wedge}$ of the finite-dimensional simple Lie algebras $\mathcal{G}$. Remarkably enough, twisted extensions also arise, in the context of $D = 4$ pure supergravities [11]. We list here the findings of [11]; N is the number of supersymmetries, $\mathcal{A}$ is the underlying Kac-Moody algebra.

Kac-Moody algebras of pure $D = 4$ supergravities

$$
\begin{array}{cc}
N & \mathcal{A} \\
 & \\
N=1 & A_1^{\wedge\wedge} \\
N=2 & A_2^{(2)\wedge} \\
N=3 & A_2^{(2)\wedge} \\
N=4 & C_2^{\wedge\wedge} \\
N=5 & A_4^{(2)\wedge} \\
N=6 & F_4^{\wedge\wedge} \\
N=8 & E_8^{\wedge\wedge}
\end{array}
\tag{20}
$$

We see that the twisted version $A_2^{(2)\wedge}$, with Cartan matrix

$$
\begin{pmatrix}
2 & -4 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\tag{21}
$$

appears for both $N=2$ and $N=3$, $D=4$ pure supergravities, while $A_4^{(2)\wedge}$,

$$
\begin{pmatrix}
2 & -2 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\tag{22}
$$

controls the asymptotic dynamics of $N=5$, $D=4$ pure supergravity. [$A_2^{(2)\wedge}$ is obtained by adding to the Dynkin diagram of the twisted affine algebra $A_2^{(2)}$ (in Kac's notations [12]) a root attached to the long root with a single link; $A_4^{(2)\wedge}$ is obtained by attaching with a single link a root to the longest root of the Dynkin diagram of the twisted affine algebra $A_4^{(2)}$.]

9 Conclusion

One sees from the above list that all the overextensions of finite dimensional simple Lie algebras appear. Some twisted extensions appear also. It would be of interest to investigate whether Lagrangians can be constructed that yield other twisted Lorentzian algebras, or whether all hyperbolic algebras can be generated.

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