Physics AUC, vol. 12 (part I), 53-60 (2002)

PHYSICS AUC

Numerical simulations of turbulent systems in hydrodynamics and magnetohydrodynamics

D. Carati, O. Debliquy and B. Knaepen

Physique Statistique et Plasmas, Association Euratom-Etat Belge, CP231

Université Libre de Bruxelles, B-1050 Brussels, Belgium

Abstract

Turbulence is one of the most challenging problems of classical physics. With the emergence of super-computers, numerical experiments are now available, giving some insights in this fascinating subject. However, despite the rapidly growing computational power, simulations of realistic fully turbulent flows in magnetohydrodynamics as well as in hydrodynamics will remain out of reach of foreseeable computers. Alternative approaches that mix numerical experiments and modelling strategies have thus been designed and applied with some success.

1 Introduction

The description of fluid turbulence has been a source of continuous "frustration" for physicists, mathematicians and engineers for more than a century. Indeed, the famous Navier-Stokes equation for the evolution of a fluid is known since the works of the French engineer Claude Navier and of the Irish mathematician George Stokes in the 19th century. Nevertheless, solving this equation when the fluid is in a turbulent state is recognised as one of the most complex mathematical problems faced by today's scientists. When turbulence phenomena in plasma physics are considered, the situation is even more complex for at least two reasons. First, instabilities are observed in both the microscopic and the macroscopic scales of the plasma. Second, when considering the fluid limit, turbulence appears to affect both the velocity and the electromagnetic fields.

Here, we will focus our discussion on the problems met when considering turbulence phenomena in the magnetohydrodynamic limit of an incompressible plasma. In that case, the electric field may be eliminated from the description and the equations, written with the magnetic field b_i expressed in Alfvén-speed units and the constant mass density rescaled to unity, read,

$$\partial_t v_i = -\partial_j(v_i v_j - b_i b_j) + \nu \Delta v_i - \partial_i p\,, \quad (1)$$

$$\partial_t b_i = -\partial_j(v_j b_i - v_i b_j) + \eta \Delta b_i\,, \quad (2)$$

$$\partial_i v_i = \partial_i b_i = 0\,, \quad (3)$$

where p is the sum of thermodynamic and magnetic pressures. It is obtained by imposing the incompressibility of the velocity field v_i; ν is the kinematic viscosity and η is the magnetic diffusivity. For magnetohydrodynamic (MHD) turbulence the numerical approach is of particular importance. Indeed, most turbulent plasmas are either situated beyond direct experimental reach, e.g., star-forming clouds and the earth's liquid core, or require rather expensive setups like in fusion devices. The rapid growth of available computer power has long been regarded as an alternative to the analytical approach of the Navier-Stokes and of the MHD equations. However, it will be discussed in section II that the computer time and the amount of memory required for simulating all the details of turbulent systems are so prohibitive that only weakly turbulent flows have been computed on available computers. Also, in Section III, it will be pointed out that these difficulties have prompted the development of approximate numerical methods in which modelling plays an important role.

2 Direct Numerical Simulation of MHD turbulence

Considering the difficulty to obtain exact or even approximate solutions to the MHD equations in the turbulent regime, the numerical simulation is regarded as an alternative to the mathematical study. Unfortunately, so-called numerical experiments are themselves rather limited on today's computer. This can be easily understood when considering the basic requirement for an accurate direct numerical simulation (DNS) of the MHD equations. It is well known that one of the effects of the viscosity is to prevent the development of structures with a characteristic scale much smaller than the viscous length

ℓ_ν. Variations of the velocity on scales smaller than ℓ_ν can thus be neglected. Similarly, variations of the fluids on time scales much smaller than t_ν (the characteristic time associated to the smaller scales) can also be neglected.

Without introducing any concept from the numerical analysis of the velocity equation, we can already understand that a numerical experiment for a conducting fluid will be accurate under, at least, two conditions. First, the variables have to be known on locations separated by a distance of the order of ℓ_ν. The number of points necessary for describing accurately the system will thus be of the order of $N \approx (L/\ell_\nu)^3$ where L is the characteristic length of the physical domain that we want to simulate. Second, for each of these points, the variables must be known every time step t_ν. The total number of time steps is thus given by $N' \approx T/t_\nu$ where T is the total duration of the experiment.

A simple dimensional analysis can be used to relates N and N' to the Reynolds number $R_e = LU/\nu = L^2/(T\nu)$ which quantifies the ratio between the nonlinear convective term and the linear dissipative term in the velocity equation. Here, U is a typical large scale velocity in the flow. The following approximations can then be derived for Navier-Stokes turbulence [1]:

$$N \approx R_e^{9/4} \,, \tag{4}$$
$$N' \approx R_e^{3/4} \,. \tag{5}$$

As a consequence, the total number of numerical operations needed in a simulation increases according to $N \times N' = R_e^3$, which is a huge number for any realistic experiment. Turbulent flows commonly reach R_e much larger than 10^5. For such a value, the number of points in the simulation is already $N \approx 10^{11}$ and the number of values required in a computation is even 10 times larger. Hence, about 10^{12} real numbers are needed for simulating a flow at $R_e = 10^5$. This corresponds to about 4 000 Gb of memory.

For conducting fluids, the ratio between the nonlinear convective term and the linear dissipative term in the induction equation (2) is expressed by the magnetic Reynolds number $R_m = LU/\eta$. Depending on the magnetic Prandtl number ν/η, the numerical simulations of MHD flows can be even more challenging for today's computers. Nevertheless, DNS of turbulent MHD flows with moderate Reynolds numbers are used more and more extensively to explore the details of nonlinear interactions between the magnetic and the velocity fields [2, 3]. When compared to real experiments, DNS have indeed the major advantages of giving access to the complete spatio-temporal details

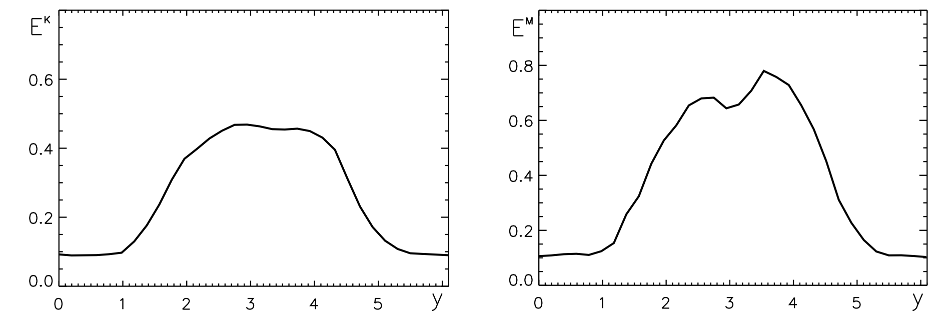

Figure 1: Typical profiles of the kinetic (left) and magnetic (right) energies along the inhomogeneous direction obtained using DNS of the MHD equations for the mixing layer.

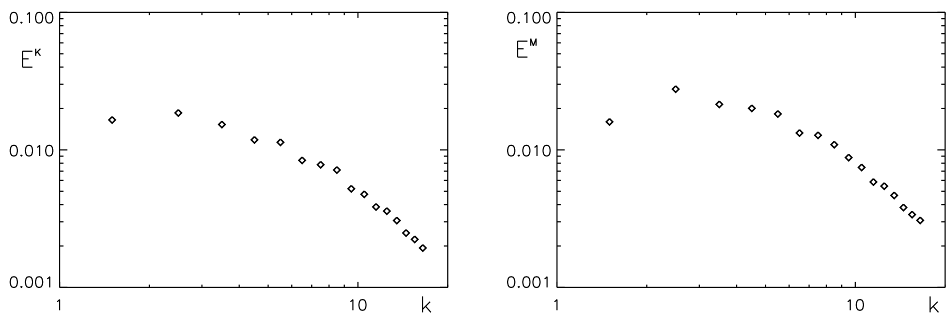

Figure 2: Typical kinetic (left) and magnetic (right) energy spectra obtained using DNS of the MHD equations for the mixing layer.

of the fields without any intrusive experimental probe. For instance, both the energy spectrum in wave space and the energy profile in the position space can be accessed simultaneously without difficulty. An example is shown below for the MHD mixing layer [4].

In this problem, the velocity and magnetic fields initially correspond to two homogeneous turbulent regions with different kinetic and magnetic energy connected through a transition region, the mixing layer. The study of this flow allows to explore a situation which is simultaneously inhomogeneous and non stationary. The DNS corresponding to Figures 1 and 2 are rather modest in size : They have been performed with 128^3 grid points. However, much bigger simulation with up to $512 \times 1024 \times 512$ grid points (i.e. more than 250 millions grid points) are under progress.

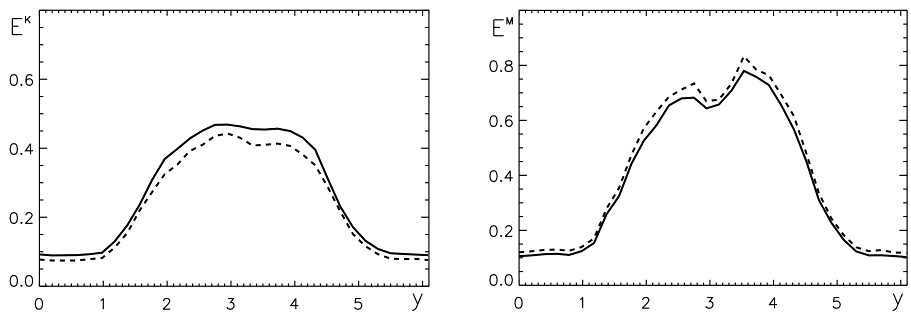

Figure 3: Comparison between profiles predicted by the DNS (diamonds) and by the LES (dashed line) for the kinetic (left) and magnetic (right) energies along the inhomogeneous direction in the mixing layer.

3 Large Eddy Simulation of MHD turbulence

From the discussion in the preceding chapter, it is obvious that high Reynolds number MHD turbulence cannot be accessed by the DNS method. This difficulty has prompted the development of approximate numerical methods in which modelling plays an important role.

Amongst these approximate numerical approaches, the large-eddy simulation (LES) is considered as one of the most promising techniques. The LES is based on a scale separation in which the variables describing the largest structures of turbulence are computed directly while the influence of the smallest structures is accounted for through a model. This scale separation is obviously motivated by the limitation of computer power. However, it also corresponds to major differences in the behaviours observed at large and small scales in a turbulent flow. Indeed, large scales usually contain the main part of the kinetic and magnetic energy carried by the medium. Their knowledge is often sufficient to characterize most of the properties of practical interest in a turbulent conducting fluid. Also, the large scales strongly depend on the geometry of the flow and, as such, would require a case-by-case modelling treatment. On the contrary, the small scales carry less energy and behave in a fairly "universal" way. Their modelling seems thus easier and more promising.

Within the framework of LES, the scale separation is achieved by applying a filter to the evolution equations for both the velocity and the magnetic fields. Traditionally, this filter is denoted by an overbar symbol and the

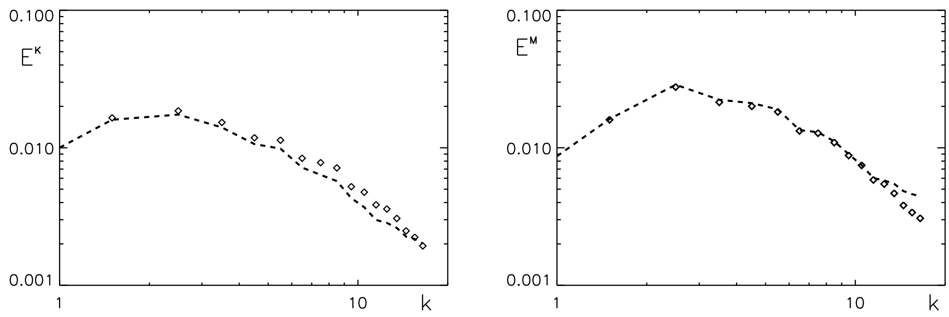

Figure 4: Comparison between energy spectra in the mixing layer predicted by the DNS (diamonds) and by the LES (dashed line) for the kinetic (left) and magnetic (right) parts.

filtered MHD equations thus read:

$$\partial_t \overline{u}_i = -\partial_j(\overline{\overline{u}_j \overline{u}_i} - \overline{\overline{b}_j \overline{b}_i}) + \nu \nabla^2 \overline{u}_i - \partial_i \overline{p} - \partial_j \overline{\tau}^u_{ji} \quad (6)$$

$$\partial_t \overline{b}_i = -\partial_j(\overline{\overline{u}_j \overline{b}_i} - \overline{\overline{b}_j \overline{u}_i}) + \eta \nabla^2 \overline{b}_i - \partial_j \overline{\tau}^b_{ij} \quad (7)$$

Due to the nonlinearities, the filtered equations are not closed and they contain terms that represent the effect of the small, filtered scales on the large and numerically resolved scales. These terms are given by: $\overline{\tau}^u_{ij} = (\overline{u_i u_j} - \overline{\overline{u}_i \overline{u}_j}) - (\overline{b_i b_j} - \overline{\overline{b}_i \overline{b}_j})$ and $\overline{\tau}^b_{ij} = (\overline{u_i b_j} - \overline{\overline{u}_i \overline{b}_j}) - (\overline{u_j b_i} - \overline{\overline{u}_j \overline{b}_i})$. In order to close equations (6) and (7) , we need to model these terms and the most traditional strategy is to use a turbulent viscosity model in the filtered velocity equation and a turbulent magnetic diffusivity in the filtered induction equation. Details about this method are given in Refs [4, 5].

In Figures 3 and 4 we compare numerical predictions from LES and DNS. Without entering the details of the physics of the MHD mixing layer, we can see that the agreement between the two numerical approaches is very satisfactory taking into account that the DNS is more than hundred times more time and memory consuming than the LES. It is also important to note that the agreement is obtained for both physical space (energy profiles) and wave space (energy spectra) predictions. According to the experience gained in LES of Navier-Stokes turbulences, the LES could be considered as a perfect tool for exploring high Reynolds number MHD turbulence.

4 Conclusion

We have briefly reviewed two numerical strategies for exploring MHD turbulence. The first one, usually referred to as the direct numerical simulation, is based on the time and space discretization of the evolution equations for both the velocity and the magnetic fields. As long as the numerics is correct, a DNS provides a perfect tool for exploring all the details of the turbulent fields. However, due to the very wide range of wavelengths that are excited in turbulent flows, accurate DNS can only be achieved for moderate Reynolds number flows.

An alternate approach, based on a scale separation between the energy containing scales and the fluctuating small scales has been proposed for exploring large number Reynolds flows. This approach, referred to as large eddy simulation, has been developed mainly in the context of the Navier-Stokes equation [6]. However, recent applications of LES to MHD turbulence [5, 7] have proved that this method is also very well suited for exploring turbulence in conducting fluids.

It is a pleasure to acknowledge here the very positive influence of Professor Radu Balescu on the development of the research in plasma physics and, in particular, in the study of turbulent phenomena in plasmas.

References

[1] U. Frisch. *Turbulence : The Legacy of A.N. Kolmogorov*. Cambridge University Press, 1995.

[2] D. Biskamp and W.-C. Müller. Decay laws in three-dimensional magnetohydrodynamic turbulence. *Phys. Rev. Lett.*, 83:2195–2198, 1999.

[3] Y. Ponty, A.D. Gilbert, and A.M. Soward. Kinematic dynamo action in large magnetic reynolds number flows driven by shear and convection. *J. Fluid Mech.*, 435:261–287, 2001.

[4] O. Debliquy, B. Knaepen, and D. Carati. Large eddy simulation of a shear-free magnetohydrodynamic mixing layer. In C. Liu, L. Sakell, and T. Beutner, editors, *Proceedings of the third AFOSR international conference on DNS/LES*, pages 815–824, Arlington, USA, 2001.

[5] O. Agullo, W.-C. Muller, B. Knaepen, and D. Carati. Large eddy simulation for decaying magnetohydrodynamics turbulence with dynamic subgrid modelling. *Phys. Plasmas*, 7:3502–3505, 2001.

[6] R.S. Rogallo and P. Moin. Numerical simulation of turbulent flows. *Annu. Rev. Fluid Mech.*, 16:99–137, 1984.

[7] Y. Zhou and G. Vahala. Aspects of subgrid modelling and large-eddy simulation of magnetohydrodynamic turbulence. *J.Plasma Physics*, 45:239–249, 1991.